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On minimal immersions of R^2 into S^N

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§ 1. Introduction.

In this paper we treat isometric minimal immersions of the Euclidean 2-plane with a standard flat metric into the unit sphere S^N of the Euclidean space R^{N+1}.

Analytically the problem can be stated as follows. Study a surface Ψ: $R^2 \to R^{N+1}$ which is given in the form

$$\Psi(x, y)=(\Psi^1(x, y), \cdots, \Psi^{N+1}(x, y))$$

and is defined on the whole plane R^2, and has the following properties: a C^∞-mapping Ψ satisfies on R^2 the equations

$$(\Psi, \Psi)=1, \tag{1.1}$$

$$(\Psi_x, \Psi_x)=(\Psi_y, \Psi_y)=1, \quad (\Psi_x, \Psi_y)=0 \tag{1.2}$$

and

$$\Psi_{xx}+\Psi_{yy}=-2\Psi, \tag{1.3}$$

where (,) denotes the inner product of R^{N+1}. The condition (1.3) under (1.1) tells us that $\Psi(R^2)$ is a minimal surface in S^N.

In the part I of a previous paper [3], the author has proved some formulas for the Laplacians of higher fundamental tensors. In the part II of the above paper, by making use of these results and the complex function theory, we have studied a restricted class of minimal immersions of R^2 into S^N.

In the case of minimal immersions $R^2 \to S^5$ [4], we have succeeded in a generalization of our previous results.

The purpose of this paper is to give a complete description in the case of any minimal immersions $R^2 \to S^N$.

If $\Psi(R^2)$ is not contained in a linear subspace of R^{N+1}, N must be an odd integer [2], [3], say $N=2n+1$. Then we shall prove that $\Psi(R^2)$ is an orbit of an abelian Lie subgroup of $SO(2n+2)$ (Theorem 1). From this result and Hsiang's Theorem [1], we know that any isometric minimal immersion of a flat torus into S^{2n+1} must be real algebraic (Theorem 2).

Our principal aim in this paper is to give a parametrization of the set of equivalent classes of Ψ's (Theorem 3).

The method of a parametrization in this paper is different from [4]. But since our main tools are the theorems in [3], in § 2 we state the notations and results which were obtained in [3]. In § 3, we shall give the Frenet-Borůvka formula which is a generalization of § 6 in [3]. In § 4 we shall state and prove our main theorem. In the last section we shall give some examples.

§ 2. **Preliminaries** (cf. part I of [3]).

Let $\Psi : M \to X$ be an isometric minimal immersion of a 2-dimensional Riemannian manifold into a space of constant curvature X of the dimension N. For any $x \in M$ and $m=1, 2, \cdots$, we denote the osculating space of order m by $T_x^{(m)}$. Suppose x to be a regular point of order $m-1 \geqq 2$ in the following sense:

(2.1) *For $b=1, 2, \cdots, m-1$, $\dim T_x^{(b)}$ is constant in a neighborhood of x.*

We shall use the following ranges of indices:

$$
\begin{aligned}
&1 \leqq \lambda_0 = i, j, \cdots \leqq 2 \\
&3 \leqq \lambda_1 \leqq \dim T_x^{(2)}, \\
&\dim T_x^{(2)} + 1 \leqq \lambda_2 \leqq \dim T_x^{(3)} \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\dim T_x^{(m-1)} + 1 \leqq \lambda_{m-1} \leqq N .
\end{aligned} \tag{2.2}
$$

Let e_A, $1 \leqq A, B, \cdots \leqq N$, be local orthonormal frame field, such that $e_{\lambda_0}, e_{\lambda_1}, \cdots, e_{\lambda_b}$ span $T_x^{(b+1)}$, $b=0, 1, \cdots, m-2$. We then have

$$
w_{\lambda_{b-1}\lambda_{a+1}} = 0, \qquad \text{for } a \geqq b,\ b=1, 2, \cdots, m-2. \tag{2.3}
$$

Let $h_{\lambda_b i_1 \cdots i_{b+1}}$ be the $(b+1)$-th fundamental tensors for e_A. Then we have, by (2.3),

$$
\sum_{\lambda_b} h_{\lambda_b i_1 \cdots i_{b+1}} w_{\lambda_b \lambda_{b+1}} = \sum_{i_{b+2}} h_{\lambda_{b+1} i_1 \cdots i_{b+2}} w_{i_{b+2}} . \tag{2.4}
$$

