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Topology of Hopf surfaces

By Masahide KATO

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§1. Introduction.

By a Hopf surface S, we shall mean a 2-dimensional compact complex manifold of which the universal covering is $W=\boldsymbol{C}^2-\{0\}$, where $\boldsymbol{C}^2$ is the space of two complex variables (z_1, z_2) and 0 is the origin $(0, 0)$. S can be represented as a quotient space W/G with a group G generated by some biholomorphic transformations of W whose action is properly discontinuous and free.

Let B be a closed spherical set in $\boldsymbol{C}^2$ determined by the inequality: $|z_1|^2+|z_2|^2 \leqq 1$. A complex analytic automorphism g of $\boldsymbol{C}^2$ is called a contraction if $g^n(B)$ converges to 0 for $n\to\infty$.

By Kodaira [4], G has the following properties;

(1) *G contains a contraction g, and the infinite cyclic subgroup generated by g has a finite index in G,*

(2) *if G is non-abelian, then by a proper choice of global coordinates of $\boldsymbol{C}^2$, G appears as a subgroup of $GL(2, \boldsymbol{C})$.*

The purpose of this paper is to classify all Hopf surfaces by diffeomorphisms (or equivalently, homeomorphisms). (See, Theorems 9, 10 and 12.)

§2. Classification of G in the case $G\subset GL(2, \boldsymbol{C})$.

First we define subgroups H and K of G as follows;

$$H=\{x\in G : |\det x|=1\},$$

$$K=\{x\in G : \det x=1\}\subset SL(2, \boldsymbol{C}).$$

Clearly, $G\triangleright H\triangleright K$ and $G\triangleright K$. In what follows H and K are assumed to satisfy these conditions. Let G_2 be a subgroup of a group G_1. We denote by $[G_1 : G_2]$ the index of G_2 in G_1. If G_2 is generated by some elements $h_1, \cdots, h_r$ of G_1, we sometimes write $\{h_1, \cdots, h_r\}$ instead of G_2.

LEMMA 1. *An element x of G is a contraction if and only if $|\det x|<1$.*

PROOF. We denote by α, β the eigenvalues of $x\in G$. If x is a contraction, then $|\alpha|<1$, $|\beta|<1$. Hence $|\det x|<1$. Conversely, let x be an element of G such that $|\det x|<1$. If $|\alpha|=|\beta|$, then x is a contraction by the inequality $|\alpha|^2=|\det x|<1$. If $|\alpha|\neq|\beta|$, then x is a contraction except the cases $|\alpha|\geqq 1>|\beta|$ and $|\beta|\geqq 1>|\alpha|$. Hence we may assume that $|\alpha|\geqq 1>|\beta|$. Then x is of the infinite order and not a contraction. Let g be a contraction in G. Then we have $\{x\}\cap\{g\}=\{1\}$. This implies that $[G:\{g\}]=\infty$. This contradicts the property (1) of G. Q. E. D.

LEMMA 2. *There exists an infinite cyclic subgroup Z of G such that G is the semi-direct product $Z\cdot H$ of Z and H.*

PROOF. Define a group homomorphism $f: G\to \boldsymbol{R}$ by $f(x)=-\log|\det x|$ $(x\in G)$. By the property (1) of G, we can choose a contraction g_1 in G such that $[f(G):\{f(g_1)\}]=d<+\infty$. Hence $f(g_1)/d\in f(G)$ is a minimum positive element of $f(G)$. Letting g be an element of G such that $f(g)=f(g_1)/d$, we define $Z=\{g\}$. Then it is clear that $G=Z\cdot H$. Q. E. D.

By the properties (1), (2) of G and Lemma 2, H and K are finite subgroups of $GL(2, \boldsymbol{C})$. By Lemma 1, we can choose a generator g of Z so that g is a contraction. We call Z the *infinite part* of G and H the *finite part* of G. If G can be expressed as a direct product of the infinite part and the finite part, then we say that G is *decomposable*. If not, we say that G is *indecomposable*.

Taking a suitable conjugate subgroup to G in $GL(2, \boldsymbol{C})$, we may assume that K is one of the following finite subgroups of $SU(2)=U(2)\cap SL(2, \boldsymbol{C})$ (F. Blichfeldt [**1**]).

1. $K=\{I\}$, $\quad I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$;

2. $K=\{\pm I\}$;

3. $K=A_m=\left\{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\right\}$, the cyclic group of order m,

   $a=$ a primitive m-th root of 1, $m\geqq 3$;

4. $K=B_n=\left\{\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}, \begin{pmatrix}\rho_n & 0\\ 0 & \rho_n^{-1}\end{pmatrix}\right\}$, the dihedral group of order $4n$,

   $i=\sqrt{-1}$, $\quad \rho_n=\exp\frac{\pi}{n}\sqrt{-1}$;

5. $K=C=\left\{\begin{pmatrix}\zeta^2 & 0\\ 0 & \zeta^{-2}\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}\zeta^3 & \zeta^3\\ \zeta & -\zeta\end{pmatrix}\right\}$, the tetrahedral group, order$=24$,

   $\zeta=\exp\frac{\pi}{4}\sqrt{-1}$;

6. $K=D=\left\{\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}\zeta^3 & \zeta^3\\ \zeta & -\zeta\end{pmatrix}\right\}$, the octahedral group, order$=48$,

$$\zeta=\exp\frac{\pi}{4}\sqrt{-1}\,;$$

7. $K=E=\left\{\begin{pmatrix}\varepsilon^3 & 0\\ 0 & \varepsilon^2\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix}\varepsilon^4-\varepsilon & \varepsilon^2-\varepsilon^3\\ \varepsilon^2-\varepsilon^3 & \varepsilon-\varepsilon^4\end{pmatrix}\right\}$, the icosahedral group, order=120,

$$\varepsilon=\exp\frac{2\pi}{5}\sqrt{-1}\,.$$

Let G_2 be a subgroup of G_1. We denote by $N_{G_1}(G_2)$ and $C_{G_1}(G_2)$ the normalizer of G_2 in G_1 and the centralizer of G_2 in G_1 respectively. The following lemma is useful. The method of proof is due to E. Ban-nai.

LEMMA 3. *If K is non-abelian, then $N_{SL(2,C)}(K)$ is a finite group.*

PROOF. Put $N=N_{SL(2,C)}(K)$ and $C=C_{SL(2,C)}(K)$. An element $n\in N$ acts on K as an inner automorphism of K in a natural way and C acts trivially on K. Since C is a normal subgroup of N, we have a following group homomorphism;

$$\begin{array}{ccc}\varphi:\ N/C & \longrightarrow & \mathrm{Aut}(K)\\ \Psi & & \Psi\\ \bar{n}=\text{the equivalent class of } n & \longmapsto & \text{the inner automorphism corresponding to } n,\end{array}$$

where Aut(K) denotes the group of all automorphisms of K. It is clear that Ker $\varphi=\{\bar{1}\}$. Hence φ is an injection. Since Aut(K) is a finite group, N/C is also a finite group. We regard $K\subsetneq GL(2, \boldsymbol{C})$ is an irreducible representation of K into $GL(2, \boldsymbol{C})$. Then, by Shur's lemma,

$$C\subset\left\{\begin{pmatrix}c & 0\\ 0 & c\end{pmatrix}:\ c\in\boldsymbol{C}^*,\ c^2=1\right\}=\left\{\pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}.$$

Hence C is a finite group. Therefore N is a finite group. Q.E.D.

