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On the Fourier coefficients of automorphic forms at various cusps and some applications to Rankin's convolution

Dedicated to Professor Tikao Tatuzawa on his 60th birthday

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For an automorphic form $f(z)$ on the upper-half-plane with respect to a Fuchsian group Γ, we can consider the Fourier expansion of $f(z)$ at each parabolic cusp of Γ. In a certain case, as we shall see in this paper, there is a simple relation among the Fourier coefficients at the various cusps. We treat the case where $\Gamma=\Gamma_0(N)$ and $f(z)$ is a new form of Neben type in the sense of Atkin, Lehner and Miyake ([**2**], [**7**]). We assume that the level N is square-free. Then, the Fourier coefficients are canonically defined, since a cusp is transformed to any other cusp by an element of the normalizer of Γ. In this situation, we can state the relation in an explicit form (Theorems 1 and 2), namely if the coefficients at one cusp are given then we can immediately know all n-th coefficients at other cusps, whether n and the level N are coprime or not.

In the latter part (§ **2**), after some preparations on Eisenstein series we shall give some applications of the above result to Rankin's convolution of Dirichlet series, namely we generalize a result of Ogg [**10**] to a case of Neben type (Theorem 3), and we also try to remove the condition of *prime* discriminant in Naganuma's work [8] (Theorem 4). On Rankin's convolution, Jacquet ([**5**]) seems to have treated in a more general point of view, and his theory may contain our results in essential.

NOTATION. As usual, by $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{C}$, we denote the ring of rational integers, the field of rational numbers and the field of complex numbers, respectively. $SL_2(A)$ is the special linear group of degree two over a ring A. We denote a linear fractional transformation by $\sigma z=(az+b)(cz+d)^{-1}$ for a real matrix $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of positive determinant, and write $f|_k\sigma=(\det\sigma)^{k/2}(cz+d)^{-k}f(\sigma z)$ for a function $f(z)$ on the upper-half-plane, while the number k may be often omitted in $f|_k\sigma$. For general notions of automorphic forms, we may refer to Shimura's book [**12**].

1.1. Let N be a square-free positive integer, and consider the group

$$\Gamma=\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\boldsymbol{Z})\,;\ c\equiv 0 \pmod N\right\}.$$

The set of all cusps of Γ is $\boldsymbol{Q}^*=\boldsymbol{Q}\cup\{\infty\}$. Every element of $\boldsymbol{Q}^*$ is uniquely expressed as a reduced fraction with positive numerator, e. g. $\infty=1/0$, with only one exception $0=0/1$, and these expressions will be kept throughout this paper. Two cusps are equivalent (relative to Γ) if and only if the denominators have the same greatest common divisor with N, so each equivalence class of cusps is in one-to-one correspondence with each ordered decomposition $N=MM_1$ of two positive divisors. We may say a cusp $\kappa=\kappa_2/\kappa_1$ belongs to M_1-class if g. c. d. of κ_1 and N is M_1, e. g. ∞ belongs to N-class. For each decomposition $N=MM_1$ and any cusp $\kappa=\kappa_2/\kappa_1$ of M_1-class, we can take a typical matrix ω_κ which transforms κ to ∞:

$$\text{(1)}\qquad \omega_\kappa=\begin{pmatrix} 1 & \\ & M \end{pmatrix}\alpha_\kappa \quad\text{with}\quad \alpha_\kappa=\begin{pmatrix} M\lambda_1 & \lambda_2 \\ -\kappa_1 & \kappa_2 \end{pmatrix}\in SL_2(\boldsymbol{Z}) \quad\text{and}\quad \lambda_i\in\boldsymbol{Z}.$$

In general, for a divisor M of N, we define the matrix W_M, which exists uniquely up to right or left Γ-multiplication, by

$$\text{(2)}\qquad W_M=\begin{pmatrix} M\xi & \eta \\ N\zeta & M\rho \end{pmatrix} \text{ with the determinant } M, \text{ and } \xi,\eta,\zeta,\rho\in\boldsymbol{Z}.$$

This W_M normalizes the group Γ and $M^{-1}W_M^2\in\Gamma$, furthermore $W_M=W_{M'}W_{M''}$ if $M=M'M''$ divides N. Since ω_κ is one of such type (2), we see that every cusp can be transformed to any other cusp by an element of the normalizer of Γ.

Let us consider a congruence equation

$$\text{(3)}\qquad u+v\equiv 1 \pmod N \quad\text{and}\quad uv\equiv 0 \pmod N.$$

Each solution $(\bmod N)$ is also in one-to-one correspondence with each ordered decomposition $N=MM_1$, in such way as $u\equiv 0 \pmod M$ and $v\equiv 0 \pmod{M_1}$. Let (u, v) be the solution of (3) corresponding to $N=MM_1$. The map $m\mapsto um+vm'$, where $mm'\equiv 1 \pmod N$, is an involutive automorphism of the group $(\boldsymbol{Z}/N\boldsymbol{Z})^\times$ of reduced residue classes mod N, which we denote by γ_M. Similarly we denote by β_M the canonical injection $m\mapsto u+vm$ of $(\boldsymbol{Z}/M\boldsymbol{Z})^\times$ into $(\boldsymbol{Z}/N\boldsymbol{Z})^\times$. The meaning of these maps is clear if we recall the isomorphism $(\boldsymbol{Z}/N\boldsymbol{Z})^\times\cong(\boldsymbol{Z}/M\boldsymbol{Z})^\times\times(\boldsymbol{Z}/M_1\boldsymbol{Z})^\times$. For an arbitrary Dirichlet character χ mod N, so a character of $(\boldsymbol{Z}/N\boldsymbol{Z})^\times$, we define

$$\text{(4)}\qquad {}^M\chi=\chi\circ\gamma_M \quad\text{and}\quad \chi_M=\chi\circ\beta_M,$$

then ${}^M\chi$ (resp. χ_M) is also a character mod N (resp. M). For example, ${}^N\chi=\bar{\chi}$, and ${}^M\chi=\chi$ if χ is real. If χ is given by Jacobi's symbol $\left(\frac{*}{N}\right)$ for odd and square-free N, χ_M is $\left(\frac{*}{M}\right)$.