Since Ψ is minimal, we have

$$
\sum_j h_{\lambda_b j j i_3 \cdots i_{b+1}} = 0, \qquad b=1, 2, \cdots, m-1 . \tag{2.5}
$$

We put

$$H_\alpha^{(b)}=h_{\alpha\underbrace{1\cdots1}_{b}}+ih_{\alpha\underbrace{1\cdots12}_{b-1}}, \qquad \alpha\geqq\mu_{b-1},\ b=2,\cdots,m,$$

where we put $\mu_{b-1}=\dim T_x^{(b-1)}+1$. Then the higher order Codazzi equation is

$$\{dH_\alpha^{(b)}+ibH_\alpha^{(b)}w_{12}+\sum_{\beta\geqq\mu_{b-1}}H_\beta^{(b)}w_{\beta\alpha}\}\wedge(w_1-iw_2)=0, \qquad \alpha\geqq\mu_{b-1}. \tag{2.6}$$

In [3], we have defined non-negative scalar invariants in a neighborhood of x.

$$K_{(b)}=\sum_{\alpha\geqq\mu_{b-1}}|H_\alpha^{(b)}|^2; \tag{2.7}$$

$$N_{(b)}=|(\sum h_{\alpha\underbrace{1\cdots1}_{b}}e_\alpha)\wedge(\sum h_{\alpha\underbrace{1\cdots12}_{b-1}}e_\alpha)|^2; \tag{2.8}$$

$$f_{(b)}=K_{(b)}^2-4N_{(b)}, \qquad b=2,3,\cdots,n. \tag{2.9}$$

For a geometric and function theoretic meaning of $f_{(b)}$, we have, by (2.6),

LEMMA 1. *Under the above notations,* $\sum_{\alpha\geqq\mu_{b-1}}(\bar{H}_\alpha^{(b)})^2(w_1+iw_2)^{2b}$ *is an abelian form of degree* $2b$. *Moreover we have*

$$f_{(b)}=|\sum_\alpha(\bar{H}_\alpha^{(b)})^2|^2. \tag{2.10}$$

For the Laplacians of $K_{(b)}$ and $f_{(b)}$, we know from [3]

$$\Delta f_{(b)}=4\{bf_{(b)}K+|A_{(b)}|^2\}; \tag{2.11}$$

$$\frac{1}{2}\Delta K_{(b)}=-2\frac{N_{(b)}}{N_{(b-1)}}K_{(b-1)}+bKK_{(b)}+2K_{(b+1)} \tag{2.12}$$

$$+2\sum_{\lambda_{b-1}}|H_{\lambda_{b-1},1}^{(b)}|^2, \qquad \text{if} \quad N_{(b-1)}\neq0,$$

where we define $N_{(1)}=1$, $K_{(1)}=2$ and K is the Gaussian curvature of M and $A_{(b)}=2\sum_\alpha H_\alpha^{(b)}H_{\alpha,1}^{(b)}$, and $H_{\alpha,k}^{(b)}=h_{\alpha1\cdots1,k}+ih_{\alpha1\cdots12,k}$.

§ 3. Frenet-Borůvka formula.

Let $M=R^2$ be an oriented Euclidean 2-plane with a standard flat metric and $X=S^N$. Then we have, by the Gauss equation,

$$K=0 \quad \text{and} \quad K_{(2)}=1. \tag{3.1}$$

By (2.9), (2.11) and (3.1), $f_{(2)}$ is bounded and subharmonic on R^2. Hence $f_{(2)}$ must be a constant function on R^2 and we have $A_{(2)}=0$. (2.9) and (3.1) tell us that $N_{(2)}$ is also constant. Thus *the 2nd osculating space is globally defined on* R^2. Since R^2 has the standard coordinate system of R^2, orthonormal tangent vector fields e_1 and e_2 are globally defined. Hence $\sum_{\alpha\geqq3}(\bar{H}_\alpha^{(2)})^2$ is also globally defined for any fixed e_1 and e_2 and holomorphic by (2.6). By (2.10) its norm is constant. Hence $\sum(\bar{H}_{\alpha}^{(2)})^2$ must be a constant by Liouville's theorem. If $N_{(2)}=0$, $\Psi(R^2)$ is contained in a 3-dimensional totally geodesic S^3 in S^N (cf. [3]). If $N_{(2)}\neq 0$, then we have

LEMMA 2. *There exist globally defined orthonormal vectors e_3, e_4 and (locally) defined orthonormal vectors e_α, $\alpha\geqq 5$, such that*

$$(3.2)\qquad \begin{cases} h_{311}>0\,, & h_{312}=0\,;\\ h_{411}\geqq 0\,, & h_{412}\neq 0\,;\\ h_{\alpha ij}=0\,, & \alpha\geqq 5\,, \end{cases}$$

where $h_{\alpha ij}$, $3\leqq\alpha\leqq 4$, are also globally defined constant functions on R^2.