LEMMA 4. *The following equalities hold;*

$$N_{SL(2,C)}(A_m)=\left\{\begin{pmatrix}s & 0\\ 0 & s^{-1}\end{pmatrix}, \begin{pmatrix}0 & t^{-1}\\ -t & 0\end{pmatrix}:\ s,\ t\in\boldsymbol{C}^*\right\}\quad(m\geqq 3),$$

$$N_{SL(2,C)}(B_n)=B_{2n}\qquad(n\geqq 2),$$

$$N_{SL(2,C)}(C)=D\,,$$

$$N_{SL(2,C)}(D)=D\,,$$

$$N_{SL(2,C)}(E)=E\,.$$

PROOF. We denote by $|G_1|$ the order of the group G_1. Considering the order of groups, we have $N_{SL(2,C)}(D)=D$, because $|D|=48$ is a divisor of $|N_{SL(2,C)}(D)|$ and it is clear that D is not contained in B_{2n}. Similarly we have

$N_{SL(2,C)}(E)=E$. Since $\{C, u\}=D$, where $u=\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}\in N_{SL(2,C)}(C)$, it follows that $D\subset N_{SL(2,C)}(C)$. Hence $|N_{SL(2,C)}(C)|$ is a multiple of 48. Thus we have $N_{SL(2,C)}(C)=D$.

Now we consider the case $N=N_{SL(2,C)}(B_n)$ $(n\geqq 2)$. Put $x=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in N$ $(\alpha\delta-\beta\gamma=1)$, $\sigma=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$ and $\tau=\begin{pmatrix}\rho_n & 0\\ 0 & \rho_n^{-1}\end{pmatrix}$. Since the eigenvalues of x are invariant under inner automorphisms, one of the following two equalities holds;

$$\text{(a)}\ \begin{cases} x\tau x^{-1}=\tau \\ x\sigma x^{-1}=\sigma\tau^{\lambda}\end{cases}\qquad \text{(b)}\ \begin{cases} x\tau x^{-1}=\tau^{-1} \\ x\sigma x^{-1}=\sigma\tau^{\lambda}\end{cases}$$

where λ is a suitable integer. By the direct calculation of matrices, we have

(a) $\alpha=\delta\rho_n^{-\lambda},\ \ \alpha\delta=1,\ \ \beta=\gamma=0$,

(b) $\beta=\gamma\rho_n^{-\lambda},\ \ \gamma\beta=-1,\ \ \alpha=\delta=0$,

that is

$$\text{(a)}\quad x=\pm\begin{pmatrix}\rho_{2n} & 0\\ 0 & \rho_{2n}^{-1}\end{pmatrix}^{-\lambda},$$

$$\text{(b)}\quad x=\pm\,\sigma\begin{pmatrix}\rho_{2n} & 0\\ 0 & \rho_{2n}^{-1}\end{pmatrix}^{\lambda}.$$

Then, in both cases, we have $x\in B_{2n}$. Conversely $x\in B_{2n}$ leads to $x\in N$. Hence we obtain $N=B_{2n}$.

In the similar manner, we can prove the case $N=N_{SL(2,C)}(A_m)$ $(m\geqq 3)$.

Q. E. D.

By Lemma 4 and the direct calculation, we have the following

LEMMA 5.

$$N_{GL(2,C)}(A_m)=\boldsymbol{C}^*I\cdot\left\{\begin{pmatrix}s & 0\\ 0 & s^{-1}\end{pmatrix},\begin{pmatrix}0 & t^{-1}\\ t & 0\end{pmatrix}:\ s,\ t\in\boldsymbol{C}^*\right\}\quad(m\geqq 3),$$

$$N_{GL(2,C)}(B_n)=\boldsymbol{C}^*I\cdot B_{2n}\quad(n\geqq 2),$$

$$N_{GL(2,C)}(C)=\boldsymbol{C}^*I\cdot D=\boldsymbol{C}^*I\cdot\left\{C,\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}\right\},$$

$$N_{GL(2,C)}(D)=\boldsymbol{C}^*I\cdot D,$$

$$N_{GL(2,C)}(E)=\boldsymbol{C}^*I\cdot E.$$

REMARK (a). We have fixed the generators of K all of which are elements of $U(2)$. There exists an inner automorphism of $GL(2,\boldsymbol{C})$ which sends H into $U(2)$ leaving K fixed. In fact, this is clear for $K=\{I\}$, $\{\pm I\}$, B_n, C, D and E by Lemma 5. In the case $K=A_m$ $(m\geqq 3)$, H is generated by $\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}$,

$\begin{pmatrix} c_l s_l & 0 \\ 0 & c_l s_l^{-1} \end{pmatrix}$ $(l=1, 2, \cdots, h)$ and $\begin{pmatrix} 0 & d_j t_j^{-1} \\ d_j t_j & 0 \end{pmatrix}$ $(j=1, 2, \cdots, k)$, where $|c_i|=|d_i|=1$. Since the order of $\begin{pmatrix} c_i s_i & 0 \\ 0 & c_i s_i^{-1} \end{pmatrix}$ is finite, we have $|c_i s_i|=|c_i s_i^{-1}|=1$. For any i, j, we have $|t_i|=|t_j|$, because $\begin{pmatrix} 0 & d_i t_i^{-1} \\ d_i t_i & 0 \end{pmatrix}\begin{pmatrix} 0 & d_j t_j^{-1} \\ d_j t_j & 0 \end{pmatrix}=\begin{pmatrix} d_i d_j t_i^{-1} t_j & 0 \\ 0 & d_i d_j t_i t_j^{-1} \end{pmatrix}$ has a finite order. Now put $\theta=\begin{pmatrix} 0 & r^{-1} \\ r & 0 \end{pmatrix}$, where $r=|t_j|^{\frac{1}{2}}$. Then it is easy to check that $\theta^{-1}H\theta \subset U(2)$ and $\theta^{-1}K\theta=K$. We assume for the remainder of the paper that H is a subgroup of $U(2)$.

LEMMA 6. *If G is indecomposable and $K \neq \{I\}$, $\{\pm I\}$, then $K \neq D$, E. Furthermore (in the cases $K=A_m$ $(m \geqq 3)$, $K=B_n$ $(n \geqq 2)$ and $K=C$), we can choose a generator g of the infinite part of G expressed in the form $g=cu$ where $c \in \boldsymbol{C}^*$ and $u \in GL(2, \boldsymbol{C})$ having the following form;*

Case 1 $K=A_m$: $u=\begin{pmatrix} 0 & t^{-1} \\ t & 0 \end{pmatrix}$ *for some* $t \in \boldsymbol{C}^*$,

Case 2 $K=B_n$: $u=\begin{pmatrix} \rho_{2n} & 0 \\ 0 & \rho_{2n}^{-1} \end{pmatrix}$,

Case 3 $K=C$: $u=\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$.

