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Uniform vector bundles on a projective space

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Introduction.

It is well known [1] that a vector bundle E on P^1 is isomorphic to a direct sum of line bundles $\mathcal{O}_{P^1}(a_1)\oplus\cdots\oplus\mathcal{O}_{P^1}(a_p)$ where $a_1, \cdots, a_p(a_1\geqq\cdots\geqq a_p)$ are uniquely determined, and we say that E is of type $(a_1, \cdots, a_p)$.

Then according to Schwarzenberger, we have the following notion:

DEFINITION. A vector bundle E on P^n is called a uniform vector bundle if the type of $i_l^*(E)$ is independent of the choice of a line l in P^n, where i_l is the natural immersion: $i_l: P^1\cong l\hookrightarrow P^n$. Furthermore in relation to a uniform vector bundle on P^n, we have another notion.

DEFINITION. A vector bundle on P^n is called homogeneous if it is invariant with respect to any automorphism of P^n.

Obviously, a homogeneous vector bundle is uniform. Conversely, is a uniform vector bundle on P^n homogeneous? Van de Ven [9] proved that every uniform vector bundle of rank 2 on P^n ($n\geqq 2$) is isomorphic to one of $\mathcal{O}_{P^n}(a)\oplus\mathcal{O}_{P^n}(b)$ and $T_{P^2}\otimes\mathcal{O}_{P^2}(c)$ in the complex case, where T_{P^2} is the tangent bundle of P^2. Consequently every uniform vector bundle is homogeneous in this case.

The aim of this paper is to generalize the above result to higher dimension. Our main theorem which will be proved in § 2 is as follows:

MAIN THEOREM. *Assume that E is a uniform vector bundle on P^n of type $(a_{11}, \cdots, a_{1r_1}, a_{21}, \cdots, a_{2r_2}, \cdots, a_{\alpha 1}, \cdots, a_{\alpha r_\alpha})$ with $n\geqq 2$, $r=\sum_{i=1}^{\alpha} r_i\geqq 2$, $a_1>a_2>\cdots>a_\alpha$, and $a_{ij}=a_i$ $(j=1, \cdots, r_i)$. Then we have the following:*

1) *If $n>r$, then E is isomorphic to $\bigoplus_{i=1}^{\alpha}\mathcal{O}_{P^n}(a_i)^{\oplus r_i}$.*

2) *If $n=r$, we have two cases as follows:*

(i) *If $r_i\geqq 2$ for $i=1, \alpha$ and if n is either 2 or odd, then E is isomorphic to $\bigoplus_{i=1}^{\alpha}\mathcal{O}_{P^n}(a_i)^{\oplus r_i}$.*

(ii) *If either r_1 or r_α is 1, and if the characteristic of the ground field is zero, then E is isomorphic to one of $T_{P^n}\otimes\mathcal{O}_{P^n}(a)$, $\Omega^1_{P^n}\otimes\mathcal{O}_{P^n}(b)$ and $\bigoplus_{i=1}^{\alpha}\mathcal{O}_{P^n}(a_i)^{\oplus r_i}$ with some integers a, b where T_{P^n} and $\Omega^1_{P^n}$ are the tangent bundle and the cotangent bundle of P^n, respectively.*

Consequently, under the assumption of this theorem a uniform vector bundle is homogeneous.

If the characteristic of the ground field is positive in (2) (ii), there are many uniform vector bundles other than those mentioned there (Remark 2.2).

According to S. Mari, there are uniform but non-homogeneous vector bundles of rank r on P^n if $r>n$.

Notation. Throughout this paper k is an algebraically closed field of characteristic p ($\geqq 0$). A variety S is a reduced and irreducible algebraic k-scheme. We use the terms "vector bundle" and "locally free sheaf" interchangeably. Furthermore, $\mathcal{O}_{P^n}(1)$ is the line bundle corresponding to the divisor class of hyperplanes in the n-dimensional projective space P^n. If E is a vector bundle on S, the $P(E)$ denotes $\mathrm{Proj}\,(S(E))$, where $S(E)$ is the $\mathcal{O}_S$-symmetric algebra of E. $Gr(n, d)$ denotes the Grassmann variety parametrizing d-dimensional linear subspaces of the n-dimensional projective space P^n. $E(n, d)$ (resp. $Q(n, d)$) denotes the universal subbundle (resp. universal quotient bundle) over $Gr(n, d)$. If l is a line in P^n and E is a vector bundle on P^n then we use the notation $E|_l$ instead of $i_l^*(E)$ where i_l is the natural immersion $i_l: P^1 \cong l \hookrightarrow P^n$.

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§ 1. Preliminaries and a theorem of Tango.

In order to prove our theorem, the following easy proposition plays an important role.