PROOF. Since $N_{(2)}\neq 0$, vectors $\{e_1, e_2, \sum_{\alpha\geqq 3} h_{\alpha 11}e_\alpha, \sum_{\alpha\geqq 3} h_{\alpha 12}e_\alpha\}$ are basis of $T_x^{(2)}$ at each point of R^2. We set

$$(3.3)\qquad e_4^* = \frac{\sum_{\alpha\geqq 3} h_{\alpha 12}e_\alpha}{\sqrt{\sum h_{\alpha 12}^2}}\,.$$

By the Gram-Schmidt's orthogonalization in $T_x^{(2)}$ and any other normal vectors in the orthogonal complement of $T_x^{(2)}$, we have, then $h_{312}^*=0$ and $h_{\alpha ij}^*=0$, $\alpha\geqq 5$. By definition, e_3^* and e_4^* are globally defined, and h_{3ij}^*, h_{4ij}^* are globally defined on R^2. By taking a suitable sign of e_3^* and e_4^*, we may assume $h_{311}^*\geqq 0$ and $h_{411}^*\geqq 0$. We know $N_{(2)}=(h_{311}^*h_{412}^*)^2\neq 0$. Since $K_{(2)}$, $N_{(2)}$ and $\sum(\bar{H}_{\alpha}^{(2)})^2$ are constant, $h_{\alpha ij}^*$'s are also constant. q. e. d.

We drop the asterisks. Since we know $w_{12}=0$, we have $w_{34}=0$ by (2.6). Since $N_{(2)}\neq 0$, we get $H_{\lambda_1,1}^{(2)}=0$, by virtue of $dK_{(2)}=0$ and $dA_{(2)}=0$ (cf. part II of [3]). The formula (2.12) says that

$$(3.4)\qquad K_{(3)}=2N_{(2)}\,.$$

By (2.9), (2.11) and (3.4), $f_{(3)}$ is bounded and subharmonic. Since the 3rd fundamental tensors are globally defined, $f_{(3)}$ must be a constant function on R^2 and we have $A_{(3)}=0$. As $N_{(3)}$ is also constant, *the 3rd osculating space is globally defined on R^2*. Since $\sum_{\alpha\geqq 5}(\bar{H}_{\alpha}^{(3)})^2$ is also globally defined and holomorphic by (2.6), this is constant by (2.10). If $N_{(3)}=0$, $\Psi(R^2)$ is contained in a 5-dimensional totally geodesic $S^5\subset S^N$. If $N_{(3)}\neq 0$, then we have

LEMMA 3. *There exist globally defined orthonormal vectors e_5, e_6 and (locally defined) orthonormal vectors e_α, $\alpha\geqq 7$, such that*

$$(3.5)\qquad \begin{cases} h_{5111}>0\,, & h_{5112}=0\,;\\ h_{6111}\geqq 0\,, & h_{6112}\neq 0\,;\\ h_{\alpha ijk}=0\,, & \alpha\geqq 7\,, \end{cases}$$

where $h_{\alpha ijk}$, $\alpha=5, 6$, *are globally defined constant functions on* R^2.

Since the proof of Lemma 3 are almost same as its of Lemma 2, we shall omit it. Since $w_{12}=w_{34}=0$, we have $w_{56}=0$ by (2.6). Continuing in this way, if $N_{(b)}$ is defined on R^2 and non-zero constant on R^2, we must have $K_{(b+1)}=K_{(b-1)}N_{(b)}/N_{(b-1)}$ and $N_{(b+1)}$ is a constant. Therefore $\Psi(R^2)$ must be contained in an odd dimensional totally geodesic $S^{2n+1}\subset S^N$ such that $N_{(2)}\cdots N_{(n)}\neq 0$ and $N_{(n+1)}=0$ (cf. [3]). Thus we have