For a decomposition $N=MM_1$, consider (2) and (3) simultaneously, then it can be seen that $u=M\xi\rho$ and $v=-M_1\eta\zeta$. This implies the following relation:

$$(5)\qquad \pi(W_M\sigma W_M^{-1})=\gamma_M(\pi(\sigma)) \qquad \text{for every } \sigma\in\Gamma.$$

Here π denotes the canonical homomorphism of Γ to $(\boldsymbol{Z}/N\boldsymbol{Z})^\times$ given by $\pi(\sigma)=d$ for $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$.

1.2. We here quote a result by Miyake ([7]) with some generalization. $\mathcal{S}_k(N,\chi)$ and $\mathcal{S}_k^0(N,\chi)$ will denote the space of integral cusp forms of Neben type χ, of weight k with respect to $\Gamma=\Gamma_0(N)$, and the subspace of its essential part, respectively. In particular, $f|_k\sigma=\chi(\pi(\sigma))f$ for $f\in\mathcal{S}_k(N,\chi)$ and every $\sigma\in\Gamma$.

LEMMA 1. *By mapping f to $f|_kW_M$, we have $\mathcal{S}_k(N,\chi)\cong\mathcal{S}_k(N,{}^M\chi)$ and $\mathcal{S}_k^0(N,\chi)\cong\mathcal{S}_k^0(N,{}^M\chi)$.*

PROOF. By virtue of (5) and the involutive property of W_M, the first isomorphism is clear. For the second isomorphism, it is enough to show the same on each complementary space. Namely we show $f|W_M$ is an old form if $f=g|\begin{pmatrix} m & \\ & 1\end{pmatrix}$ for some $g\in\mathcal{S}_k(N_0,\chi)$, $N_0|N$, $N_0\neq N$ and $m|(N/N_0)$ (so that χ is defined mod N_0). Let us define $N=N_0mm_1$, $(N_0,M)=M_0$, $(m,M)=m_3$, $(m_1,M)=m_4$, $M=M_0m_3m_4$, $m=m_3\mu$ and $m_1=m_4\mu_1$, then for $W_M=\begin{pmatrix} M\xi & \eta \\ N\zeta & M\rho\end{pmatrix}$, $\begin{pmatrix} m & \\ & 1\end{pmatrix}W_M=W_{M_0}^{(N_0)}\begin{pmatrix} m_4\mu & \\ & 1\end{pmatrix}\begin{pmatrix} m_3 & \\ & m_3\end{pmatrix}$, where $W_{M_0}^{(N_0)}=\begin{pmatrix} M_0\xi' & \eta' \\ N_0\zeta' & M_0\rho'\end{pmatrix}$ with $\xi'=m_3\xi$, $\eta'=\mu\eta$, $\zeta'=\mu_1\zeta$ and $\rho'=m_4\rho$. Thus $f|W_M=(g|W_{M_0}^{(N_0)})|\begin{pmatrix} m_4\mu & \\ & 1\end{pmatrix}$ and here $g|W_{M_0}^{(N_0)}\in\mathcal{S}_k(N_0,{}^M\chi)$. The last relation follows from the fact that the solution of (3) corresponding to $M|N$ is also the solution corresponding to a divisor M_0 of the lower level N_0. q.e.d.

1.3. For a prime number p, we mean by Hecke operator $T(p,\chi)$ the endomorphism of $\mathcal{S}_k(N,\chi)$ given by

$$f|T(p,\chi)=p^{k/2-1}\left\{\chi(p)f\Big|\begin{pmatrix} p & \\ & 1\end{pmatrix}+\sum_{j=0}^{p-1}f\Big|\begin{pmatrix} 1 & j \\ & p\end{pmatrix}\right\}.$$

The following relation is elementary:

LEMMA 2. *For each decomposition $N=MM_1$,*

$$(6)\qquad T(p,\chi)\circ W_M=\chi_M(p)W_M\circ T(p,{}^M\chi) \qquad \textit{for every prime } p\nmid M.$$

PROOF. The relation does not depend on a choice of W_M of (2), hence we may add the condition $\eta\equiv\zeta\equiv 0 \pmod p$. This is possible, since $(M, M_1p^2)=1$. First, we can see $\begin{pmatrix} p & \\ & 1\end{pmatrix}W_M\begin{pmatrix} p & \\ & 1\end{pmatrix}^{-1}W_M^{-1}=\sigma_1\in\Gamma$ and $\pi(\sigma_1)=M\xi\rho-M_1\eta\zeta p^{-1}$, so we have

$$\chi(p)f\Big|\begin{pmatrix} p & \\ & 1\end{pmatrix}W_M=\chi_M(p)^M\chi(p)f|W_M\begin{pmatrix} p & \\ & 1\end{pmatrix}.$$

Next, for $0\leqq j\leqq p-1$, let us determine $0\leqq l\leqq p-1$ by $j\rho\equiv l\xi \pmod p$ (this is one-to-one correspondence), then $\begin{pmatrix} 1 & j\\ & p\end{pmatrix}W_M\begin{pmatrix} 1 & l\\ & p\end{pmatrix}^{-1}W_M^{-1}=\sigma_2\in\Gamma$ and $\pi(\sigma_2)=M\xi\rho-M_1\eta\zeta p$, so we have

$$f\Big|\begin{pmatrix} 1 & j\\ & p\end{pmatrix}W_M=\chi_M(p)f\Big|W_M\begin{pmatrix} 1 & l\\ & p\end{pmatrix}.$$

q. e. d.