PROPOSITION 1. *For a point p of the n-dimensional projective space ($n\geqq 2$), consider the monoidal transformation $\varphi: X\to P^n$ with center p. Then X is isomorphic to the P^1-bundle $\pi: \mathrm{Proj}\,(\mathcal{O}_{P^{n-1}}\oplus\mathcal{O}_{P^{n-1}}(1))\to P^{n-1}$. Moreover the fibers of the bundle are in one to one correspondence via φ with the lines going through the point p.*

PROOF. Let m_p be the sheaf of ideals defining the point p in P^n. Then we have the following exact sequence: $\mathcal{O}_{P^n}^{\oplus n}\to m_p\otimes\mathcal{O}_{P^n}(1)\to 0$. By taking φ^*, we get an exact sequence: $\mathcal{O}_X^{\oplus n}\to\varphi^*(m_p)\otimes\varphi^*\mathcal{O}_{P^n}(1)\to 0$. Now we know easily that $\varphi^*(m_p)$ is isomorphic to the line bundle $\mathcal{O}_X(-\varphi^{-1}(p))$ where $\mathcal{O}_X(-\varphi^{-1}(p))$ is the sheaf of ideals defining the exceptional divisor $\varphi^{-1}(p)$ of φ. Let L be $\varphi^*(m_p)\otimes\varphi^*(\mathcal{O}_{P^n}(1))$. The line bundle L on X induces a morphism $\pi: X\to P^{n-1}$. By the construction of φ, any fiber of π is P^1. Moreover the exceptional divisor $\varphi^{-1}(p)$ of φ induces a section of π. So π is a P^1-bundle [**3**]. Moreover X is isomorphic to $P(E)$ where E is a vector bundle of rank 2 on P^{n-1} ([**4**], Lemma 1.2). For $n=2$, $E\cong\mathcal{O}_{P^1}(a)\oplus\mathcal{O}_{P^1}(b)$ because E is a vector bundle of rank 2 on P^1 [**1**]. E has a quotient line bundle corresponding to the section of π mentioned above. So E is isomorphic to a direct sum of line bundles, because $H^1(P^{n-1}, M)=0$ for any line bundle M on P^{n-1} $(n\geqq 3)$. Therefore, we see that X is isomorphic to Proj $(\mathcal{O}_{P^{n-1}}\oplus\mathcal{O}_{P^{n-1}}(a))$ for $n\geqq 2$, where a is a non-negative integer. Let $\bar{L}$ be the linebundle corresponding to the exceptional divisor $\varphi^{-1}(p)$ of φ. It is easy to see that restriction of $\bar{L}$ to the $\varphi^{-1}(p)$ $(\cong P^{n-1})$ is isomorphic to $\mathcal{O}_{P^{n-1}}(-1)$. Therefore we obtain $a=1$. The latter half of this proposition is obvious. q. e. d.

The next proposition, which can be thought of as a universal version of Proposition 1 with p running over all points of P^n, is useful for our proof of (2) (ii) of the Main Theorem.

PROPOSITION 2. *Let Δ be the diagonal of $P^n\times P^n$ $(n\geqq 2)$ and let $\bar{\varphi}:\bar{X}\to P^n\times P^n$ be the monoidal transformation with center Δ. Then there exists a morphism $\bar{\pi}:\bar{X}\to P(\Omega^1_{P^n})$ which induces an isomorphism $\bar{\varphi}^{-1}(\Delta)\simeq P(\Omega^1_{P^n})$ such that $\bar{\pi}$ is a P^1-bundle. If we denote by q the canonical morphism $P(\Omega^1_{P^n})\to P^n$ and by p_1 the first projection $P^n\times P^n\to P^n$, we have the following commutative diagram:*

$$\begin{array}{ccc} P^n\times P^n & \xleftarrow{\ \bar{\varphi}\ } & \bar{X} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \bar{\pi}} \\ P^n & \xleftarrow{\ q\ } & P(\Omega^1_{P^n}) \end{array}$$

Moreover we see that

1) *the canonical immersion $\bar{\varphi}^{-1}(\Delta)\hookrightarrow\bar{X}$ induces a section of $\bar{\pi}$,*

2) *for every point t of P^n, $\bar{\varphi}^{-1}[p_1^{-1}(t)]=\bar{\pi}^{-1}(q^{-1}(t))$ where $\bar{\varphi}^{-1}[p_1^{-1}(t)]$ is the proper transform of $p_1^{-1}(t)$ by $\bar{\varphi}$, and $\bar{\varphi}^{-1}|_{\bar{\varphi}^{-1}[p_1^{-1}(t)]}:\bar{\varphi}^{-1}[p_1^{-1}(t)]\to p_1^{-1}(t)$ is the monoidal transformation with center $t\times t$.*