PROPOSITION 1. *Let* $\Psi: R^2\to S^N$ *be an isometric minimal immersion such that the image is not contained in a linear subspace of* R^{N+1}. *Then*

i) *N must be an odd integer,* $N=2n+1$;

ii) *Any* $x\in R^2$ *is a regular point of order* n;

iii) $K_{(m)}$, $N_{(m)}$, $m=2, 3, \cdots, n+1$, *are constant functions on* R^2 *and*

$$K_{(m)}=\frac{K_{(m-2)}}{N_{(m-2)}}N_{(m-1)}\,; \tag{3.6}$$

iv) *With respect to any fixed* e_1 *and* e_2, *the globally defined complex function* $\sum_{\alpha\geqq 2b-1}(\bar{H}_\alpha^{(b)})^2$ *is also constant on* R^2.

We have also

PROPOSITION 2. *Let* $\Psi: R^2\to S^{2n+1}$ *be an isometric minimal immersion such that the image is not contained in a linear subspace of* R^{2n+2}. *Then there exist globally defined orthonormal vector fields* e_i *and* e_α *such that higher fundamental tensors satisfy*

$$\begin{cases} h_{(2b-1)\underbrace{1\cdots1}_{b}}>0\,, \quad h_{(2b-1)\underbrace{1\cdots12}_{b-1}}=0\,; \quad b=2,3,\cdots,n+1\,, \\ h_{(2b)\underbrace{1\cdots1}_{b}}\geqq 0\,, \quad h_{(2b)\underbrace{1\cdots12}_{b-1}}\neq 0\,; \quad b=2,3,\cdots,n\,, \\ h_{\alpha\underbrace{1\cdots1}_{b}}=h_{\alpha\underbrace{1\cdots12}_{b-1}}=0\,, \quad \alpha\geqq 2b+1\,, \quad b=2,3,\cdots,n\,. \end{cases} \tag{3.7}$$

Moreover, $h_{(2b)1\cdots1}$, $h_{(2b)1\cdots12}$, $h_{(2b-1)1\cdots1}$ *and* $h_{(2n+1)1\cdots1}$ *are globally defined constant functions and we have*

$$\begin{cases} h_{(2b-1)\underbrace{1\cdots1}_{b}}=\sqrt{\dfrac{K_{(b-2)}}{N_{(b-2)}}N_{(b-1)}-h^2_{(2b)\underbrace{1\cdots1}_{b}}-h^2_{(2b)\underbrace{1\cdots12}_{b-1}}}\;; \\ h_{(2n+1)\underbrace{1\cdots1}_{n+1}}=\sqrt{\dfrac{K_{(n-1)}}{N_{(n-1)}}N_{(n)}}\;, \end{cases} \tag{3.8}$$

$$w_{12}=w_{34}=\cdots=w_{2n-1,2n}=0\,.$$

PROOF. From the preceding discussion, (3.7) with $b\leqq n$ is valid for any orthonormal vectors e_1 and e_2. We can choose suitable e_1 and e_2 such that $h_{(2n+1)1\cdots12}=0$. If necessary we take $-e_{2n+1}$, we may assume $h_{(2n+1)1\cdots1}>0$. q.e.d.

We call such a frame the Frenet-Borůvka frame. From these results we have a system of Pfaff for Ψ by making use of the formulas in pp. 477 and 478 of the part I of [3]:

We put

$$
(3.9)\quad \begin{cases} \alpha_k = \dfrac{h_{(2k+2)\overbrace{1\cdots1}^{k+1}}}{\sqrt{N_{(k)}}}, & \beta_k = \dfrac{h_{(2k+2)\overbrace{1\cdots1}^{k}2}}{\sqrt{N_{(k)}}} \\ \delta_k = \dfrac{h_{(2k+1)\overbrace{1\cdots1}^{k+1}}}{\sqrt{N_{(k)}}}, & k=1, 2, \cdots, n-1. \end{cases}
$$