PROOF. First we show that for non-abelian K any generator g of the infinite part of G can not belong to $\boldsymbol{C}^*I \cdot K$. In fact, $g \in \boldsymbol{C}^*I \cdot K$ implies the existence of $c \in \boldsymbol{C}^*$ and $g_0 \in K$ such that $g=cg_0$, so we have the direct product splitting $G=\{cI\} \times H$ which leads to the contradiction to the indecomposability of G. On the other hand $G \subset N_{GL(2,C)}(K)$. Now applying Lemma 5, we conclude that K is one of the groups A_m, B_n and C. In the cases $K=B_m$ and $K=C$, g can be written in the form $g=cug_0$ with $c \in \boldsymbol{C}^*$, $g_0 \in K$. Hence we have $g^2 \in \boldsymbol{C}^*I \cdot K$ and $G=\{cu\} \cdot H$. If K and H are abelian, then H is a group generated by the elements of K and some matrices of the form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$, since $K \neq \{I\}$, $\{\pm I\}$. On the other hand, we have $g=c\begin{pmatrix} 0 & t^{-1} \\ t & 0 \end{pmatrix}$ or $c\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$ by Lemma 5. If g is of the latter form, G is abelian. This contradicts the assumption that G is indecomposable. Hence we have $g=c\begin{pmatrix} 0 & t^{-1} \\ t & 0 \end{pmatrix}$ as desired. If K is abelian and H is non-abelian, then H contains an element of the form $\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$ $(|q|=|r|=1)$. Now we express G as a semi-direct product $G=\{g\} \cdot H$ where $g=c\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$. Then we define an element g' of G by

$$g'=g\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}=c\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}=c(rq)^{\frac{1}{2}}\begin{pmatrix} 0 & t^{-1} \\ t & 0 \end{pmatrix}$$

where $t=(rq^{-1})^{\frac{1}{2}}s^{-1}$. It is easy to check that $G=\{g\}\cdot H=\{g'\}\cdot H$ and g' is an element of the required form. We note that, in each of the cases, we have $g^2\in \boldsymbol{C}^*I\cdot K$. Q. E. D.

LEMMA 7. *If G is indecomposable and $K\neq\{I\}$, $\{\pm I\}$, then $K\neq D$, E and G can be expressed as follows;*

$$G=G_0\cup g\cdot G_0 \qquad (\textit{coset decomposition}),$$

where $G_0=\{c^2I\}\times H$, $g=cu$. The element u of $GL(2, \boldsymbol{C})$ is defined as follows;

Case 1_a $K=A_m$ $(m\geqq 3)$, $H=$*abelian,*

$$u=\begin{pmatrix}0 & t^{-1}\\ t & 0\end{pmatrix} \qquad (t\in \boldsymbol{C}^*);$$

Case 1_b $K=A_m$ $(m\geqq 3)$, $H=$*non-abelian,*

$$u=\begin{pmatrix}0 & t^{-1}\\ t & 0\end{pmatrix} \qquad (|t|=1);$$

Case 2 $K=B_n$ $(n\geqq 2)$,

$$u=\begin{pmatrix}\rho_{2n} & 0\\ 0 & \rho_{2n}^{-1}\end{pmatrix}\in SU(2);$$

Case 3 $K=C$,

$$u=\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}\in SU(2).$$

We note that the case 1_a really happens if and only if the element $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ is contained in $N_{GL(2,\boldsymbol{C})}(H)$ and that in the cases 1_b, 2 and 3, u is a unitary matrix.

PROOF. Let $g=cu$ be the generator of the infinite part of G which is defined in Lemma 6 and we express G as a semi-direct product $G=\{g\}\cdot H$. Since $g^2=c^2u^2\in \boldsymbol{C}^*I\cdot K$, we have a subgroup $G_0=\{c^2I\}\times H$ of G of index 2. G can be expressed as $G=G_0\cup g\cdot G_0$. Now examining the condition that g must be contained in $N_{GL(2,\boldsymbol{C})}(H)$, we shall derive some restrictions to the form of u. If $K=A_m$ and H is abelian, then each element of H is of the form $\begin{pmatrix}* & 0\\ 0 & *\end{pmatrix}$. Hence $g=c\begin{pmatrix}0 & t^{-1}\\ t & 0\end{pmatrix}$ is contained in $N_{GL(2,\boldsymbol{C})}(H)$ if and only if $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\in N_{GL(2,\boldsymbol{C})}(H)$. If $K=A_m$ and H is non-abelian, H contains an element of the form $\begin{pmatrix}0 & q\\ r & 0\end{pmatrix}$ $(|q|=|r|=1)$. Then by $g^{-1}\begin{pmatrix}0 & q\\ r & 0\end{pmatrix}g=\begin{pmatrix}0 & t^{-2}r\\ t^2q & 0\end{pmatrix}\in H$, $|t|=1$ follows. If $K=B_n$, then H is generated by the elements of B_n, scalar matrices and some elements of the form $\lambda\begin{pmatrix}\rho_{2n} & 0\\ 0 & \rho_{2n}^{-1}\end{pmatrix}$ $(|\lambda|=1)$. Thus it is clear that $g=cu=c\begin{pmatrix}\rho_{2n} & 0\\ 0 & \rho_{2n}^{-1}\end{pmatrix}\in N_{GL(2,\boldsymbol{C})}(H)$. If $K=C$, then H is generated by the elements of C, scalar matrices and some elements of the form $\lambda\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}$ $(|\lambda|=1)$. Thus we have $g=c\begin{pmatrix}\zeta & 0\\ 0 & \zeta^{-1}\end{pmatrix}\in N_{GL(2,C)}(H)$. Q. E. D.

PROPOSITION 8. *G is conjugate in $GL(2, \boldsymbol{C})$ to one of the following groups,*
Notation: α, β, c, $t\in \boldsymbol{C}^*$: constants satisfying

$$0<|\alpha|<1, \quad 0<|\beta|<1 \quad \textit{and} \quad 0<|c|<1,$$

$m=$ the order of H,
$a=$ a primitive m-th root of 1,
$(m, n)=$ the greatest common divisor of integers m, n.
Case I $K=\{I\}$,

(i) $\left\{\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right\}\times H, \quad H=\left\{\begin{pmatrix}a & 0\\ 0 & a^n\end{pmatrix}\right\}, \quad (m, n)=1, (m, n+1)=1;$

(ii) $\left\{\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}\right\}\times H, \quad H=\left\{\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}\right\};$

Case II $K=\{\pm I\}$,

(i) $\left\{\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right\}\times H, \quad H=\left\{\begin{pmatrix}a & 0\\ 0 & a^n\end{pmatrix}\right\}, \quad (m, n)=1,\ m\equiv n+1\equiv 0 \pmod 2,\ \left(\frac{m}{2}, \frac{n+1}{2}\right)=1;$

(ii) $\left\{\begin{pmatrix}\alpha & 1\\ 0 & \alpha\end{pmatrix}\right\}\times H, \quad H=\left\{\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}\right\}, \quad m\equiv 0 \pmod 2;$