In a particular case $M=N$, (6) is the well known formula:

(7) $$T(p, \chi)\circ W_N=\chi(p)W_N\circ T(p, \bar\chi) \qquad \text{for every prime } p\nmid N.$$

Now, let $f\in\mathcal{S}_k^0(N, \chi)$ be a new form, that is, a common eigen function of all $T(p, \chi): f|T(p, \chi)=a_pf$ for each prime p. Put $g=f|W_N$, then by (7) and the well known relation (see [12], p. 87): $\bar\chi(p)a_p=\bar a_p$ for $p\nmid N$, we have $g|T(p, \bar\chi)=\bar a_pg$ for $p\nmid N$. On the other hand, for the operator $f|K=\overline{f(-\bar z)}$, it holds that $T(p, \chi)\circ K=K\circ T(p, \bar\chi)$. So the theory of new form ([7], p. 188) combined with the above fact implies that g coincides with $f|K$ up to a constant multiple. Namely, we have

(8) $$g|T(p, \bar\chi)=\bar a_pg \qquad \text{for } \textit{every} \text{ prime } p.$$

This classical result by Hecke can be generalized in the following way.

For each decomposition $N=MM_1$, in addition to the relation (6), the complementary relation

(9) $$T(p, \bar\chi)\circ W_{M_1}=\bar\chi_{M_1}(p)W_{M_1}\circ T(p, {}^M\chi) \qquad \text{for every prime } p\nmid M_1,$$

holds (note ${}^M\chi={}^{M_1}\bar\chi$). On one hand, (6) implies

$$(f|W_M)|T(p, {}^M\chi)=\bar\chi_M(p)a_p(f|W_M),$$

for $p\nmid M$. On the other hand, since $f|W_M$ coincides with $g|W_{M_1}$ up to a constant multiple, we obtain, by (8) and (9),

$$(f|W_M)|T(p, {}^M\chi)=\chi_{M_1}(p)\bar a_p(f|W_M),$$

for $p\nmid M_1$. Here it should be remarked that $\bar\chi_M(p)a_p=\chi_{M_1}(p)\bar a_p$ if $p\nmid N$. Thus we have proved the following

THEOREM 1. *Let $f(z)$ be a new form of $\mathcal{S}_k(N, \chi)$ and $f|T(p, \chi)=a_pf$ for every prime p. For each decomposition $N=MM_1$, put $f_M=f|W_M$. Then, f_M is* *also a new form of* $\mathcal{S}_k(N, {}^M\chi)$, *and* $f_M|T(p, {}^M\chi)=a_p^{(M)}f_M$ *for every prime* p. *The eigen value* $a_p^{(M)}$ *is given by*

$$a_p^{(M)}=\begin{cases}\bar{\chi}_M(p)a_p & \text{if } p\nmid M,\\ \chi_{M_1}(p)\bar{a}_p & \text{if } p\nmid M_1.\end{cases}$$

1.4. We now treat the Fourier coefficients of new forms. Let $f(z)$ be again a new form of $\mathcal{S}_k(N, \chi)$ with the Fourier expansion

$$f(z)=\sum_{n=1}^{\infty}a_n e^{2\pi i n z} \qquad (a_1=1),$$

so that we have

$$f|T(p, \chi)=a_p f \qquad \text{for every } p.$$

For each decomposition $N=MM_1$, let us define $a_n^{(M)}$ by

$$(10)\qquad \begin{cases} a_n^{(M)}=\bar{\chi}_M(n)a_n & \text{if } (n, M)=1,\\ a_n^{(M)}=\chi_{M_1}(n)\bar{a}_n & \text{if } (n, M_1)=1,\\ a_{nm}^{(M)}=a_n^{(M)}a_m^{(M)} & \text{if } (n, m)=1,\end{cases}$$

and put

$$(11)\qquad f^{(M)}(z)=\sum_{n=1}^{\infty}a_n^{(M)}e^{2\pi i n z},$$

then, by virtue of Theorem 1, we can easily see

$$(12)\qquad f|W_M=\lambda f^{(M)}$$

with some constant λ. It should be remarked that the relation (10) is compatible with

$$\sum_{n=1}^{\infty}a_n^{(M)}n^{-s}=\prod_p(1-a_p^{(M)}p^{-s}+{}^M\chi(p)p^{k-1-2s})^{-1}.$$

By definition, the Fourier coefficients of $f(z)$ at a cusp $\kappa=\kappa_2/\kappa_1$ of M_1-class are those of $f|\omega_\kappa$ at the cusp ∞, where ω_κ is given by (1). Obviously, the coefficients do not depend on a choice of ω_κ. In view of Theorem 1, together with (10) and (11), the problem is reduced to a computation of the value of λ in (12).

In the case of prime level N and primitive character χ, the value of λ is known, e.g. due to Hecke or Miyake in [8] p. 553, and the latter method is applicable to our case when the divisor M is prime. Namely,

LEMMA 3. *For a decomposition* $N=qQ$ *with a prime factor* q, *let* $W_q=\begin{pmatrix} q & 1\\ N\zeta & q\rho\end{pmatrix}$ *with the determinant* q *and* $\zeta, \rho\in\boldsymbol{Z}$. *Then* $f|W_q=\lambda_q f^{(q)}$ *and the value* λ_q *is given by*

$$\lambda_q = \begin{cases} C(\chi_q)q^{-k/2}\bar{a}_q & \text{if } \chi_q \text{ is primitive}, \\ -q^{1-k/2}\bar{a}_q & \text{if } \chi_q \text{ is principal}, \end{cases} \tag{13}$$

where $C(\chi_q)=\sum_{h(\mathrm{mod}\, q)}\chi_q(h)e^{2\pi i(h/q)}$. *In either case,* $|\lambda_q|=1$ *and in the latter case* $\lambda_q^2=\bar{\chi}_Q(q)$.