$$
(3.10)\quad \begin{cases} w_1 = dx, \quad w_2 = dy, \quad w_3 = \cdots = w_{2n+1} = 0, \\ w_{2k-1,2k+1} = \sqrt{N_{(k-1)}}\,\delta_k(\beta_{k-1}dx + \alpha_{k-1}dy), \\ w_{2k-1,2k+2} = \sqrt{N_{(k-1)}}\,(\beta_{k-1}\alpha_k - \alpha_{k-1}\beta_k)dx + \sqrt{N_{(k-1)}}\,(\beta_{k-1}\beta_k + \alpha_{k-1}\alpha_k)dy, \\ w_{2k,2k+1} = -\sqrt{N_{(k-1)}}\,\delta_{k-1}\delta_k dy, \\ w_{2k,2k+2} = \sqrt{N_{(k-1)}}\,\delta_{k-1}(\beta_k dx - \alpha_k dy), \\ w_{12} = \cdots = w_{2n-1,2n} = w_{2k-1,\alpha} = w_{2k,\alpha} = 0, \qquad \alpha \geqq 2k+3, \\ w_{2n-1,2n+1} = \sqrt{N_{(n-1)}}\,\delta_n(\beta_{n-1}dx + \alpha_{n-1}dy), \\ w_{2n,2n+1} = -\sqrt{N_{(n-1)}}\,\delta_{n-1}\delta_n dy, \end{cases}
$$

where we set $N_{(0)}=K_{(0)}=1$, $\alpha_0=0$, $\beta_0=\delta_0=1$ and we have

$$
(3.11)\quad \begin{cases} \alpha_1^2+\beta_1^2+\delta_1^2=1, \\ \alpha_m^2+\beta_m^2+\delta_m^2 = \dfrac{N_{(m-2)}}{N_{(m-1)}}(\alpha_{m-2}^2+\beta_{m-2}^2+\delta_{m-2}^2), \qquad m=2, 3, \cdots, n-1, \\ N_{(m-1)} = (N_{(m-2)}\beta_{m-2}\delta_{m-2})^2, \\ \delta_n = \sqrt{\dfrac{1}{N_{(n-1)}}(\alpha_{n-2}^2+\beta_{n-2}^2+\delta_{n-2}^2)}\ . \end{cases}
$$

The Pfaff system (3.10) is completely integrable if and only if (3.11) is valid. From (3.10) we can get a system of total differential equations with constant coefficients for $M(x, y)={}^t(\Psi(x, y), e_1(x, y), \cdots, e_{(2n+1)}(x, y))$, where we consider $\Psi(x, y)$, $e_i(x, y)$, $e_\alpha(x, y)$ as row vectors in R^{2n+2}. We put

$$
(3.12)\qquad dM(x, y) = (Pdx+Qdy)M(x, y),
$$

where P and Q are constant skew symmetric matrices which are determined by $h_{\alpha i_1\cdots i_b}$. Since (3.10) is completely integrable, we have $PQ=QP$. We can seek the solution of (3.12) as follows: Let T be an orthogonal matrix such that

$$T^{-1}PT=\begin{pmatrix} P_1 & & 0 \\ & \ddots & \\ 0 & & P_{n+1} \end{pmatrix}, \quad T^{-1}QT=\begin{pmatrix} Q_1 & & 0 \\ & \ddots & \\ 0 & & Q_{n+1} \end{pmatrix},$$

$$P_i=\begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}, \quad Q_i=\begin{pmatrix} 0 & \mu_i \\ -\mu_i & 0 \end{pmatrix}, \quad i=1, 2, \cdots, n+1,$$

and $\pm\sqrt{-1}\,\lambda_i$ (resp. $\pm\sqrt{-1}\,\mu_i$) are the eigenvalues of P (resp. Q). We set $T^{-1}M=(f_{AB})$. Since we have $d(T^{-1}M)=(T^{-1}PTdx+T^{-1}QTdy)(T^{-1}M)$, (3.12) is equivalent to

(3.13)
$$\frac{\partial(f_{(2k-1)B}+\sqrt{-1}f_{(2k)B})}{\partial x}=-\sqrt{-1}\,\lambda_k(f_{(2k-1)B}+\sqrt{-1}f_{(2k)B}),$$
$$\frac{\partial(f_{(2k-1)B}+\sqrt{-1}f_{(2k)B})}{\partial y}=-\sqrt{-1}\,\mu_k(f_{(2k-1)B}+\sqrt{-1}f_{(2k)B}),$$

where $k=1, 2, \cdots, n+1$, $B=1, 2, \cdots, 2n+2$.