(iii) $\{c^2I\}\times H\cup(cu)\cdot(\{c^2I\}\times H)$ *(coset decomposition)*,

$$H=\left\{\begin{pmatrix}a & 0\\ 0 & -a\end{pmatrix}\right\}, \quad u=\begin{pmatrix}0 & t^{-1}\\ t & 0\end{pmatrix}, \quad m\equiv 0 \pmod 2;$$

Case III $K\neq\{I\}, \{\pm I\}$,

(i) $\left\{\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right\}\times H, \quad H=$ *abelian*;

(ii) $\{cI\}\times H$;

(iii) $\{c^2I\}\times H\cup(cu)\cdot(\{c^2I\}\times H)$ *(coset decomposition)*;

where, in the case III (iii), *$H\subset U(2)$ is a finite group acting on S^3 freely, u is determined according to Lemma 7 and K is neither D nor E.*

PROOF. Define a group homomorphism d by

$$\begin{array}{cccl} d: H & \longrightarrow & U(1)=\{z\in \boldsymbol{C}^*: |z|=1\}. \\ \cup\!\!\!\shortmid & & \cup\!\!\!\shortmid \\ x & \longmapsto & \det x. \end{array}$$

Then $\operatorname{Ker} d = K$ and $d(H) \cong \boldsymbol{Z}_{m'}$ for some positive integer m'.

If G is indecomposable and $K \neq \{I\}$, $\{\pm I\}$, then there occurs the case III (iii) by Lemma 7.

If G is decomposable and $K \neq \{I\}$, $\{\pm I\}$, we have two cases.

(a) $H =$ non-abelian: In this case, the generator of the infinite part of G is a scalar matrix cI with $c \in \boldsymbol{C}^*$. So the case is III (ii).

(b) $H =$ abelian, $K = A_m$ $(m \geqq 3)$: The case is III (i).

Now there remain two cases I and II.

Case I $K = \{I\}$. Since d is an injection, we have $m' = m$ and $H \cong \boldsymbol{Z}_{m'}$. Since H acts on S^3 freely, we may assume that H is generated by an element $h = \begin{pmatrix} a & 0 \\ 0 & a^n \end{pmatrix}$ with some integer n such that $(m, n) = 1$. In addition, the condition $K = \{I\}$ implies $(m, n+1) = 1$. We express G as $\{g\} \cdot H$ (semi-direct product). Since $g^{-1}hg \in H$ and $d(g^{-1}hg) = d(h)$, it follows that $g^{-1}hg = h$. Thus if $n \not\equiv 1 \pmod{m}$, then g has the form $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, and G is the case I (i). If $n \equiv 1 \pmod{m}$, then g is an arbitrary contraction and consequently the case I (ii) may occur.

Case II $K = \{\pm I\}$. In this case, we have $2m' = m$. Take an element $h \in H$ so that $d(h)$ is a generator of $d(H)$. Then $h^{m'}$ is either I or $-I$. If $h^{m'} = I$ and m' is odd, the order of $-h \in H$ is $2m' = m$. Hence we have $H \cong \boldsymbol{Z}_m$. Let $h^{m'} = I$ and m' is even, i. e., $m' = 2m''$. Letting ε_1, ε_2 be the eigenvalues of h, we have $\varepsilon_1^{m''} = \pm 1$, $\varepsilon_2^{m''} = \pm 1$. Since $h^{m''} \neq I$ and $h^{m''}$ has no fixed points, we have $\varepsilon_1^{m''} = \varepsilon_2^{m''} = -1$. Consequently $h^{m''} = -I$. But this contradicts the assumption that $d(h)$ is a generator of $d(H)$. Hence m' is odd. If $h^{m'} = -I$, then h has the order $2m' = m$. This implies $H \cong \boldsymbol{Z}_m$. We express G as $\{g\} \cdot H$ (semi-direct product). Let $h = \begin{pmatrix} a & 0 \\ 0 & a^n \end{pmatrix}$ be a generator of H. Since a^{n+1} is a primitive $\frac{m}{2}$-th root of 1, we have $n+1 \equiv 0 \pmod{2}$. Let $e = \left(\frac{m}{2}, \frac{n+1}{2}\right)$, $\frac{m}{2} = ek$ and $\frac{n+1}{2} = el$ where $k, l \in \boldsymbol{Z}$. Then $(a^{n+1})^k = a^{2elk} = (a^m)^l = 1$. Thus k is a multiple of $\frac{m}{2}$ and we have $e = 1$. Since $g^{-1}hg \in H$ and $d(g^{-1}hg) = d(h)$, it follows that $g^{-1}hg = h$ or $-h$. If $g^{-1}hg = h$, then g is of the form $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. So the cases II (i) and II (ii) may occur by the similar argument to case I. If $g^{-1}hg = -h$, then we have $a^n = -a$, since h has the same eigenvalues as $-h$. And moreover g has the form $\begin{pmatrix} 0 & \gamma \\ \delta & 0 \end{pmatrix}$. Hence, letting $c = (\gamma\delta)^{\frac{1}{2}}$, $t = (\delta\gamma^{-1})^{\frac{1}{2}}$ and $u = \begin{pmatrix} 0 & t^{-1} \\ t & 0 \end{pmatrix}$, this case is reduced to the case II (iii). Q. E. D.

REMARK (b). All the types of finite subgroups H of $U(2)$ which acts on S^3 freely are determined by Threlfall-Seifert [**6**].

REMARK (c). Any finite subgroups of $SU(2)$ acts on S^3 freely, since any elements of them except the identity matrix never have 1 as their eigenvalues.

§ 3. Topology of Hopf surfaces.

Let $S=W/G$ be a Hopf surface. If G is abelian, it may happen that G is not contained in $GL(2, \boldsymbol{C})$ (cf. the property (2) of G). If G is abelian, G is expressed as $G\cong \boldsymbol{Z}\oplus \mathrm{Torsion}(G)$. Let g be a contraction corresponding to a generator of $\boldsymbol{Z}$. Then we may assume that g has the form

$$g:(z_1, z_2)\longmapsto (z_1', z_2')=(\alpha z_1+\lambda z_2^n, \beta z_2),$$

where n is a certain positive integer and α, β and λ are constants satisfying $(\alpha-\beta^n)\lambda=0$, $0<|\alpha|\leqq|\beta|<1$. Let h be an arbitrary element of G which has a finite order. Now we determine the form of h by a similar method to the proof of Theorem 47, Kodaira [3].