PROOF. Since $Q\zeta\equiv -1 \pmod q$, there exists l such that $l(1+jQ\zeta)\equiv 1 \pmod q$ if $j\not\equiv 1 \pmod q$. Thus each $0\leqq j\leqq q-1$ $(j\neq 1)$ is in one-to-one correspondence with each $1\leqq l\leqq q-1$. And then we can see $\begin{pmatrix}1 & j\\ & q\end{pmatrix}W_q=\sigma_1\begin{pmatrix}1 & l\\ & q\end{pmatrix}\begin{pmatrix}q & \\ & 1\end{pmatrix}$ with $\sigma_1\in\Gamma$ and $\pi(\sigma_1)=q\rho-Q\zeta l$, so that $\chi(\pi(\sigma_1))=\chi_q(l)$. If $j=1$, $\begin{pmatrix}1 & 1\\ & q\end{pmatrix}W_q=\sigma_2 W_q\begin{pmatrix}q & \\ & 1\end{pmatrix}$ with $\sigma_2\in\Gamma$ and $\pi(\sigma_2)=q^2\rho-Q\zeta$, so that $\chi(\pi(\sigma_2))=\chi_Q(q)$. Hence we have

$$\begin{aligned} f|T(q,\chi)\circ W_q &= q^{k/2-1}\sum_{j=1}^{q-1} f\Big|\begin{pmatrix}1 & j\\ & q\end{pmatrix}W_q \\ &= q^{k/2-1}\sum_{n=1}^{\infty}a_n\{\sum_{l=1}^{q-1}\chi_q(l)e^{2\pi i(nl/q)}\}e^{2\pi inz} \\ &\quad +q^{k-1}\chi_Q(q)\lambda_q\sum_{n=1}^{\infty}a_n^{(q)}e^{2\pi inqz}. \end{aligned} \tag{14}$$

We must here consider two cases. If χ_q is a primitive character mod q (i.e. q divides the conductor of χ), then $\sum_{l=1}^{q-1}\chi_q(l)e^{2\pi i(nl/q)}=C(\chi_q)\bar{\chi}_q(n)$, so that the right-hand-side of (14) becomes

$$q^{k/2-1}C(\chi_q)\sum_{n=1}^{\infty}\bar{\chi}_q(n)a_n e^{2\pi inz}+q^{k-1}\chi_Q(q)\lambda_q\sum_{n=1}^{\infty}a_n^{(q)}e^{2\pi inqz}. \tag{15}$$

If χ_q is a principal character mod q (i.e. q does not divide the conductor of χ), then $\sum_{l=1}^{q-1}\chi_q(l)e^{2\pi i(nl/q)}=q-1$ or -1 according as $q|n$ or not, so that (14) is equal to

$$-q^{k/2-1}\sum_{n=1}^{\infty}a_n e^{2\pi inz}+\sum_{n=1}^{\infty}\{q^{k/2}a_{nq}+q^{k-1}\chi_Q(q)\lambda_q a_n^{(q)}\}e^{2\pi inqz}. \tag{16}$$

On the other hand,

$$f|T(q,\chi)\circ W_q=a_q f|W_q=a_q\lambda_q\sum_{n=1}^{\infty}a_n^{(q)}e^{2\pi inz}. \tag{17}$$

Comparing (15) and (16) with (17) in the coefficients of $e^{2\pi iz}$ and also $e^{2\pi iqz}$, we obtain (13) and so forth. q.e.d.

For a general $W_q=\begin{pmatrix}q\xi & \eta\\ N\zeta & q\rho\end{pmatrix}$, it holds that $W_q=\sigma\begin{pmatrix}q & 1\\ N\eta\zeta & q\xi\rho\end{pmatrix}$ with $\sigma\in\Gamma$, and $\pi(\sigma)=q\rho-Q\zeta$, hence it can be seen that the value λ in $f|W_q=\lambda f^{(q)}$ is equal to $\chi(q\rho-Q\zeta)\lambda_q$, where λ_q is given by (13). Moreover, for a general divisor M of N, we can compute the value λ of (12) inductively by means of the relation $W_M=W_{M'}W_{M''}$ for $M=M'M''$. In fact, we can prove the following by induction with respect to the number of prime factors of M, but we shall omit the detail.

THEOREM 2. *Let* $f(z)=\sum_{n=1}^{\infty}a_n e^{2\pi i n z}$ $(a_1=1)$ *be a new form of* $\mathcal{S}_k(N, \chi)$. *For each decomposition* $N=MM_1$, *define* W_M *and* $f^{(M)}(z)$ *by* (2) *and* (11), *respectively. Then* $f|W_M=\lambda f^{(M)}$ *and* λ *is given by*

$$\lambda=\chi(M\rho-M_1\zeta)\prod_{q\mid M}\{\chi_q(M/q)\lambda_q\}\ ,$$

where λ_q *is the value given by* (13).

If we take ω_κ of (1) instead of W_M, we can put $\chi_M(\kappa_1)\chi_{M_1}(M\kappa_2)$ for $\chi(M\rho-M_1\zeta)$. Thus, Theorems 1 and 2 altogether give a rule to compute the Fourier coefficients at various cusps for a new form. A similar problem for a modular form given by a theta series of a positive definite quadratic form has been treated by Kitaoka [**6**].