PROOF. Let $p_2: P^n\times P^n\to P^n$ be the second projection. Put $f_i=p_i\bar{\varphi}$ for $i=1, 2$ and $L=f_2^*\mathcal{O}_{P^n}(1)\otimes\mathcal{O}_{\bar{X}}(-\bar{\varphi}^{-1}(\Delta))$ where $\mathcal{O}_{\bar{X}}(-\bar{\varphi}^{-1}(\Delta))$ is the sheaf of ideals defining $\bar{\varphi}^{-1}(\Delta)$ in $\bar{X}$. Since f_1 is flat, L is f_1-flat. On the other hand, we obtain that $f_1^{-1}(s)\cong s\times\varphi_s^{-1}(s)$ and $L|_{f_1^{-1}(s)}=\mathcal{O}_{P^n}(1)\otimes\varphi_s^{-1}(m_s)$, where $\varphi_s:\varphi_s^{-1}(P^n)\to P^n$ is the monoidal transformation with center s and m_s is the sheaf of ideals defining a point s in P^n. By virtue of these facts and Proposition 1, $\dim_{k(s)}H^0(f_1^{-1}(s), L|_{f_1^{-1}(s)})=n$ for every point s of P^n. Hence by the base change theorem of Grothendieck [**6**], $f_{1*}L$ is a vector bundle of rank n on P^n. Furthermore it is easy to see that there is a surjective homomorphism; $f_1^*f_{1*}L\to L\to 0$. So we have a canonical closed immersion $\varphi:\bar{X}(\cong P(L))\to P(f_1^*f_{1*}L)$. On the other hand there is a canonical projection $\psi: P(f_1^*f_{1*}L)\to P(f_{1*}L)$ because $P(f_1^*f_{1*}L)\cong f_1^*P(f_{1*}L)$ by virtue of the functoriality of Proj. Put $\psi\varphi=\bar{\pi}$. Let us show that $\bar{\pi}:\bar{X}(\cong P(L))\to P(f_{1*}L)$ is a P^1-bundle. In the first place, by restricting the surjective homomorphism; $f_1^*f_{1*}L\to L\to 0$ to every fiber of f_1, we obtain the following commutative diagram:

$$\begin{array}{ccccc} \bar{X} & \overset{\varphi}{\hookrightarrow} & P(f_1^* f_{1*}L) & \overset{\psi}{\longrightarrow} & P(f_{1*}L) \\ \uparrow\cup & & \uparrow\cup & & \uparrow\cup \\ X_s & \hookrightarrow & P^{n-1}\times X_s & \longrightarrow & P^{n-1} \end{array}$$

(s is a point of P^n and $X_s=f_1^{-1}(s)$).

By virtue of the above diagram and Proposition 1, we know that every fiber of $\bar{\pi}$ is P^1. Also we obtain that $\bar{\varphi}^{-1}(\Delta)\cap X_s$ induces a section of P^1-bundle: $\bar{\pi}|_{X_s}: X_s\to P^{n-1}$ by Proposition 1. So noting that $\mathcal{O}_{\bar{X}}(\bar{\varphi}^{-1}(\Delta))$ induces a sheaf of hyperplane in $\bar{\pi}^{-1}(q)$ for every point q of $P(f_{1*}L)$, we know that $\bar{\pi}$: $X\to P(f_{1*}L)$ is a P^1-bundle ([3]) and the canonical immersion $\bar{\varphi}^{-1}(\Delta)\hookrightarrow X$ induces a section of $\bar{\pi}$. On the other hand, since $\bar{\varphi}^{-1}(\Delta)\cong P(I/I^2)$ where I is the sheaf of ideals defining Δ in $P^n\times P^n$, $\bar{\varphi}^{-1}(\Delta)\cong P(\Omega^1_{P^n})$ by virtue of the definition of $\Omega^1_{P^n}$. So $\bar{\pi}: \bar{X}\to P(\Omega^1_{P^n})$ ($\cong P(f_{1*}L)$) is a P^1-bundle. (2) is obvious by virtue of the above facts. q. e. d.

If we lift a uniform vector bundle on P^n to X in Proposition 1, then it contains a subbundle. In fact,

PROPOSITION 3. *Let E be a uniform vector bundle of rank r on P^n such that $E|_l$ is isomorphic to $\mathcal{O}_{P^1}^{\oplus r_1}\oplus(\bigoplus_{i=2}^{\alpha}\mathcal{O}_{P^1}(a_i)^{\oplus r_i})$ for all lines l with $0>a_2>\cdots>a_\alpha$. Then $\pi^*\pi_*\varphi^*E$ is a subbundle of φ^*E of rank r_1, where φ and π are the same as in Proposition* 1.