By the Hartogs' theorem, there exist holomorphic functions ϕ, ψ on $\boldsymbol{C}^2$ such that

$$h: z=(z_1, z_2)\longmapsto h(z)=(\phi(z), \psi(z)), \ \phi(0)=\psi(0)=0.$$

By $g\cdot h=h\cdot g$, we have

$$\phi(\alpha z_1+\lambda z_2^n, \beta z_2)=\alpha\phi(z_1, z_2)+\lambda\psi(z_1, z_2)^n, \tag{3-1}$$

$$\psi(\alpha z_1+\lambda z_2^n, \beta z_2)=\beta\psi(z_1, z_2). \tag{3-2}$$

If $\lambda=0$, then we have by (3-1), (3-2),

$$\phi(\alpha z_1, \beta z_2)=\alpha\phi(z_1, z_2),$$

$$\psi(\alpha z_1, \beta z_2)=\beta\psi(z_1, z_2).$$

Hence we have $\phi(z_1, z_2)=az_1$, $\psi(z_1, z_2)=bz_2$ where a, b are certain constants, and so $h\in GL(2, \boldsymbol{C})$. Consequently we have $G\subset GL(2, \boldsymbol{C})$. If $\lambda\neq 0$, then $\alpha=\beta^n$. Applying $\frac{\partial}{\partial z_1}$ to (3-2), we get

$$\alpha\frac{\partial\psi}{\partial z_1}(gz)=\beta\frac{\partial\psi}{\partial z_1}(z).$$

Hence we obtain

$$\frac{\partial\psi}{\partial z_1}(z)=\lim_{\nu\to+\infty}\left(\frac{\alpha}{\beta}\right)^{\nu}\frac{\partial\psi}{\partial z_1}(g^{\nu}z)=\lim_{\nu\to+\infty}\left(\frac{\alpha}{\beta}\right)^{\nu}\frac{\partial\psi}{\partial z_1}(0).$$

This shows that

$$\frac{\partial\psi}{\partial z_1}(z)=\begin{cases}\frac{\partial\psi}{\partial z_1}(0), & \text{if } \alpha=\beta, \\ 0, & \text{if } \alpha\neq\beta.\end{cases}$$

If $\alpha=\beta$, then $n=1$. Applying $\frac{\partial}{\partial z_2}$ to (3-2), we have

$$\lambda\frac{\partial\psi}{\partial z_1}(0)+\beta\frac{\partial\psi}{\partial z_2}(0)=\beta\frac{\partial\psi}{\partial z_2}(0),$$

and $\frac{\partial\psi}{\partial z_1}(0)=0$ follows. Thus, in both cases, we have $\frac{\partial\psi}{\partial z_1}(z)=0$. Applying $\frac{\partial}{\partial z_2}$ to (3-2), we get

$$\frac{\partial\psi}{\partial z_2}(gz)=\frac{\partial\psi}{\partial z_2}(z).$$

Therefore $\frac{\partial\psi}{\partial z_2}$ is reduced to constant. Denote this constant by $a(h)$. Then we have $\psi(z)=a(h)z_2$. Now (3-1) is reduced to

$$\phi(\alpha z_1+\lambda z_2^n, \beta z_2)=\alpha\phi(z_1, z_2)+\lambda a(h)^n z_2^n.$$

Applying $\frac{\partial}{\partial z_1}$, we obtain

$$\frac{\partial\phi}{\partial z_1}(gz)=\frac{\partial\phi}{\partial z_1}(z).$$

Therefore $\frac{\partial\phi}{\partial z_1}$ is a constant and we denote this constant by $b(h)$. Then $\phi(z)$ is reduced to the form

$$\phi(z)=b(h)z_1+\eta(z_2), \qquad \eta(0)=0,$$

where $\eta(z_2)$ is a holomorphic function of z_2. By the equality (3-1), we have

$$\alpha\eta(z_2)-\eta(\beta z_2)=\lambda(b(h)-a(h)^n)z_2^n.$$

Letting $\eta(z_2)=\sum_{k\geqq 1}c_k z_2^k$, we get

$$c_k=0 \qquad \text{for all} \quad k\neq n,$$
$$a(h)^n=b(h).$$

Since h has a finite order, c_n vanishes and $a(h)^{m_1}=1$ for some integer m_1. Hence h has the form

$$h:(z_1, z_2)\longmapsto(a(h)^n z_1, a(h)z_2), \qquad a(h)^{m_1}=1.$$

Clearly the mapping $h\mapsto a(h)$ is a group isomorphism from Torsion (G) into $U(1)$. Therefore we conclude that Torsion $(G)\cong \boldsymbol{Z}_m$ for a certain positive integer m such that $(m, n)=1$, since h has no fixed points.

The above argument shows that, if $G\not\subset GL(2, \boldsymbol{C})$, then $G\cong \boldsymbol{Z}\oplus\boldsymbol{Z}_m$ and that G is generated by the following two elements;

$$g_\lambda:(z_1, z_2)\longmapsto(\beta^n z_1+\lambda z_2^n, \beta z_2) \qquad 0<|\beta|<1,$$

$$h:(z_1, z_2)\longmapsto(a^n z_1, a z_2) \qquad (m, n)=1,$$

where a is a primitive m-th root of 1 and λ is a fixed constant. Let $\tilde{\omega}: W \to W/\{h\}$ be the canonical projection. We define a holomorphic automorphism φ of $(W/\{h\}) \times \boldsymbol{C}$ by

$$\begin{array}{ccc} \varphi: (W/\{h\}) \times \boldsymbol{C} & \longrightarrow & (W/\{h\}) \times \boldsymbol{C} \\ \Downarrow & & \Downarrow \\ (\tilde{\omega}(z), \lambda) & \longmapsto & (\tilde{\omega}(g_\lambda z), \lambda) \end{array}$$

where $\boldsymbol{C}$ denotes the plane of complex numbers. Since $g_\lambda \cdot h = h \cdot g_\lambda$, φ is well-defined. The infinite cyclic group $\{\varphi\}$ acts on $(W/\{h\}) \times \boldsymbol{C}$ properly-discontinuously and freely. Hence $\mathcal{M} = ((W/\{h\}) \times \boldsymbol{C})/\{\varphi\}$ is a family of deformations of Hopf surfaces parametrized by λ. Let p be the natural projection $\mathcal{M} \to \boldsymbol{C}$. Each $p^{-1}(\lambda)$ $(\lambda \in \boldsymbol{C})$ is diffeomorphic to $p^{-1}(0)$. Note that $p^{-1}(0)$ is a quotient space of W by a subgroup G of $GL(2, \boldsymbol{C})$. Hence, in order to classify Hopf surfaces by diffeomorphisms, it is sufficient to consider only the case where G are subgroups of $GL(2, \boldsymbol{C})$.

THEOREM 9. *Each Hopf surface $S = W/G$ is diffeomorphic to one of the following types;*

(1) $S^1 \times (S^3/H)$,

(2) *(S^3/H)-bundle over S^1 whose transition function $u: S^3/H \to S^3/H$ is an involution of S^3/H.*

PROOF. (a) Cases I (i), II (i), III (i) (ii) (cf. Proposition 8).

Let S^3 be the sphere defined by $S^3 = \{(z_1, z_2) \in \boldsymbol{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$. Define a diffeomorphism $\varphi: \boldsymbol{R} \times S^3 \to W$ by

$$\begin{array}{ccc} \varphi: \boldsymbol{R} \times S^3 & \longrightarrow & W \\ \Downarrow & & \Downarrow \\ (r, (z_1, z_2)) & \longmapsto & (\alpha^r z_1, \beta^r z_2), \end{array}$$

where $\boldsymbol{R}$ is the real line. Let h be an element of G belonging to $H \subset U(2)$ and g a contraction of G of the form $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. Now we define automorphisms $f(h)$, f of $\boldsymbol{R} \times S^3$ by

$$f(h): (r, z) \longmapsto (r, hz),$$

$$f: (r, z) \longmapsto (r+1, z).$$

Then we have the following two commutative diagrams of diffeomorphisms;

$$\begin{array}{ccc} \boldsymbol{R} \times S^3 & \xrightarrow{\varphi} & W \\ \downarrow f(h) & & \downarrow h \\ \boldsymbol{R} \times S^3 & \xrightarrow{\varphi} & W, \end{array} \qquad \begin{array}{ccc} \boldsymbol{R} \times S^3 & \xrightarrow{\varphi} & W \\ \downarrow f & & \downarrow g \\ \boldsymbol{R} \times S^3 & \xrightarrow{\varphi} & W. \end{array}$$

Hence we have a diffeomorphism $S^1 \times (S^3/H) \to W/G = S$ induced from φ.