At the end of this section we add an immediate consequence of Lemma 3:

COROLLARY. *Let* a_q *be an eigen value of* $T(q, \chi)$ *on* $\mathcal{S}_k^0(N, \chi)$ *for each prime factor* q *of* N. *Then* $|a_q|^2$ *is* q^{k-1} *or* q^{k-2} *according as* q *divides the conductor of* χ *or not. In the latter case,* $a_q^2=\chi'(q)q^{k-2}$, *where* χ' *is the primitive character associated with* χ.

A similar result of this can be found in Ogg [**9**].

2.1. For later use, we here introduce the Eisenstein (and Epstein) series, and we deal only with the case of $\Gamma=\Gamma_0(N)$ with a square-free level N, so that the situation is the same as in 1.1. We also use the following notations: $y(\sigma z)=\mathrm{Im}\,\sigma z=(ad-bc)y|cz+d|^{-2}$, $J(\sigma, z)=e^{i\arg(cz+d)}$ for $z=x+iy$ $(y>0)$, and a real matrix $\sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ of positive determinant. Let χ be a character mod N, then χ is naturally regarded as a character of Γ by $\chi(\sigma)=\chi(\pi(\sigma))$. Let r be an integer with the same parity of χ, i. e. $\chi(-1)=(-1)^r$. For a cusp $\kappa=\kappa_2/\kappa_1$ of M_1-class, an Eisenstein series at κ is defined by

$$E_\kappa(z, s, r, \chi)_N=\sum_{\sigma\in\Gamma_\kappa\backslash\Gamma}\chi(\sigma)J(\alpha_\kappa\sigma, z)^r y(\alpha_\kappa\sigma z)^s\ , \tag{18}$$

where α_κ is defined in (1), $s\in\boldsymbol{C}$ with $\mathrm{Re}\,s>1$ and Γ_κ denotes the stabilizer of κ in Γ. For abbreviation we use the notation:

$$E_\kappa(z, s, r, \chi)_N|\sigma=J(\sigma, z)^r E_\kappa(\sigma z, s, r, \chi)_N\ ,$$

for a real matrix σ of positive determinant. We have

$$E_\kappa(z, s, r, \chi)_N|\sigma=\bar{\chi}(\sigma)E_\kappa(z, s, r, \chi)_N\ , \tag{19}$$

for every $\sigma\in\Gamma$, and

(20) $$E_{\kappa}(z, s, r, \chi)_N = M^s E_{\infty}(z, s, r, {}^M\chi)_N \mid \omega_{\kappa}\,,$$

where we should recall that ω_{κ} of (1) normalizes Γ and its determinant is M, as well as the relations (4) and (5). On the other hand, let us define another function by

(21) $$E_M(z, s, r, \chi)_N = y^s \sum_{\nu/\mu} \bar{\chi}_M(\mu)\chi_{M_1}(\nu)(\mu z+\nu)^r \mid \mu z+\nu\mid^{-(2s+r)}\,,$$

where in the summation ν/μ runs over all cusps of M_1-class, i. e. $\nu/\mu \in \boldsymbol{Q}^*$ such that $(\mu, N)=M_1$. Then by a simple computation we have

(22) $$E_M(z, s, r, \chi)_N = \bar{\chi}_M(-\kappa_1)\chi_{M_1}(\kappa_2)E_{\kappa}(z, s, r, \chi)_N\,.$$

Consequently we obtain

(23) $$E_M(z, s, r, \chi)_N = \bar{\chi}_M(M_1\zeta)\chi_{M_1}(\rho)M^s E_1(z, s, r, {}^M\chi)_N \mid W_M\,,$$

for an arbitrary W_M of type (2).

In order to get the functional equation of $E_M(z, s, r, \chi)_N$, we first treat a case of primitive character. To avoid confusion, we replace the letters N, M and χ by A, B and φ, respectively.

LEMMA 4. *Let φ be a primitive character* mod A *and let $r \in \boldsymbol{Z}$ such that $\varphi(-1)=(-1)^r$. For a coprime decomposition $A=BB_1$, put*

(24) $$\begin{aligned} &E_B^*(z, s, r, \varphi)_A \\ &\quad = \varphi_B(-1)C(\varphi_B)B^{-s-1/2}A^{(3/2)s}\pi^{-s}\Gamma\Big(s+\frac{|r|}{2}\Big)L(2s, \varphi)E_B(z, s, r, \varphi)_A\,, \end{aligned}$$

where L is the Dirichlet L-function. Then $E_B^*(z, s, r, \varphi)_A$ can be analytically continued to the whole complex s-plane and is entire if either $A\neq 1$ or $r\neq 0$. In the case $A=1$ and $r=0$, it has two simple poles at $s=1$ and 0. Moreover, it satisfies the functional equation

(25) $$E_B^*(z, s, r, \varphi)_A = E_{B_1}^*(z, 1-s, r, \varphi)_A\,.$$

PROOF. There are several ways to prove this, and it is rather well known in a special case of $B=A$ and $B_1=1$ (e. g. [13]). So we assume this case, then the general case follows immediately by operating W_B (of level A) to both sides. q. e. d.

Now we return to the case of N and χ. Let A be the conductor of χ, so that χ_A is the primitive character (mod A) associated with χ. Then we can easily show by definition

(26) $$E_B(z, s, r, \chi_A)_A = \sum_{B\mid M,\ B_1\mid M_1} E_M(z, s, r, \chi)_N$$

for a decomposition $A=BB_1$. In this sum, M runs over all factors of N with the condition that M and the cofactor M_1 are divisible by B and B_1, respectively. The number of such M's is always that of positive divisors of N/A. Thus we have

LEMMA 5. *The functional equation* (25), *for* $\varphi=\chi_A$, *in Lemma* 4 *is also valid when* E_B *in* (24) *is replaced by the right-hand-side of* (26).