PROOF. Let s be a point of P^{n-1}. By Proposition 1, $\varphi^*E|_{\pi^{-1}(s)}$ is isomorphic to $\mathcal{O}_{P^1}^{\oplus r_1}\oplus(\bigoplus_{i=2}^{\alpha}\mathcal{O}_{P^1}(a_i)^{\oplus r_i})$. Thus for every point s of P^{n-1}, we have $H^0(\pi^{-1}(s), \mathcal{O}_{P^1}^{\oplus r_1}\oplus(\bigoplus_{i=2}^{\alpha}\mathcal{O}_{P^1}(a_i)^{\oplus r_i}))\cong k^{\oplus r_1}$. Hence by the base change theorem of Grothendieck [6], $\pi_*\varphi^*E$ is a vector bundle of rank r_1 on P^{n-1} and for every point s of P^{n-1}, we have $\pi_*\varphi^*E\otimes k(s)\simeq H^0(\pi^{-1}(s), \varphi^*E|_{\pi^{-1}(s)})$. This means that $\pi^*\pi_*\varphi^*E$ is a subbundle of rank r_1 of φ^*E. q. e. d.

In the sequel we denote the vector bundle $\varphi^*E/\pi^*\pi_*\varphi^*E$ on X by F.

REMARK 1.1. The conclusion of Proposition 3 holds good under a weaker assumption on E that $E|_l$ is isomorphic to $\mathcal{O}_{P^1}^{\oplus r_1}\oplus(\bigoplus_{i=2}^{\alpha}\mathcal{O}_{P^1}(a_i)^{\oplus r_i})$ for all lines l in P^n going through a fixed point p.

REMARK 1.2. Let D_p be the exceptional variety $\varphi^{-1}(p)$ of φ. Then the exact sequence;

$$0\longrightarrow \pi^*\pi_*\varphi^*E\longrightarrow \varphi^*E\longrightarrow F\longrightarrow 0$$

obtained in Proposition 3 gives rise to an exact sequence;

$$0\longrightarrow \pi_*\varphi^*E\longrightarrow \mathcal{O}_{P^{n-1}}^{\oplus r}\longrightarrow F|_{P^{n-1}}\longrightarrow 0$$

on D_p $(\cong P^{n-1})$ $(r=\sum_{i=1}^{\alpha} r_i)$. It follows that there is a morphism $f: P^{n-1}\to Gr(r-1, r-r_1-1)$ such that

$$0 \longrightarrow \pi_*\varphi^*E \longrightarrow \mathcal{O}_{P^{n-1}}^{\oplus r} \longrightarrow F|_{P^{n-1}} \longrightarrow 0$$

is isomorphic to the pull back of

$$0 \longrightarrow E(r-1, r-r_1-1) \longrightarrow \mathcal{O}_{Gr(r-1,r-r_1-1)}{}^{\oplus r} \longrightarrow Q(r-1, r-r_1-1) \longrightarrow 0$$

by f.

The following theorem due to Tango ([7], [8]) is used essentially in the proof of (1) and (2) (i) of our Main Theorem.

THEOREM OF TANGO. *Let f be a morphism from P^N to $Gr(m, d)$ with $m-1>d>0$. If* 1) *$N>m$ or if* 2) *$N=m$ and md is even except the case $m=5$ and $d=2$, then f is a constant map.*

The following proposition gives us a sufficient condition for a vector bundle to be generated by its global sections. We shall make use of this in our proof of (2) (ii) of our Main Theorem.

PROPOSITION 5. *Let S be a variety and E a vector bundle of rank r on S. Assume that V is an m-dimensional linear subspace of $H^0(S, E)$ with $m\geqq r+1$. If moreover $\{x\in S \mid s(x)=0\}\cap\{x\in S \mid s'(x)=0\}=\emptyset$ for every pair of elements s, s' in V which are linearly independent over k, then the vector bundle E is generated by elements of V.*

PROOF. Let $s_1, \cdots, s_m$ be a basis for V. Assume that for some closed point x of S, $E\otimes k(x)$ cannot be generated by the m elements $s_1(x), \cdots, s_m(x)$.

