

## On some improvements of the Brun-Titchmarsh theorem

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(Received Nov. 28, 1972)

(Revised Feb. 1, 1973)

### §1. Introduction.

The so-called Brun-Titchmarsh theorem states that there is an absolute constant  $C_0$  such that

$$(1.1) \quad \pi(x; q, l) \leq C_0 \frac{x}{\varphi(q) \log \frac{x}{q}},$$

where  $\pi(x; q, l)$  is defined as usual to be the number of primes not exceeding  $x$  that are congruent to  $l \pmod{q}$ . This estimation holds uniformly for all  $q < x$  with  $x/q$  sufficiently large and for all  $l \pmod{q}$  with  $(q, l) = 1$ .

The most important feature of this theorem is that it holds for a quite wide range of  $q$  and in this respect it surpasses any results obtained by analytic methods. Actually this inequality may be the strongest tool to attack various problems in the theory of numbers, apart from the mean value theorem of Bombieri.

Although the asymptotic formula of  $\pi(x; q, l)$ , which holds uniformly for smaller  $q$ , can be obtained by analytic methods, the result (also its aesthetic value) is mared by the exceptional zero of Dirichlet's  $L$ -functions. The elimination of this zero remains still one of the deepest problems in the theory of numbers.

In this respect the reduction of the value of the constant  $C_0$  of (1.1) has a very significant meaning, for if we could show that

$$(1.2) \quad C_0 \leq 2 - \eta$$

with an effectively calculable positive  $\eta$  for at least  $\frac{\varphi(q)}{2} + 1$  reduced residue classes  $l \pmod{q}$  under the condition

$$q \leq \exp\left(c_1 \frac{\log x}{\log \log x}\right)$$

then we would be able to prove the extremely valuable inequality

$$\beta \leq 1 - \frac{c_2}{\log(q+3) \log \log(q+3)}$$

for the exceptional zero  $\beta$  of Dirichlet's  $L$ -functions mod  $q$  with the aid of Rodoskii-Tatuzawa's theorem [10, p. 314]. Here  $c_1$  and  $c_2$  are effectively calculable positive constants.

Hence considerable efforts have been made to improve the Brun-Titchmarsh theorem.

By the direct application of the Selberg sieve we can show that

$$(1.3) \quad \pi(x; q, l) \leq 2 \frac{x}{\varphi(q) \log \frac{x}{q}} \left( 1 + O \left( \frac{\log \log \frac{x}{q}}{\log \frac{x}{q}} \right) \right),$$

which is due to Klimov [6]. By a more careful treatment of the remainder term in the Selberg sieve, van Lint and Richert [8] improved this to

$$(1.4) \quad \pi(x; q, l) \leq 2 \frac{x}{\varphi(q) \log \frac{x}{q}} \left( 1 + O \left( \frac{1}{\log \frac{x}{q}} \right) \right).$$

Later the strong improvement initiated by Bombieri of the large sieve method enabled Bombieri and Davenport [2] to give a second proof of (1.3), and further Bombieri [1] proved (1.4) on the same line. Recently Montgomery [9] has elaborated their proof and obtained the neat

$$(1.5) \quad \pi(x; q, l) \leq 2 \frac{x}{\varphi(q) \log \frac{x}{q}},$$

and so it turns out that

$$C_0 \leq 2.$$

More recently towards the inequality (1.2) a progress has been made by Hooley [5], who has proved that

$$\pi(x; q, l) \leq (2 + \delta_1) \frac{x}{\varphi(q) \log \frac{x}{\sqrt{q}}} \quad \text{if } 1 \leq q \leq x^{\frac{2}{3}},$$

and

$$\pi(x; q, l) \leq (1 + \delta_1) \frac{x}{\varphi(q) \log \frac{x}{q}} \quad \text{if } x^{\frac{2}{3}} \leq q \leq x^{1-\delta_2}$$

for almost all  $l \bmod q$ , where  $\delta_1$  and  $\delta_2$  are arbitrarily small positive numbers.