Furthermore, we can get the complete system of the functional equations for the Eisenstein series E_M's in the case of $\Gamma_0(N)$, i. e. the equation of matric type whose size is the number of positive divisors of N, by operating W_L (of level N), $L|(N/A)$, to both sides of (26), (24) and (25). For later use we restate Lemma 5, operated by W_N, in the convenient form in the case of the trivial character i. e. $A=1$.

LEMMA 5′. *Let* $\Gamma=\Gamma_0(N)$ *with the square-free level* N, *and define*

$$E_1(z, s, r)_N=\sum_{\sigma\in\Gamma_\infty\backslash\Gamma} J(\sigma, z)^r y(\sigma z)^s,$$

and put

$$E^*(z, s, r)_N=(N/\pi)^s\Gamma\left(s+\frac{|r|}{2}\right)\zeta(2s)\{\sum_{M|N} M^{-s}E_1(z, s, r)_N|W_M\}$$

with the Riemann zeta function ζ, *then* $E^*(z, s, r)_N$ *can be analytically continued to the whole complex s-plane and is entire for every non-zero integer* r. *If* $r=0$, *it has two simple poles at* $s=1$ *and* 0. *Moreover, it satisfies the functional equation:* $E^*(z, s, r)_N=E^*(z, 1-s, r)_N$.

2.2. It seems that so-called Rankin's method (or convolution), as well as Mellin's transform, has nowadays become one of the most fundamental ways of treating Dirichlet series and automorphic forms (see [**1**], [**3**], [**8**], [**13**], and [**14**], for example). Rankin ([**11**]) has treated originally the cases of $SL_2(\boldsymbol{Z})$ and the principal congruence subgroups. The case of $\Gamma_0(N)$ is considered by Ogg ([**10**]) on Haupt type. As an application of our argument in the preceding sections we can now deal with such a general case as $f(z)$ and $g(z)$ are new forms of arbitrarily different levels, weights and characters, under only one condition that the least common multiple of two levels is square-free. We, however, explain only a typical case as an example in this section.

Let N be a square-free, positive integer, χ be a real, primitive character mod N, and $f(z)=\sum_{n=1}^{\infty} a_n e^{2\pi inz}$ $(a_1=1)$ be a common eigen cusp form of all Hecke operators on $\mathcal{S}_k(N, \chi)$. Let us put

$$(27)\qquad A_p=a_{p^2}-\chi(p)p^{k-1} \qquad \text{for every prime } p\nmid N,$$

and define an Euler product $\phi(s)$ by

$$(28)\qquad \phi(s)=\prod_p \phi_p(s)\,;$$

$$\psi_p(s)=\begin{cases}(1-A_p p^{-s}+p^{2k-2-2s})^{-1}(1-\chi(p)p^{k-1-s})^{-2} & \text{if } p\nmid N,\\ (1-a_p^2 p^{-s})^{-1}(1-\bar{a}_p^2 p^{-s})^{-1} & \text{if } p\,|\,N,\end{cases}$$

where p runs over all primes, and $s\in\boldsymbol{C}$ with $\operatorname{Re}s>k$. Further we put

$$\psi^*(s)=N^s(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+1)\psi(s)\,, \tag{29}$$

then we have the following

THEOREM 3. *$\psi^*(s)$ can be analytically continued to the whole complex s-plane and is entire if $N\neq 1$. In the case of $N=1$, it has two simple poles at $s=k$ and $k-1$. Moreover, it satisfies the functional equation $\psi^*(s)=\psi^*(2k-1-s)$.*

PROOF. Let $g(z)=\overline{f(-\bar{z})}=\sum_{n=1}^{\infty}\bar{a}_n e^{2\pi i n z}$, and consider an integral

$$J(s_1)=\int_{\mathscr{D}} y^k f(z)\overline{g(z)}E^*(z,\,s_1,\,0)_N y^{-2}dx\,dy\,,$$

where $z=x+iy$ $(y>0)$, $\mathscr{D}$ is a fundamental domain of $\Gamma=\Gamma_0(N)$ and $E^*(z,\,s_1,\,0)_N$ is given in Lemma 5′. The integral is well defined independently of a choice of $\mathscr{D}$, and is absolutely convergent for all $s_1\in\boldsymbol{C}$ with possible simple poles at $s_1=1$ and 0 (see [**11**] or [**13**]). In fact, $J(s_1)$ is entire if $N\neq 1$, because $\int_{\mathscr{D}} y^{k-2}f(z)\overline{g(z)}dx\,dy=0$, while it has simple poles at $s_1=1$ and 0 if $N=1$. Also by means of Lemma 5′, we have

$$J(s_1)=J(1-s_1)\,. \tag{30}$$

On the other hand, $J(s_1)=(N/\pi)^{s_1}\Gamma(s_1)\zeta(2s_1)\sum_{M|N}I_M(s_1)$ with

$$I_M(s_1)=M^{-s_1}\int_{\mathscr{D}} y^k f(z)\overline{g(z)}\{E_1(z,\,s_1,\,0)_N\,|\,W_M\}y^{-2}dx\,dy\,.$$

Since W_M normalizes Γ, we have

$$I_M(s_1)=M^{-s_1}\int_{\mathscr{D}} y^k f_M(z)\overline{g_M(z)}E_1(z,\,s_1,\,0)_N y^{-2}dx\,dy\,,$$

where we put $f_M=f\,|\,W_M$, and $g_M=g\,|\,W_M$ (note $(f\bar{g})\,|\,W_M=(f\bar{g})\,|\,W_M^{-1}$). So, for s_1 with $\operatorname{Re}s_1>1$,

$$\begin{aligned}I_M(s_1)&=M^{-s_1}\int_{\mathscr{D}} y^k f_M(z)\overline{g_M(z)}\sum_{\sigma\in\Gamma_\infty\backslash\Gamma} y(\sigma z)^{s_1}y^{-2}dx\,dy\\ &=M^{-s_1}\int_0^\infty\Big\{\int_0^1 f_M(z)\overline{g_M(z)}dx\Big\}y^{s_1+k-2}dy\,.\end{aligned}$$