From this we derive a contradiction. In the first place, $\sum_{i=1}^{m} ks_i(x)$ is an l-dimensional linear subspace of $E\otimes k(x)$ with $l\leqq r-1$. We may assume that $s_1(x), \cdots, s_l(x)$ are linearly independent over k in $E\otimes k(x)$. Since $s_1(x), \cdots, s_l(x), s_{l+1}(x)$ (resp. $s_1(x), \cdots, s_l(x), s_{l+2}(x)$) are linearly dependent over k, there are $l+1$ elements $\lambda_1, \cdots, \lambda_l, \lambda_{l+1}$ in k (resp. $\lambda'_1, \cdots, \lambda'_l, \lambda'_{l+2}$) such that $\sum_{i=1}^{l+1}\lambda_i s_i(x)=0$ (resp. $\sum_{i=1, \neq l+1}^{l+2}\lambda'_i s_i(x)=0$). Since $s_1(x), \cdots, s_l(x)$ are linearly independent, we get $\lambda_{l+1}\neq 0$ (resp. $\lambda'_{l+2}\neq 0$). This and the assumption that $s_1, \cdots, s_m$ form a basis for V imply that $\sum_{i=1}^{l+1}\lambda_i s_i$, $\sum_{i=1, \neq l+1}^{l+2}\lambda'_i s_i$ are linearly independent over k. On the other hand $\{y\in S \mid \sum_{i=1}^{l+1}\lambda_i s_i(y)=0\}\cap\{y\in S \mid \sum_{i=1, \neq l+1}^{l+2}\lambda'_i s_i(y)=0\}$ is non-empty, because it contains x. But this is a contradiction to the assumption. Thus we complete our proof.

§ 2. Proof of the Main Theorem.

First note that, for every uniform vector bundle $\bar{E}$, we can express $\bar{E}$ or $\bar{E}^{V}$ as $E(a)$ with an integer a and a uniform vector bundle E which enjoys the properties that $a_1=0$ and $r_1 \leqq r_\alpha$ in the notation in our Main Theorem.

Therefore we may assume that $a_1=0$ and $r_1 \leqq r_\alpha$. With these notation and assumptions, we have only to prove the following:

1) If $n>r$, then $E \cong \bigoplus_{i=1}^{\alpha} \mathcal{O}_{P^n}(a_i)^{\oplus r_i}$.

a) If $n=r$, we have two cases as follows:

(i) If $r_1 \geqq 2$ and if n is either 2 or odd, then $E \cong \bigoplus_{i=1}^{\alpha} \mathcal{O}_{P^n}(a_i)^{\oplus r_i}$.

(ii) If $r_1=1$ and if the characteristic is zero, then $E \cong T_{P^n}(-2)$ or $\bigoplus_{i=1}^{\alpha} \mathcal{O}_{P^n}(a_i)^{\oplus r_i}$.

First, let us prove (1) and (2) (i) of our Main Theorem. We employ the notation of Proposition 1, and we shall prove these by induction on α. When $\alpha=1$ (i.e. $E|_l=\mathcal{O}_{P^1}^{\oplus r_1}$), $\pi^*\pi_*\varphi^*E \cong \varphi^*E$ by Proposition 3. Restricting it to D_p ($\cong P^{n-1}$), we know $\pi_*\varphi^*E \cong \mathcal{O}_{P^{n-1}}^{\oplus r_1}$. So $\varphi^*E \cong \mathcal{O}_X^{\oplus r_1}$. Hence we conclude $E=\varphi_*\varphi^*E = \varphi_*\mathcal{O}_X^{\oplus r_1}=\mathcal{O}_{P^n}^{\oplus r_1}$ by the projection formula and $\varphi_*\mathcal{O}_X=\mathcal{O}_{P^n}$. If $\alpha \geqq 2$, we have the following exact sequence by Remark 1.2:

$$0 \longrightarrow \pi_*\varphi^*E \longrightarrow \mathcal{O}_{P^{n-1}}^{\oplus r} \longrightarrow F|_{P^{n-1}} \longrightarrow 0$$

which provides us with a morphism $f: P^{n-1} \to Gr(r-1, r-r_1-1)$ such that $\pi_*\varphi^*E = f^*(E(r-1, r-r_1-1))$. By virtue of the theorem of Tango, f is a constant map, whence $\pi_*\varphi^*F \cong \mathcal{O}_{P^{n-1}}^{\oplus r_1}$. Thus we have the following exact sequence: $0 \to \mathcal{O}_X^{\oplus r_1} \to \varphi^*E \to F \to 0$. By the canonical isomorphism; $H^0(P^n, E) \cong H^0(X, \varphi^*E)$, we can find r_1 elements of $H^0(P^n, E)$, $s_1, \cdots, s_{r_1}$ such that for every point u in P^n, $s_1(u), \cdots, s_{r_1}(u)$ are linearly independent over $k(u)$. This implies that $s_1, \cdots, s_{r_1}$ generate a trivial subbundle $E'(\cong \mathcal{O}_{P^n}^{\oplus r_1})$ of rank r_1 of E. Now we claim that the quotient bundle $F'=E/E'$ is a uniform vector bundle with the property that for all lines $l \subset P^n$, $F'|_l \cong \bigoplus_{i=2}^{\alpha} \mathcal{O}_{P^1}(a_i)^{\oplus r_i}$. Indeed, we have the following exact sequence for all lines $l \subset P^n$;

$$0 \longrightarrow \mathcal{O}_{P^1}^{\oplus r_1} \longrightarrow \bigoplus_{i=1}^{\alpha} \mathcal{O}_{P^1}(a_i)^{\oplus r_i} \longrightarrow F'|_l \longrightarrow 0 .$$