(b) Cases I (ii), II (ii).

Using the deformation argument, we can reduce these cases to case (a). Hence we have $S\cong S^1\times(S^3/H)$.

(c) Case III (iii) and $H=$non-abelian.

Let $w=(w_1, w_2)$ be a point of W and define a differentiable mapping $\phi_1: W\to \boldsymbol{R}$, $\phi_2: W\to S^3$ by

$$\phi_1(w)=r=\frac{\log s}{\log|c|^2},$$

$$\phi_2(w)=(c^{-\frac{\log s}{\log|c|}}w_1, c^{-\frac{\log s}{\log|c|}}w_2),$$

where $s=(|w_1|^2+|w_2|^2)^{\frac{1}{2}}$. Then it is easy to check that $\phi=(\phi_1, \phi_2): W\to\boldsymbol{R}\times S^3$ is a diffeomorphism. Now we have the following two commutative diagram of diffeomorphisms;

$$\begin{array}{ccc} W & \xrightarrow{\phi} & \boldsymbol{R}\times S^3 \\ \downarrow h & & \downarrow f(h) \\ W & \xrightarrow{\phi} & \boldsymbol{R}\times S^3, \end{array} \qquad \begin{array}{ccc} W & \xrightarrow{\phi} & \boldsymbol{R}\times S^3 \\ \downarrow c^2 I & & \downarrow f \\ W & \xrightarrow{\phi} & \boldsymbol{R}\times S^3, \end{array}$$

where $f(h)$, f are diffeomorphisms defined in case (a). Thus we have $W/G_o\cong S^1\times(S^3/H)$ where $G_o=\{c^2I\}\times H$. Since H is non-abelian, u is a unitary matrix by Lemma 7. Therefore u acts on S^3 naturally and we obtain

$$\phi(cu(w))=(\phi_1(w)+\frac{1}{2}, u(\phi_2(w))).$$

Hence we infer that $S=W/G=(W/G_o)/\{cu\}$ is an (S^3/H)-bundle over S^1 of which the transition function is u. Since $u^2\in K\lhd H$, u is an involution of S^3/H.

(d) Cases II (iii), III (iii) and $H=$abelian.

Put $G_o=\{c^2I\}\times H$, $u=u_t=\begin{pmatrix}0 & t^{-1}\\ t & 0\end{pmatrix}$, $G_t=\{G_o, cu_t\}$ and $W/G_o=S_o$. Since, in this case, $N_{GL(2,\boldsymbol{C})}(H)$ contains $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, W/G_t defines a Hopf surface for arbitrary $t\in\boldsymbol{C}^*$. Let φ be a holomorphic automorphism of $S_o\times\boldsymbol{C}^*$ defined by

$$\begin{array}{rccc} \varphi: & S_o\times\boldsymbol{C}^* & \longrightarrow & S_o\times\boldsymbol{C}^* \\ & \cup\!\!\!\!\,| & & \cup\!\!\!\!\,| \\ & (x, t) & \longmapsto & (cu_t(x), t). \end{array}$$

Then φ is a fixed point free involution of $S_o\times\boldsymbol{C}^*$. Hence $\mathcal{M}=S_o\times\boldsymbol{C}^*/\{\varphi\}=\bigcup_{t\in\boldsymbol{C}^*}(W/G_t)$ is a family of deformations of Hopf surfaces parametrized by t. Since $u_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\in U(2)$, W/G_1 is an (S^3/H)-bundle over S^1 with transition function u_1 by the similar argument to case (c). Moreover, for each $t\in\boldsymbol{C}^*$, $W/G_t\cong W/G_1$. Q.E.D.

Let $S=W/G$ and $S'=W/G'$ be arbitrary two Hopf surfaces. We may assume that both G and G' are subgroups of $GL(2, \boldsymbol{C})$. Note that H (H') is the set of elements of G (G' resp.) of which the orders are finite.

THEOREM 10. *If G and G' are isomorphic as abstract groups, and if $H(\cong H')$ is not a (finite) cyclic group, then S is diffeomorphic to S'.*

PROOF. The group H is unique up to conjugate in $GL(2, \boldsymbol{C})$, since H is not cyclic (Brieskorn [2], Threlfall-Seifert [6]). Hence there exists $\theta\in GL(2, \boldsymbol{C})$ such that $\theta^{-1}H\theta=H'$. It is clear that $\theta^{-1}K\theta=K'$. Hence we may assume that $H=H'\subset U(2)$ and $K=K'$.

If G is decomposable, then by Proposition 8 and Theorem 9, we have $S\cong S'\cong S^1\times(S^3/H)$ (Proposition 8, cases III (i), III (ii)).

Suppose that G is indecomposable. Since we have assumed that H is not a cyclic group, G is of the case III (iii). Hence, in the case $K=B_n$ or C, $S\cong S'$ follows because u is determined uniquely by K. If $K=A_m$ ($m\geqq 3$) and H is abelian, then, as in the proof of Theorem 9 (d), we may assume that $u=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Hence we obtain $S\cong S'$. Next we consider the case where $K=A_m=\left\{k=\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\right\}$ ($m\geqq 3$) and H is non-abelian. Let $d: H\rightarrow U(1)$ be a group homomorphism defined by $d(x)=\det x$. Choose $h\in H$ such that $d(h)$ is a generator of $d(H)\cong \boldsymbol{Z}_n$. Then H is generated by k and h. Since $h\in N_{U(2)}(A_m)$ and H is non-abelian, h has the form $\begin{pmatrix}0 & q\\ r & 0\end{pmatrix}$, where $|q|=|r|=1$. Now we shall determine the generator $g=c\begin{pmatrix}0 & t^{-1}\\ t & 0\end{pmatrix}$ of the infinite part of G. Since $g^{-1}hg$ has a finite order, we can express this element as $g^{-1}hg=h^{\lambda}k^{\mu}$, where λ, μ are integers. Note that λ is clearly odd: $\lambda=2\nu+1$. By the direct calculation, we obtain

$$g^{-1}hg=\begin{pmatrix}0 & t^{-2}r\\ t^2q & 0\end{pmatrix},$$

$$h^{\lambda}k^{\mu}=\begin{pmatrix}0 & a^{-\mu}(qr)^{\nu}q\\ a^{\mu}(qr)^{\nu}r & 0\end{pmatrix},$$