Thus we need the Fourier coefficients of $f(z)$ and $g(z)$ not only at the infinity but also at all cusps, and they can be obtained by means of Theorems 1 and 2. Namely, we get by putting $s=s_1+k-1$,

$$I_M(s_1)=(4\pi)^{-s}\Gamma(s)M^{-s}\bar{a}_M^2$$

$$\times\prod_{p\mid M}\{\sum_{n=0}^{\infty}\bar{a}_{pn}^2p^{-ns}\}\prod_{p\mid M_1}\{\sum_{n=0}^{\infty}a_{pn}^2p^{-ns}\}\prod_{p\nmid M}\{\sum_{n=0}^{\infty}a_{pn}^2p^{-ns}\},$$

where a_{pn}^2 in the last factor may be replaced by $\bar{a}_{pn}^2$. Since $a_{pn}=a_p^n$ for $p\mid N$,

$$\sum_{M\mid N}M^{-s}\bar{a}_M^2\{\prod_{p\mid M}[\sum_{n=0}^{\infty}\bar{a}_{pn}^2p^{-ns}]\prod_{p\mid M_1}[\sum_{n=0}^{\infty}a_{pn}^2p^{-ns}]\}$$
$$=\sum_{M\mid N}M^{-s}\bar{a}_M^2\{\prod_{p\mid M}(1-\bar{a}_p^2p^{-s})^{-1}\prod_{p\mid M_1}(1-a_p^2p^{-s})^{-1}\}$$
$$=\prod_{p\mid N}(1-p^{-2s_1})(1-a_p^2p^{-s})^{-1}(1-\bar{a}_p^2p^{-s})^{-1}.$$

For $p\nmid N$, by easy computation,

$$\sum_{n=0}^{\infty}a_{pn}^2p^{-ns}=(1-p^{-2s_1})(1-A_pp^{-s}+p^{-2s_1})^{-1}(1-\chi(p)p^{-s_1})^{-2}.$$

Consequently, we have

$$\psi^*(s)=(N/\pi)^{k-1}J(s_1).$$

This combined with (30) completes the proof.

We would like to add an corollary of the above theorem which follows immediately by virtue of the functional equation of Dirichlet's $L(s,\chi)$. Let us define

(31) $$D(s)=\prod_p D_p(s);$$

$$D_p(s)=\begin{cases}(1-A_pp^{-s}+p^{2k-2-2s})^{-1}(1-\chi(p)p^{k-1-s})^{-1} & \text{if } p\nmid N,\\ (1-a_p^2p^{-s})^{-1}(1-\bar{a}_p^2p^{-s})^{-1} & \text{if } p\mid N,\end{cases}$$

and put

$$R(s)=N^{s/2}\pi^{-(3/2)s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s-k+1+\varepsilon}{2}\right)D(s)$$

with $\varepsilon=(1+\chi(-1))/2$.

COROLLARY. *$R(s)$ can be analytically continued to the whole complex s-plane, and satisfies the functional equation $R(s)=R(2k-1-s)$.*

On the holomorphy of $D(s)$, there is Shimura's work [**14**], though the functional equation and the Euler factors at $p\mid N$ are not mentioned there.

2.3. Another example is an application to the case of Naganuma [8]. Let F be a real quadratic number field with the discriminant N. We assume that N is odd, while Naganuma has treated the case of prime N. Let χ denote the character given by Jacobi's symbol $\left(\frac{*}{N}\right)$, and $\mathfrak{o}$ be the ring of integers. Let ξ be a grössen-character of F with trivial conductor. Define $\kappa=\pi/(\rho\log\varepsilon_0)$, where ε_0 denotes the fundamental unit of F, $\varepsilon_0>1$, and $\rho=1$ or 2 according to $N\varepsilon_0=1$ or -1, then at a principal ideal $\mathfrak{a}=(\alpha)$ of $\mathfrak{o}$, $\xi(\mathfrak{a})$ is given by $\operatorname{sgn}(N\alpha)^l|\alpha/\alpha'|^{im\kappa}$ for some $l=0$ or 1 and $m\in \boldsymbol{Z}$ such that $l\equiv m \pmod{\rho}$. The function of Maass type corresponding to the zeta function $\zeta_F(s, \xi)=\sum \xi(\mathfrak{a})N\mathfrak{a}^{-s}$ is given by

$$g(z, \xi)=C_\xi y^{1/2}+\sum_{\mathfrak{a}\neq 0}\xi(\mathfrak{a})y^{1/2}K_{im\kappa}(2\pi N\mathfrak{a}y)\{e^{2\pi iN\mathfrak{a}x}+(-1)^l e^{-2\pi iN\mathfrak{a}x}\},$$

where the constant C_ξ is zero unless $m=0$, and $\mathfrak{a}$ runs over all non-zero integral ideals of $\mathfrak{o}$. We write $g|\sigma=g(\sigma z, \xi)$ for a real matrix σ of positive determinant. Then we can see that $g|\sigma=\chi(\sigma)g$ for every $\sigma\in\Gamma=\Gamma_0(N)$ and $g|W_N=g$ for a suitably chosen W_N. To prove this, there are at least two different ways: by theta series or by Weil's criterion*⁾.