Since $H^0(P^1, \mathrm{Hom}(\mathcal{O}_{P^1}, \mathcal{O}_{P^1}(b)))=0$ for $b<0$ and since $a_i<0$ for $i \geqq 2$ by our assumption, we see that $F'|_l \cong \bigoplus_{i=2}^{\alpha} \mathcal{O}_{P^1}(a_i)^{\oplus r_i}$ for all lines $l \subset P^n$, that is, F' is a uniform vector bundle. Hence by applying the induction assumption to $F'(-a_2)$, we see that $F' \cong \bigoplus_{i=2}^{\alpha} \mathcal{O}_{P^n}(a_i)^{\oplus r_i}$. Since $H^1(P^n, L)=0$ for $n \geqq 2$ and for any line bundle L, we conclude that $E\cong\bigoplus_{i=1}^{\alpha}\mathcal{O}_{P^n}(a_i)^{\oplus r_i}$. Consequently, we get (1) and (2) (i) of our Main Theorem.

REMARK 2.1. As was pointed out in Remark 1.1 these conclusions (1) and (2) (i) hold good under the assumption that $E|_l$ is isomorphic to $\bigoplus_{i=1}^{\alpha}\mathcal{O}_{P^1}(a_i)^{\oplus r_i}$ for every line l in P^n through a fixed point p.

Next, we shall prove (2) (ii) of our Main Theorem. We maintain the notation of Proposition 2. Let s be a point of $P(\Omega^1_{P^n})$. Using (2) of Proposition 2 for $q(s)=t(\in P^n)$, we obtain that $\bar{\varphi}^*p_1^*E|_{\bar{\pi}^{-1}(s)}\cong\mathcal{O}_{P^1}\bigoplus_{i=2}^{\alpha}\mathcal{O}_{P^1}(a_i)^{\oplus r_i}$. Since $H^0(\bar{\pi}^{-1}(s), \bar{\varphi}^*p_1^*E|_{\bar{\pi}^{-1}(s)})\cong k$, we get the following exact sequence in the same way as in Proposition 3;

$$0\longrightarrow\bar{\pi}^*\bar{\pi}_*\bar{\varphi}^*(p_1^*E)\longrightarrow\bar{\varphi}^*(p_1^*E)\longrightarrow Q\longrightarrow 0$$

with a vector bundle Q on $\bar{X}$. This gives rise to a closed immersion $i: P(Q)\hookrightarrow P(\bar{\varphi}^*p_1^*E)$. On the other hand, $\bar{\varphi}$ induces an isomorphism: $\bar{X}-\bar{\varphi}^{-1}(\Delta)\cong P^n\times P^n-\Delta$. Hence if Y is the closure of $P(Q)|_{\bar{X}-\bar{\varphi}^{-1}(\Delta)}$ in $P(p_1^*E)$, then we get a commutative diagram.

$$\begin{array}{ccccc} P(Q) & \subset\!\longrightarrow & P(\bar{\varphi}^*(p_1^*E)) & & \\ \downarrow & & \downarrow\bar{\varphi} & & \\ Y & \subset\!\longrightarrow & P(p_1^*E)\cong P^n\times P(E) & \xrightarrow{p_1} & P^n \end{array}$$

Now, we put $Y_t=Y|_{p_2^{-1}(t)}$ and $t=p$ in Proposition 3. Then we see that Y_t is the closure of $P(F)|_{X-\varphi^{-1}(t)}$ in $P(E)$ ([5] Lecture 7, corollary 2). Also Y_t is an effective divisor. Furthermore if $\{U_\lambda\}$ is a sufficiently small open covering of P^n, $Y_t|_{U_\lambda}$ can be expressed as $\sum_{i=0}^{n-1}g_i^\lambda X_i^\lambda=0$, where $X_0^\lambda, \cdots, X_{n-1}^\lambda$ is a homogeneous coordinate system of $U_\lambda\times P^{n-1}\cong P(E)|_{U_\lambda}$ and $g_i^\lambda\in\Gamma(U_\lambda, \mathcal{O}_{P^n})$ for $0\leqq i\leqq n-1$. Therefore the fibre of Y_t at $t\times t$ is either P^{n-2} or P^{n-1}.

$$\begin{array}{ccc} P(F) & \subset\!\longrightarrow & P(\varphi^*E) \\ \downarrow & & \downarrow \\ Y_t & \subset\!\longrightarrow & P(E) \end{array}$$

Now we have two cases.