(3.13) can be easily solved:

(3.14)
$$f_{(2k+1)B}(x, y)=C_{(2k-1)B}\cos(\lambda_k x+\mu_k y)+C_{(2k)B}\sin(\lambda_k x+\mu_k y),$$
$$f_{(2k)B}(x, y)=-C_{(2k-1)B}\sin(\lambda_k x+\mu_k y)+C_{(2k)B}\cos(\lambda_k x+\mu_k y),$$

where $C_{(2k-1)B}$ and $C_{(2k)B}$ are constants.
Then under the initial condition $M(0, 0)=T$, we have $M(x, y)=TG$, where

(3.15)
$$G=\begin{pmatrix} G_1 & & 0 \\ & \ddots & \\ 0 & & G_{n+1} \end{pmatrix}, \quad G_i=\begin{pmatrix} \cos(\lambda_i x+\mu_i y) & \sin(\lambda_i x+\mu_i y) \\ -\sin(\lambda_i x+\mu_i y) & \cos(\lambda_i x+\mu_i y) \end{pmatrix}.$$

We set

(3.16)
$$\mathfrak{G}=\{T^{-1}(xP+yQ)T : (x, y)\in R^2\}.$$

$\mathfrak{G}$ is an abelian Lie subalgebra of the Lie algebra of $SO(2n+2)$ and we have $G=\exp\mathfrak{G}$. Hence we have

THEOREM 1. *Let* $\Psi: R^2\to S^{2n+1}$ *be an isometric minimal immersion such that* $\Psi(R^2)$ *is not contained in a linear subspace of* R^{2n+2}. *Then* $\Psi(R^2)$ *is an orbit of an abelian Lie subgroup of* $SO(2n+2)$.

By Theorem 1 and a Hsiang's Theorem [1], we get

THEOREM 2. *Let* $\Psi: T^2\to S^{2n+1}$ *be an isometric minimal immersion of a flat torus into* S^{2n+1} *such that the image is not contained in a linear subspace of* R^{2n+2}. *Then* Ψ *is real algebraic.*

PROOF. We set $T^2=R^2/\Gamma$, where Γ of translations of R^2 is generated by

2 linearly independent vectors and $\pi: R^2 \to R^2/\Gamma$ is the natural projection. Let $\tilde{\Psi}$ be an isometric minimal immersion of R^2 into S^{2n+1} with $\tilde{\Psi}(u, v)=\Psi\circ\pi(u, v)$. There exist some isometry, A, of R^2 such that $\tilde{\Psi}\circ A(x, y)=(v_1, \cdots, v_{2n+2})G$ by Theorem 1. Then there is a set of points (a, c), (b, d) such that

$$\lambda_i a+\mu_i c=2\pi p_i, \quad \lambda_i b+\mu_i d=2\pi q_i, \quad i=1, \cdots, n+1, \tag{3.14}$$

where $ad-bc\neq 0$ and $\{p_i, q_i\}$ are integers. Hence G is the closed Lie subgroup of $SO(2n+2)$. By [1], Ψ is real algebraic. q. e. d.

§ 4. Parametrization.

Let us say that two minimal immersions $\Psi_1, \Psi_2: R^2 \to S^{2n+1}$ are equivalent if there is an isometry F of S^{2n+1} and an orientation preserving isometry A of R^2 so that $F\cdot\Psi_1=\Psi_2\cdot A$. By $[\Psi]$ we denote the equivalence class of Ψ's. We denote the set of $[\Psi]$'s such that $\Psi(R^2)$ is not contained in a linear subspace of R^{2n+2} by Σ. *Then Σ can be written as a disjoint union of $\Sigma(K_{(2)}, \cdots, K_{(n)})$'s, where $\Sigma(K_{(2)}, \cdots, K_{(n)})$ is the set of $[\Psi]\in\Sigma$ such that $\sum_{\alpha\geq 2s+1}(h^2_{\alpha 1\cdots 1}+h^2_{\alpha 1\cdots 12})=K_{(s+1)}$, $s=1, \cdots, n-1$ and $h_{\alpha i_1\cdots i_{s+1}}$'s are the higher fundamental tensors of Ψ.* We remark that any immersions in $\Sigma(K_{(2)}, \cdots, K_{(n)})$ have the same quantities $N_{(3)}, \cdots, N_{(n-1)}$ by (3.6).