## §2. Main results.

The purpose of the present paper is to show that the recent developments mainly due to Gallagher [4] concerning the hybrid mean values of character-

sums and integrals of finite Dirichlet series can yield rather strong results not only in the theory of the density of the zeros of Dirichlet's  $L$ -functions but also in the field of the Selberg sieve.

THEOREM 1. *If*

$$1 \leq q \leq x^{1-\varepsilon},$$

*then we have*

$$\pi(x; q, l) \leq 2(1+2\varepsilon) \frac{x}{\varphi(q) \log \frac{x}{\sqrt{q}}},$$

*save for at most  $q^{1-\frac{\varepsilon}{5}}$  residue classes  $l \pmod{q}$ .*

This is an improvement of Hooley's result and may be the first one in this field that maintains its significance still for  $q$  close to  $x$ .

But, compared with Hooley's method, the essential novelty of our method is that it enables us to prove results which hold uniformly for all  $l \pmod{q}$ ,  $(q, l) = 1$ , though for  $q$  in the restricted range.

THEOREM 2. *We have uniformly for all  $l \pmod{q}$  with  $(q, l) = 1$*

$$(i) \quad \pi(x; q, l) \leq 2 \frac{x}{\varphi(q) \log \frac{x}{q}} \quad \text{if } x^{\frac{3}{7}} \leq q < x,$$

$$(ii) \quad \pi(x; q, l) \leq \frac{x}{\varphi(q) \log \frac{x}{q^{\frac{5}{3}}}} \left(1 + O\left(\frac{\log \log x}{\log x}\right)\right) \quad \text{if } x^{\frac{6}{17}} \leq q \leq x^{\frac{3}{7}},$$

$$(iii) \quad \pi(x; q, l) \leq 2 \frac{x}{\varphi(q) \log \frac{x}{\sqrt{q}}} \left(1 + O\left(\frac{\log \log x}{\log x}\right)\right) \quad \text{if } x^{\frac{4}{17}} \leq q \leq x^{\frac{6}{17}},$$

$$(iv) \quad \pi(x; q, l) \leq (1+\varepsilon) \frac{x}{\varphi(q) \log \frac{x^{\frac{7}{11}}}{q^{\frac{73}{88}}}} \quad \text{if } x^{\frac{56}{249}} \leq q \leq x^{\frac{4}{17}},$$

$$(v) \quad \pi(x; q, l) \leq 2(1+\varepsilon) \frac{x}{\varphi(q) \log \frac{x^{\frac{25}{56}}}{q^{\frac{56}{56}}}} \quad \text{if } 1 \leq q \leq x^{\frac{56}{249}}.$$

It is easy to see that the results (ii)-(v) are all improvements of Montgomery's inequality (1.5).

If we assume the natural extension to Dirichlet's  $L$ -functions of the Lindelöf hypothesis concerning the size of the Riemann zeta-function along the critical line, further improvements are possible.

THEOREM 3. *If we assume the Lindelöf hypothesis*

$$\left| L\left(\frac{1}{2} + it, \chi\right) \right| \ll (q(|t|+1))^{\frac{\varepsilon}{24}},$$

then we have, when  $1 \leq q \leq x^{1-\varepsilon}$ ,

$$\pi(x; q, l) \leq 2(1+5\varepsilon) \frac{x}{\varphi(q) \log x},$$

save for at most  $q^{1-\varepsilon}$  residue classes  $l \pmod{q}$ .