1) Assume that there is a point $t\in P^n$ such that the fiber of Y_t at $t\times t$ is P^{n-2}. Then Y_t is a P^{n-2}-bundle over P^n in $P(E)$. This implies that there is an exact sequence;

$$0\longrightarrow L\longrightarrow E\longrightarrow F_t\longrightarrow 0$$

where F_t is of rank $n-1$ and $P(F_t)=Y_t$. When $n=2$, L and F_t are line bundles on P^2. Consequently E is isomorphic to $L\oplus F_t$ because $H^1(P^2, M)=0$ for any line bundle M on P^2. By virtue of the assumption that $E|_l\cong\mathcal{O}_{P^1}\oplus\mathcal{O}_{P^1}(a_2)$ for all lines l, we have $E\cong\mathcal{O}_{P^2}\oplus\mathcal{O}_{P^2}(a_2)$.

When $n\geqq 3$, it is easy to see that $\varphi^{-1}(P(F_t))\supset P(F)$. On the other hand, both $\varphi^{-1}(P(F_t))$ and $P(F)$ are P^{n-2}-bundles in $P(\varphi^*E)$. Hence $\varphi^{-1}(P(F_t))=P(F)$. By the functoriality of Proj, $\varphi^{-1}(P(F_t))=P(\varphi^*F_t)$, that is, $P(\varphi^*F_t)=P(F)$. This means that $F\cong\varphi^*F_t\otimes L$ for some line bundle L on X. Now every line bundle on X is isomorphic to $L_0^{\otimes b}\otimes\pi^*\mathcal{O}_{P^{n-1}}(a)$ where L_0 is a line bundle corresponding to the divisor $\varphi^{-1}(t)$ ($\cong P^{n-1}$) and a, $b\in Z$.

Let us show that $a=b=0$. Let S ($\cong P^{n-1}$) be a section of π such that $S\cap\varphi^{-1}(t)=\emptyset$ (in fact, such an S exists because $X\cong P(\mathcal{O}_{P^{n-1}}\oplus\mathcal{O}_{P^{n-1}}(1))$. Since $F|_S\cong\varphi^*F_t|_S$ by a property of φ, we have $L|_S=\mathcal{O}_S$. Thus we have $a=0$, because $L|_S=L_0^{\otimes b}|_S\otimes\pi^*\mathcal{O}_{P^{n-1}}(a)|_S=\mathcal{O}_S(a)$. On the other hand, the exact sequence on $X: 0\to\pi^*\pi_*\varphi^*E\to\varphi^*E\to F\to 0$ gives rise to the exact sequence on the section $\varphi^{-1}(t)$ ($\cong P^{n-1}$) $0\to\pi_*\varphi^*E\to\mathcal{O}_{P^{n-1}}^{\oplus r}\to F|_{P^{n-1}}\to 0$. Hence by virtue of the fact that $F\cong\varphi^*F_t\otimes L_0^{\otimes b}$, we have an exact sequence:

$$0\longrightarrow\pi_*\varphi^*E\longrightarrow\mathcal{O}_{P^{n-1}}^{\oplus r}\longrightarrow\mathcal{O}_{P^{n-1}}^{\oplus r-1}\otimes\mathcal{O}_{P^{n-1}}(-b)\longrightarrow 0\,.$$

Consequently $b=0$. Thus we have $\pi_*\varphi^*E\cong\mathcal{O}_{P^{n-1}}$. This provides us with an exact sequence,

$$0\longrightarrow\mathcal{O}_X\longrightarrow\varphi^*E\longrightarrow F\longrightarrow 0\,.$$

Then we obtain the following exact sequence in the same way as in the proof of (1) and (2) (i);

$$0\longrightarrow\mathcal{O}_{P^n}\longrightarrow E\longrightarrow F'\longrightarrow 0\,.$$

Similarly we see that F' is a uniform vector bundle of rank $n-1$. So $F'\cong\bigoplus_{i=2}^{\alpha}\mathcal{O}_{P^n}(a_i)^{\oplus r_i}$. Since $H^1(P^n, L')=0$ for $n\geqq 2$ and for any line bundle L', we know that E is isomorphic to $\mathcal{O}_{P^n}\bigoplus_{i=2}^{\alpha}\mathcal{O}_{P^n}(a_i)^{\oplus r_i}$.