and consequently,

$$t^2=\frac{r}{q}(qr)^{\nu}a^{\mu}=\frac{r}{q}(qr)^{-\nu}a^{\mu}.$$

Hence we get $(qr)^{\nu}=\pm 1$ and

$$g^{-1}hg=hk^{\mu}, \qquad \text{if } t^2=\frac{r}{q}a^{\mu}, \tag{3-3}$$

$$g^{-1}hg=-hk^{\mu}, \qquad \text{if } t^2=-\frac{r}{q}a^{\mu}. \tag{3-4}$$

If (3-3) holds, then we may assume that g is one of the following;

$$g_1=c\begin{pmatrix}0 & (qr^{-1})^{\frac{1}{2}}\\ (qr^{-1})^{-\frac{1}{2}} & 0\end{pmatrix}\qquad g_1^{-1}hg_1=h\,,$$

$$g_2=c\begin{pmatrix}0 & (qr^{-1})^{\frac{1}{2}}a^{-\frac{1}{2}}\\ (qr^{-1})^{-\frac{1}{2}}a^{\frac{1}{2}} & 0\end{pmatrix}\qquad g_2^{-1}hg_2=hk\,.$$

If (3-4) holds, then $m\equiv 0 \pmod 2$, and we may assume that g is one of the following;

$$g_3=c\begin{pmatrix}0 & (-qr^{-1})^{\frac{1}{2}}\\ (-qr^{-1})^{-\frac{1}{2}} & 0\end{pmatrix}\qquad g_3^{-1}hg_3=-h\,,$$

$$g_4=c\begin{pmatrix}0 & (-qr^{-1})^{\frac{1}{2}}a^{-\frac{1}{2}}\\ (-qr^{-1})^{-\frac{1}{2}}a^{\frac{1}{2}} & 0\end{pmatrix}\qquad g_4^{-1}hg_4=-hk\,.$$

Let $G_i=\{h, k, g_i\}$ $(i=1, 2, 3, 4)$. Note that H is a subgroup of G_i consisting of all elements of which the orders are finite.

Step 1. It is easy to check that $[H, H]=\{k^2\}$, $G_i\triangleright[H, H]$ and $g_i^{-1}kg_i=k^{-1}$ for any i, where $[H, H]$ denotes the commutator subgroup of H.

Step 2. If $m\equiv 1 \pmod 2$, then $-I\notin K$. This implies that equalities $g_3^{-1}hg_3=-h$ and $g_4^{-1}hg_4=-hk$ never hold. Since m is odd, there exists $\lambda\in \boldsymbol{Z}$ such that $a^{2\lambda+1}=1$. Then we obtain

$$g_2k^{\lambda}=c\begin{pmatrix}0 & (qr^{-1})^{\frac{1}{2}}a^{-\frac{1}{2}}\\ (qr^{-1})^{-\frac{1}{2}}a^{\frac{1}{2}} & 0\end{pmatrix}\begin{pmatrix}a^{\lambda} & 0\\ 0 & a^{-\lambda}\end{pmatrix}=c\kappa\begin{pmatrix}0 & (qr^{-1})^{\frac{1}{2}}\\ (qr^{-1})^{-\frac{1}{2}} & 0\end{pmatrix}$$
$$=\kappa g_1\,,\qquad (\kappa=\pm 1)\,.$$

This shows that if $m\equiv 1 \pmod 2$, then G and G' are $\{h, k, \kappa g_1\}$ $(\kappa=\pm 1)$. Hence we have $S\cong S'$.

Step 3. If $m\equiv 0 \pmod 2$ and $m\not\equiv 0 \pmod 4$, then there exists $\lambda\in\boldsymbol{Z}$ such that $a^{2\lambda+1}=-1$. Hence we get

$$g_2k^{\lambda}=c\begin{pmatrix}0 & (qr^{-1})^{\frac{1}{2}}a^{-\frac{1}{2}}\\ (qr^{-1})^{-\frac{1}{2}}a^{\frac{1}{2}} & 0\end{pmatrix}\begin{pmatrix}a^{\lambda} & 0\\ 0 & a^{-\lambda}\end{pmatrix}=c\begin{pmatrix}0 & (-qr^{-1})^{\frac{1}{2}}\\ (-qr^{-1})^{-\frac{1}{2}} & 0\end{pmatrix}$$
$$=\kappa g_3\,,$$

$$g_4k^{\lambda}=c\begin{pmatrix}0 & (-qr^{-1})^{\frac{1}{2}}a^{-\frac{1}{2}}\\ (-qr^{-1})^{-\frac{1}{2}}a^{\frac{1}{2}} & 0\end{pmatrix}\begin{pmatrix}a^{\lambda} & 0\\ 0 & a^{-\lambda}\end{pmatrix}=c\kappa\begin{pmatrix}0 & (qr^{-1})^{\frac{1}{2}}\\ (qr^{-1})^{-\frac{1}{2}} & 0\end{pmatrix}$$

$$=\kappa g_1 .$$

This shows that G and G' are G_1 or G_3.

Now we shall show that G_1 and G_3 are not isomorphic as abstract groups. This implies that $S\cong S'$. In $G_1/[H, H]$, we have $\bar{g}_1^{-1}\bar{k}\bar{g}_1=\bar{k}^{-1}=\bar{k}$, $\bar{g}_1^{-1}\bar{h}\bar{g}_1=\bar{h}$ and $\bar{h}\bar{k}=\bar{k}\bar{h}$. This shows that $G_1/[H, H]$ is abelian. On the other hand, by $m\not\equiv 0 \pmod 4$, we get $-I\notin [H, H]$. Hence $\bar{g}_3^{-1}\bar{h}\bar{g}_3=-\bar{h}\neq\bar{h}$ in $G_3/[H, H]$. This implies that $G_3/[H, H]$ is non-abelian. Hence $G_1\not\cong G_3$.

Step 4. If $m\equiv 0 \pmod 4$, then there exists $\lambda\in \boldsymbol{Z}$ such that $a^{2\lambda}=-1$. Hence we get

$$g_3k^{\lambda}=c\begin{pmatrix} 0 & (-qr^{-1})^{\frac{1}{2}} \\ (-qr^{-1})^{-\frac{1}{2}} & 0 \end{pmatrix}\begin{pmatrix} a^{\lambda} & 0 \\ 0 & a^{-\lambda} \end{pmatrix}=c\kappa\begin{pmatrix} 0 & (qr^{-1})^{\frac{1}{2}} \\ (qr^{-1})^{-\frac{1}{2}} & 0 \end{pmatrix}$$
$$=\kappa g_1 ,$$
$$g_4k^{\lambda}=c\begin{pmatrix} 0 & (-qr^{-1})^{\frac{1}{2}}a^{-\frac{1}{2}} \\ (-qr^{-1})^{-\frac{1}{2}}a^{\frac{1}{2}} & 0 \end{pmatrix}\begin{pmatrix} a^{\lambda} & 0 \\ 0 & a^{-\lambda} \end{pmatrix}$$
$$=c\kappa\begin{pmatrix} 0 & (qr^{-1})^{\frac{1}{2}}a^{-\frac{1}{2}} \\ (qr^{-1})^{-\frac{1}{2}}a^{\frac{1}{2}} & 0 \end{pmatrix}=\kappa g_2 .$$

This shows that G and G' are G_1 or G_2. Then $S\cong S'$ follows from the fact that G_1 and G_2 are not isomorphic. In fact, $\bar{g}_2^{-1}\bar{h}\bar{g}_2=\bar{h}\bar{k}\neq\bar{h}$ holds, since $k\notin [H, H]$. Hence $G_2/[H, H]$ is non-abelian, while $G_1/[H, H]$ is abelian. This completes the proof of the theorem.