Furthermore by a similar argument to § 1 (Theorems 1 and 2), or by using the transformation formula of theta series, we can obtain the following

LEMMA 6. *For a suitably chosen W_M for each decomposition $N=MM_1$, it holds that*

$$g(z, \xi)|W_M=\varepsilon_M g^{(M)}(z, \xi),$$

where $\varepsilon_M=1$ or i according to $M\equiv 1$ or 3 (mod 4), and $g^{(M)}$ is given by

$$g^{(M)}(z, \xi)=C_{\xi\psi_M}y^{1/2}+\sum_{\mathfrak{a}\neq 0}\psi_M\xi(\mathfrak{a})y^{1/2}K_{im\kappa}(2\pi N\mathfrak{a}y)\Big\{e^{2\pi iN\mathfrak{a}x}+\Big(\frac{-1}{M}\Big)(-1)^l e^{-2\pi iN\mathfrak{a}x}\Big\}.$$

In the above, we denote by $\psi_M(\mathfrak{a})$ the genus character corresponding to $N=MM_1$, that is, $\psi_M(\mathfrak{p})$ is a non-zero value of either $\Big(\frac{N\mathfrak{p}}{M}\Big)$ or $\Big(\frac{N\mathfrak{p}}{M_1}\Big)$ for each prime ideal $\mathfrak{p}$. It should be also noticed that $\psi_M(\mathfrak{p})$ is $\chi_M(N\mathfrak{p})$ if $\mathfrak{p}\nmid M$ and $\chi_{M_1}(N\mathfrak{p})$ if $\mathfrak{p}\nmid M_1$ by using the notation in 1.1.

Now we take an arbitrary cusp form $f(z)$ in $\mathcal{S}_k(N, \chi)$ and put

$$f(z)|W_M=\sum_{n=1}^{\infty}A_M(n)e^{2\pi inz}$$

for each divisor M of N, with W_M as in Lemma 6. For each integral ideal $\mathfrak{a}$ of $\mathfrak{o}$, we put

$$C(\mathfrak{a})=\sum_{M|N}C_M(\mathfrak{a});$$

$$C_M(\mathfrak{a})=\varepsilon_M^{-1}M^{(k-1)/2}\Big(\frac{M_1}{M}\Big)\psi_M(\mathfrak{a})\sum_{d|\mathfrak{a}}d^{k-1}A_M\Big(\frac{N\mathfrak{a}}{d^2M}\Big),$$

where d runs over all rational positive integers containing $\mathfrak{a}$, and $A_M(r)=0$ for non-integral r.

By these preparations and after some lengthy but quite similar computa-

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*⁾ A precise description of the proof by Weil's criterion can be found in S. Nakamoto's master thesis, Univ. of Tokyo, 1974.

tion as in 2.2., we have

THEOREM 4. *Define a Dirichlet series* $D(s, \xi)$ *over* F *by*

$$D(s, \xi)=\sum_{\mathfrak{a}\neq 0} \xi(\mathfrak{a})C(\mathfrak{a})N\mathfrak{a}^{-s},$$

then,

(i) $D(s, \xi)$ *can be continued to an entire function on the whole complex s-plane and satisfies the functional equation* $D^*(s, \xi)=D^*(k-s, \xi)$, *where we put* $D^*(s, \xi)=N^s(2\pi)^{-s}\Gamma(s+im\kappa)\Gamma(s-im\kappa)D(s, \xi)$.

(ii) *If* $f(z)$ *is a new form and* ξ *is the trivial character* ξ_0, $D(s, \xi_0)$ *splits as follows:*

$$D(s, \xi_0)=\sum_{n=1}^{\infty} A_1(n)n^{-s}\cdot \sum_{n=1}^{\infty} \overline{A_1(n)}n^{-s}.$$

By virtue of (i) we are convinced that the Dirichlet series $D(s, \xi_0)$ is associated with a Hilbert modular cusp form of weight k with respect to $GL_2(\mathfrak{o})$.

References

[1] A. N. Andrianov, Dirichlet series with Euler product in the theory of Siegel modular forms of genus 2, Proc. Steklov Inst. Math., **112** (1971), 70–93.

[2] A. O. Atkin and J. Lehner, Hecke operators on $\Gamma_0(m)$, Math. Ann., **185** (1970), 134–160.

[3] K. Doi and H. Naganuma, On the functional equation of certain Dirichlet series, Invent. math., **9** (1969), 1–14.

[4] E. Hecke, Mathematische Werke, Vandenhoech und Ruprecht.

[5] H. Jacquet, Automorphic forms on $GL(2)$, II, Lecture notes in math., **278**, Springer, 1972.

[6] Y. Kitaoka, On the transformation formula of theta-series, to appear.

[7] T. Miyake, On automorphic forms on GL_2 and Hecke operators, Ann. of Math., **94** (1971), 174–189.

[8] H. Naganuma, On the coincidence of two Dirichlet series associated with cusp forms of Hecke's Neben-type and Hilbert modular forms over a real quadratic field, J. Math. Soc. Japan, **25** (1973), 547–555.

[9] A. P. Ogg, On the eigenvalues of Hecke operators, Math. Ann., **179** (1969), 101–108.

[10] A. P. Ogg, On a convolution of L-series, Invent. Math., **7** (1969), 297–312.

[11] R. A. Rankin, Contribution to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions, I, II, Proc. Cambridge Philos. Soc., **35** (1939), 351–372.

[12] G. Shimura, Introduction to the arithmetical theory of automorphic functions, Publ. Math. Soc. Japan, No. 11.

[13] G. Shimura, On modular forms of half integral weight, Ann. of Math., **97** (1973), 440–481.

[14] G. Shimura, On the holomorphy of certain Dirichlet series, to appear.

Added in proof. After the preparation of this paper, Professor K. Doi has informed the author that a similar result of Theorem 4 has been also obtained by Don Zagier in his recent article: Modular forms associated to real quadratic fields. His method is quite different from the author's.

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