2) Assume that the fiber of Y_t at $t\times t$ is P^{n-1} for all $t\in P^n$. As was shown, Y_t is linearly equivalent to $Y_{t'}$, for all t, $t'\in P^n$, because $\{Y_t|t\in P^n\}$ is parametrized by P^n. Y_t is a divisor of $P(E)$ for all $t\in P^n$ and $Y_t\otimes\operatorname{Spec}k(x)$ is isomorphic to $P_{k(x)}^{n-2}$ for the generic point x of of the base space P^n of $P(E)$ because of the definition of Y_t. On the other hand, we know that a line bundle of $P(E)$ is expressed as $\mathcal{O}_{P(E)}(1)^{\otimes a}\otimes\sigma^*\mathcal{O}_{P^n}(b)$ where a and b are integers and σ is the canonical projection: $P(E)\to P^n$. Consequently $\{Y_t|t\in P^n\}$ induces an $(n+1)$-dimensional subspace V of $H^0(P(E), \mathcal{O}_{P(E)}(1)\otimes\sigma^*\mathcal{O}_{P^n}(h))$ for some integer h. Also we have the following isomorphism: $H^0(P(E), \mathcal{O}_{P(E)}(1)\otimes\sigma^*\mathcal{O}_{P^n}(h))\cong H^0(P^n, E(h))$ by virtue of Leray's spectral sequence. Moreover the assumption on the fiber of Y_t at $t\times t$ implies that for every element s in V, $\operatorname{supp}s$ is one point, and $\{x\in P^n|s(x)=0\}\cap\{x\in P^n|s'(x)=0\}=\emptyset$ for all pairs s, s' of V which are independent over k. By Proposition 5, we have an exact sequence

$\mathcal{O}_{P^n}^{\oplus(n+1)}\to E(h)\to 0$, which implies that there is a morphism $f: P^n\to Gr(n, n-1)$ such that the exact sequence $\mathcal{O}_{P^n}^{\oplus(n+1)}\to E(h)\to 0$ is isomorphic to the pull back of $\mathcal{O}_{Gr(n,n-1)}^{\oplus(n+1)}\to Q(n, n-1)\to 0$ by f (Remark 1.2). Since $Gr(n, n-1)=P^n$, we see that $Q(n, n-1)\cong T_{P^n}(-1)$. Thus we have $E(h)=f^*T_{P^n}(-1)$. We know that, for every section $\bar{s}$ of $H^0(P^n, T_{P^n}(-1))$, the scheme defined by $\bar{s}=0$ is isomorphic to Spec $k(x)$ which is the subscheme of P^n for some point $x\in P^n$ (Remark 2.3). On the other hand, for every section $\bar{s}$ of $H^0(P^n, T_{P^n}(-1))$, $f^*\bar{s}\in V$ and supp $f^*\bar{s}$ is one point. Since f is a proper morphism and since every fiber of f consists of one point, f is a finite birational morphism, if the characteristic of k is zero. By Zariski's Mains Theorem, f is an isomorphism, that is, $E(h)\cong T_{P^n}(-1)$. We know that $T_{P^n}(-1)|_l\cong\mathcal{O}_{P^1}(1)\oplus\mathcal{O}_{P^1}^{\oplus(n-1)}$ for all lines l in P^n. Hence $h=1$ and we get the required result that $E\cong T_{P^n}(-2)$.

REMARK 2.2. When the characteristic p of the ground field is positive, let $f: P_1^n\to P_2^n$ be the Frobenius map [2] with $P_1^n=P_2^n=P^n$. For any line l_2 in P_2^n, $f^{-1}(l_2)_{\text{red}}$ is a line of P_1^n where $f^{-1}(l_2)_{\text{red}}$ is the reduced scheme of $f^{-1}(l_2)$. On the other hand any line l_1 in P_1^n is the reduced scheme of $f^{-1}(l_2)$ for some line l_2 in P_2^n. Therefore, $f^*T_{P^n}(-1)$ is a uniform vector bundle where $f^*T_{P^n}(-1)|_l$ for any line l in P_1^n is $\mathcal{O}_{P^1}(p)\oplus\mathcal{O}_{P^1}^{\oplus(n-1)}$. It is easy to show that $f^*T_{P^n}(-1)$ is an indecomposable vector bundle.

REMARK 2.3. We will see easily that in the following canonical exact sequence;

$$0\longrightarrow \mathcal{O}_{P^n}(-1)\xrightarrow{f}\bigoplus_{i=0}^{n}\mathcal{O}_{P^n}e_i\longrightarrow T_{P^n}(-1)\longrightarrow 0,$$

$f\otimes\mathcal{O}_{P^n}(1)$ is given by $1\mapsto X_ie_i$ where $X_0,\cdots,X_n$ is a homogeneous coordinate system of P^n. From the above exact sequence, we have an isomorphism: $\bigoplus_{i=0}^{n} ke_i\cong H^0(P^n, T_{P^n}(-1))$. Hence, for a non-zero section $s=\sum_{i=0}^{n}a_ie_i$ of $T_{P^n}(-1)$, the closed subscheme defined by $s=0$ is defined by the set of equations $X_ia_j-X_ja_i$ $(i=0,\cdots,n;\ j=0,\cdots,n)$. This implies the closed subscheme is Spec $k(x)$, where $x=(a_0:\cdots:a_n)$.

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