Next we consider the case where H is a (finite) cyclic group. Let $H\cong \boldsymbol{Z}_m$ and $h=\begin{pmatrix} a & 0 \\ 0 & a^n \end{pmatrix}$ $((m, n)=1)$ the generator of H, where a is a primitive m-th root of 1. S^3/H defines a 3-dimensional lens space. Now we shall prove the following lemma.

LEMMA 11. *Let L_1 and L_2 be lens spaces. If $\boldsymbol{R}\times L_1$ and $\boldsymbol{R}\times L_2$ are diffeomorphic, then L_1 and L_2 are diffeomorphic.*

PROOF. Let $\tilde{\varphi}: \boldsymbol{R}\times L_1\rightarrow \boldsymbol{R}\times L_2$ be a diffeomorphism. Put $\tilde{\varphi}(\{0\}\times L_1)=L_1'$ and choose t large enough so that $L_1'\cap\{t\}\times L_2=\emptyset$. We identify $\{t\}\times L_2$ with L_2. Let V be a compact submanifold in $\boldsymbol{R}\times L_2$ with boundaries; $\partial V=L_1'\cup L_2$, $L_1'\cap L_2=\emptyset$. Note that L_2 is a deformation retract of V. In fact, V is a deformation retract of $\{r\leqq t\}\times L_2$, since $\tilde{\varphi}(\{r\leqq 0\}\times L_1)\cup V=\{r\leqq t\}\times L_2$. On the other hand, by the fact that L_2 is a deformation retract of $\{r\leqq t\}\times L_2$, the inclusion mapping $i_1: L_2\rightarrow V$ is a homotopy equivalence. Hence L_2 is a deformation retract of V. Similarly, L_1' is a deformation retract of V. Thus (V, L_1', L_2) is an h-cobordism. Then $L_1'\cong L_2$ follows from Corollary 12.12 of Milnor [4, p. 410]. Q. E. D.

Now we shall prove the following

THEOREM 12. *Let G and G' be isomorphic as abstract groups and H $(\cong H')$ be a (finite) cyclic group. Then,*

(1) *if G is indecomposable, $S\cong S'$,*

(2) *if G is decomposable, $S\cong S'$ if and only if $S^3/H\cong S^3/H'$.*

PROOF. If G is decomposable, the condition $S^3/H\cong S^3/H'$ is clearly sufficient for $S\cong S'$. Conversely, assume that there exists a diffeomorphism $\varphi: S\to S'$. Then we have a following commutative diagram;

$$\begin{array}{ccc} \boldsymbol{R}\times(S^3/H) & \xrightarrow{\tilde{\varphi}} & \boldsymbol{R}\times(S^3/H') \\ \downarrow{\tilde{p}} & & \downarrow{\tilde{p}} \\ S & \xrightarrow{\varphi} & S' \end{array}$$

where $\tilde{p}$ is a covering map induced from the natural covering $p:\boldsymbol{R}\to S^1$. Hence, by Lemma 11, we have $S^3/H\cong S^3/H'$. This proves (2).

If G is indecomposable, then, by Proposition 8 and the proof of Theorem 9, we may assume that

$$G=\left\{\begin{pmatrix} a & 0 \\ 0 & a^n \end{pmatrix},\ c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} \qquad n^2\equiv 1 \pmod m,\ (m, n)=1,$$

$$G'=\left\{\begin{pmatrix} a & 0 \\ 0 & a^{n'} \end{pmatrix},\ c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\} \qquad n'^2\equiv 1 \pmod m,\ (m, n')=1.$$

Then (1) follows easily from the following

LEMMA 13. *The groups G and G' are isomorphic if and only if $n\equiv n'$ $(\bmod m)$ (or equivalently $nn'\equiv 1$ $(\bmod m)$).*

PROOF. Equivalence of $n\equiv n'$ and $nn'\equiv 1$. If $n\equiv n'$, then $nn'\equiv n^2\equiv 1$. Conversely, if $nn'\equiv 1$, then $1\equiv nn'=n^2+n(n'-n)\equiv 1+n(n'-n)$. Hence, by $(m, n)=1$, we get $n'\equiv n$.

Now assume that G and G' are isomorphic. Let $\phi: G\to G'$ be an isomorphism. We have the following relations: $gh=h^n g$, $g'h'=h'^{n'}g'$. It is clear that ϕ satisfies the following conditions;

$$\begin{cases} \phi(h)=h'^{\lambda}, & (m, \lambda)=1, \\ \phi(g)=h'^{\nu}g' & \text{or} \quad h'^{\nu}g'^{-1}. \end{cases}$$

Consider the case

$$\begin{cases} \phi(h)=h'^{\lambda}, \\ \phi(g)=h'^{\nu}g'. \end{cases}$$

Then we have

$$\phi(gh)=\phi(g)\cdot\phi(h)=h'^{\nu}g'h'=h'^{\nu+\lambda n'}g',$$

$$\phi(h^n g) = \phi(h)^n \cdot \phi(g) = h'^{\lambda n} h'^{\nu} g' = h'^{\nu+\lambda n} g' .$$

By $\phi(gh) = \phi(h^n g)$, we get $\lambda n \equiv \lambda n'$. Hence, by $(m, \lambda) = 1$, we obtain $n \equiv n'$. In the case

$$\begin{cases} \phi(h) = h'^{\lambda}, \\ \phi(g) = h'^{\nu} g'^{-1}, \end{cases}$$

we get $nn' \equiv 1$ by the similar argument as above. Q. E. D.

REMARK (d). The lens spaces S^3/H and S^3/H' are diffeomorphic if and only if $n \equiv \pm n'$ or $nn' \equiv \pm 1 \pmod{m}$ (Reidemeister [5], Milnor [4]).

References

[1] F. Blichfeldt, Finite Collineation Groups, Univ. Chicago Press, Chicago, Illinois, 1917.

[2] E. Brieskorn, Rationale Singularitäten komplexer Flächen, Invent. math., **4** (1968), 336-358.

[3] K. Kodaira, On the structure of compact complex analytic surfaces, II, III, Amer. J. Math., **88** (1966), 682-721, **90** (1968), 55-83.

[4] J. Milnor, Whitehead torsion, Bull. Amer. Math. Soc., **72** (1966), 358-426.

[5] K. Reidemeister, Homotopieringe und Linsenräume, Abh. Math. Sem. Univ. Hamburg, **11** (1935), 102-109.

[6] W. Threlfall und H. Seifert, Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes (Schluß), Math. Ann., **107** (1932), 543-586.

Masahide KATO
Department of Mathematics
Faculty of Science
Rikkyo University
Nishiikebukuro, Toshima-ku
Tokyo, Japan