In this section we intend to parametrize $\Sigma(K_{(2)}, \cdots, K_{(n)})$. We consider a $[\Psi]\in\Sigma(K_{(2)}, \cdots, K_{(n)})$. Since $N_{(2)}, \cdots, N_{(n)}\neq 0$, we know

$$\Big|\sum_{\alpha=2b-1}^{\alpha=2b}(\bar{H}^{(b)}_{\alpha})^2\Big|^2=f_{(b)}<\Big(\frac{K_{(b-2)}}{N_{(b-2)}}N_{(b-1)}\Big)^2, \quad b=2, 3, \cdots, n. \tag{4.1}$$

For simplicity we write $C_{b-1}=\sum(\bar{H}^{(b)}_{\alpha})^2$, $b=2, 3, \cdots, n+1$. We remark that C_b is independent for the adapted normal vectors e_{λ_b}. Under the transformation $e_1+ie_2 \to e^{i\theta}(e_1+ie_2)=\tilde{e}_1+i\tilde{e}_2$, we have

$$\tilde{C}_{b-1}=e^{-2bi\theta}C_{b-1}, \quad b=2, 3, \cdots, n+1. \tag{4.2}$$

For an isometric minimal immersion $\Psi: R^2 \to S^{2n+1}$, the Frenet-Borůvka frame is not unique. Let $\tilde{e}_i$ and $\tilde{e}_\alpha$ be a Frenet-Borůvka frame on R^2. Since we know $h_{(2n+1)1\cdots 12}=0$ and $\tilde{h}_{(2n+1)1\cdots 12}=0$, we must have

$$C_n=(\bar{H}^{(n+1)}_{(2n+1)})^2=K^2_{(n+1)}=\tilde{C}_n$$

and hence we get $\theta=k\pi/n+1$, $k=0, 1, 2, \cdots$. Thus we have

LEMMA 4. *We set*

$$\Delta_1=\{z\in C: |z|<1\}$$

$$\Delta_b=\Big\{z\in C: |z|<\frac{K_{(b-1)}}{N_{(b-1)}}N_{(b)}\Big\}, \quad b=2, \cdots, n-1.$$

Then we have a correspondence $\chi : \Psi \to [(C_1, \cdots, C_{n-1})] \in \Delta_1 \times \cdots \times \Delta_{n-1}/\Gamma$ *where*

$$(4.3) \qquad \Gamma = \left\{ \begin{pmatrix} \exp\sqrt{-1}\,\frac{4k}{n+1}\pi & & 0 \\ & \ddots & \\ 0 & & \exp\sqrt{-1}\,\frac{2nk}{n+1}\pi \end{pmatrix}, \quad k = 0, 1, 2, \cdots \right\},$$

and $\exp\sqrt{-1}\,\frac{2bk}{n+1}\pi$ *acts on* Δ_{b-1}.

The map χ in Lemma 4 induces the one-to-one correspondence of $\Sigma(K_{(2)}, \cdots, K_{(n)})$ into $\Delta_1 \times \cdots \times \Delta_{n-1}/\Gamma$: Let Φ and Ψ be isometric minimal immersions of R^2 into S^{2n+1} such that $[\Phi]$ and $[\Psi]$ are elements of $\Sigma(K_{(2)}, \cdots, K_{(n)})$. We denote $\chi(\Psi) = [(C_1, \cdots, C_{n-1})]$ and $\chi(\Phi) = [(D_1, \cdots, D_{n-1})]$. Now we suppose $\chi(\Psi) = \chi(\Phi)$. Then we have $C_{b-1} = \exp\left(\sqrt{-1}\,\frac{2bk}{n+1}\pi\right) D_{b-1}$. Under the transformation $f_1 + \sqrt{-1} f_2 \to \exp\left(-\sqrt{-1}\,\frac{k\pi}{n+1}\right)(f_1 + \sqrt{-1} f_2) = \tilde{f}_1 + \sqrt{-1}\tilde{f}_2$, where (f_1, f_2) and $(\tilde{f}_1, \tilde{f}_2)$ are orthonormal tangent vectors of the Frenet-Borůvka frames of Φ, we have $\tilde{D}_{b-1} = \exp\left(\sqrt{-1}\,2b\frac{k\pi}{n+1}\right) D_{b-1} = C_{b-1}$ and $\tilde{k}_{(2n+1)1\cdots 12} = 0$, where $\tilde{k}_{(2n+1)1\cdots 12}$ is a component of the $(n+1)$-th fundamental tensor of Φ with respect to $\tilde{f}_i$ and $\tilde{f}_\alpha$ of the Frenet-Borůvka frame of Φ. Therefore Φ is equivalent to Ψ.

LEMMA 5 (Surjectiveness). *For any* $[(C_1, \cdots, C_{n-1})] \in \Delta_1 \times \cdots \times \Delta_{n-1}/\Gamma$ *there exists an equivalence class of a minimal immersion* $\Psi : R^2 \to S^{2n+1}$ *with* $\chi(\Psi) = [(C_1, \cdots, C_{n-1})]$.

PROOF. We can define constants α_b, β_b and δ_b such that they satisfy

$$(4.4) \qquad \begin{cases} \operatorname{Re} C_1 = 1 - 2\beta_1^2, \quad \operatorname{Im} C_1 = 2\alpha_1\beta_1, \quad \alpha_1 \geqq 0,\ \beta_1 \neq 0, \\ \delta_1 = \sqrt{1 - \alpha_1^2 - \beta_1^2}, \quad N_{(2)} = (\beta_1\delta_1)^2, \\ \operatorname{Re} C_b = \frac{N_{(b)}}{N_{(b-1)}} N_{(b-2)}(\alpha_{b-2}^2 + \beta_{b-2}^2 + \delta_{b-2}^2) - 2N_{(b)}\beta_b^2, \\ \operatorname{Im} C_b = 2N_{(b)}\alpha_b\beta_b, \quad \alpha_b \geqq 0,\ \beta_b \neq 0,\ b = 2, 3, \cdots, n-1, \\ \delta_b = \sqrt{\frac{N_{(b-2)}}{N_{(b-1)}}(\alpha_{b-2}^2 + \beta_{b-2}^2 + \delta_{b-2}^2) - \alpha_b^2 - \beta_b^2}, \\ N_{(b)} = (N_{(b-1)}\beta_{b-1}\delta_{b-1})^2, \\ \delta_n = \sqrt{\frac{1}{N_{(n-1)}}(\alpha_{n-2}^2 + \beta_{n-2}^2 + \delta_{n-2}^2)}, \\ N_{(0)} = 1, \quad \alpha_0 = 0, \quad \beta_0 = \delta_0 = 1. \end{cases}$$

For a C_b, if $\operatorname{Im} C_b \neq 0$, then α_b and β_b satisfying $\alpha_b \geqq 0$ and $\beta_b \neq 0$ is uniquely determined. In the case of $\operatorname{Im} C_b = 0$, Ψ_+ (resp. Ψ_-) denotes the minimal immersion constructed by $\alpha_b=0$ and β_b (resp. $\alpha_b=0$ and $-\beta_b$) from (3.10). Then by taking suitable transformation of e_{2b-1} and e_{2b}, we have $\Psi_+=\Psi_-$. q. e. d.

Thus we have

THEOREM 3. *There exists a* 1-1 *correspondence between* $\Sigma(K_{(2)}, \cdots, K_{(n)})$ *and* $\Delta_1\times\cdots\times\Delta_{n-1}/\Gamma$.

REMARK. In the case of $\Psi: R^2\to S^5$ with $N_{(2)}\neq 0$, we have $\Sigma=\Sigma(K_{(2)})$, because of $K_{(2)}=1$ for any Ψ and so $\Sigma=\Delta_1/\Gamma$ (cf. [4]).

§ 5. Examples.

In general it is not easy that we calculate the eigenvalues of P and Q. See [3] for the minimal immersions with $f_{(2)}=\cdots=f_{(n)}=0$ into S^{2n+1}. We shall give some another examples in this paper.

$$\Psi_s(x, y)=\frac{1}{\sqrt{(1+2s)(3-2s)}}\Big(\sqrt{s(3-2s)}\,e^{i\sqrt{2}x},\ \sqrt{(1+2s)(1-s)}e^{i\sqrt{2}y},\ e^{i\left(\sqrt{\frac{3-2s}{2}}x+\sqrt{\frac{1+2s}{2}}y\right)},\ e^{i\left(\sqrt{\frac{3-2s}{2}}x-\sqrt{\frac{1+2s}{2}}y\right)}\Big)\subset S^7\subset C^4. \tag{5.1}$$

By the direct verification, we can see the following results of (5.1): For each $0<s<1$, (5.1) is an isometric minimal immersion of R^2 into S^7 satisfying $f_{(2)}=0$, $f_{(3)}=1/4-s(1-s)\geqq 0$ and $N_{(3)}=s(1-s)/4$. Therefore, for $0<s\leqq 1/2$, these non-equivalent minimal immersions are different from one's constructed by T. Itoh [2]. Especially, in the case of $s=((6/5)^2-1)/2$, the Ψ_s induces an isometric minimal immersion of a flat torus into S^7.

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