**THEOREM 4.** *If we assume the extended Lindelöf hypothesis as in Theorem 3, we have uniformly for all  $l \pmod{q}$  with  $(q, l) = 1$*

$$\pi(x; q, l) \leq (1+7\varepsilon) \frac{x}{\varphi(q) \log \frac{x}{q^{\frac{3}{2}}}} \quad \text{if } x^{\frac{1}{3}} \leq q \leq x^{\frac{1}{2}}$$

and

$$\pi(x; q, l) \leq 2(1+\varepsilon) \frac{x}{\varphi(q) \log x} \quad \text{if } 1 \leq q \leq x^{\frac{1}{3}}.$$

In this paper we prove only Theorems 1 and 2, since the proofs of Theorems 3 and 4 are similar and also they have only theoretical interest.

The author wishes to express his indebtedness to Prof. Tatzawa for his encouragements and helpful discussions on this subject, and to Prof. Hooley for suggesting the introduction of the parameter  $z_1$  of §7, which improves the final result substantially.

Notations:  $\varphi(n)$  and  $\mu(n)$  are Euler's and Möbius' functions, respectively.  $\tau(n)$  denotes the number of divisors of  $n$ . We denote by  $(m, n)$  and  $[m, n]$  the greatest common divisor and the least common multiple of  $m$  and  $n$ , respectively.  $\chi$  is a Dirichlet character mod  $q$  and  $L(s, \chi)$  is the Dirichlet's  $L$ -function attached to  $\chi$  with the complex variable  $s = \sigma + it$ .  $\varepsilon$  is an arbitrarily small positive number, and the constants implied by the symbols " $O$ " and " $\ll$ " depend on  $\varepsilon$  at most and are effectively calculable. Finally the positive variable  $x$  is assumed to be half an odd integer and sufficiently large, depending only on  $\varepsilon$ .

### § 3. Selberg's sieve.

Let  $z$  be a positive number with

$$1 < z \leq x$$

and let  $S(x; q, l)$  denote the number of integers not exceeding  $x$  that are congruent to  $l \pmod{q}$  and are free from any prime factors  $\leq z$ . Then we have obviously

$$(3.1) \quad \pi(x; q, l) \leq S(x; q, l) + \frac{z}{q} + 1.$$

The quantity  $S(x; q, l)$  is estimated by the standard application of the Selberg sieve. Here we summarize his method: We set

$$(3.2) \quad \lambda_d = \begin{cases} \frac{1}{Y} \cdot \frac{\mu(d)d}{\varphi(d)} \sum_{\substack{(r, dq)=1 \\ r \leq z/d}} \frac{\mu^2(r)}{\varphi(r)} & \text{if } (q, d) = 1 \text{ and } d \leq z, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$Y = \sum_{\substack{(r, q)=1 \\ r \leq z}} \frac{\mu^2(r)}{\varphi(r)}.$$

Then we have

$$(3.3) \quad \frac{1}{Y} = \sum_{d_1, d_2 \leq z} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]},$$

and also, when  $z$  is sufficiently large, the inequality

$$(3.4) \quad Y \geq \frac{\varphi(q)}{q} \log z$$

is well-known. Further, to make simpler the estimations in what follows, we remark that

$$(3.5) \quad |\lambda_d| \leq 1.$$

This is proved as follows: if  $(q, d) = 1$ , then we have

$$\begin{aligned} Y &= \sum_{\delta|d} \sum_{\substack{(r, d)=\delta \\ (r, q)=1 \\ r \leq z}} \frac{\mu^2(r)}{\varphi(r)} = \sum_{\delta|d} \sum_{\substack{(r, \frac{d}{\delta}q)=1 \\ r \leq z/\delta}} \frac{\mu^2(\delta r)}{\varphi(\delta r)} \\ &\geq \sum_{\delta|d} \frac{\mu^2(\delta)}{\varphi(\delta)} \sum_{\substack{(r, dq)=1 \\ r \leq z/d}} \frac{\mu^2(r)}{\varphi(r)} = \frac{d}{\varphi(d)} \sum_{\substack{(r, dq)=1 \\ r \leq z/d}} \frac{\mu^2(r)}{\varphi(r)}, \end{aligned}$$

and by the definition of  $\lambda_d$  we get (3.5).

Now as for  $S(x; q, l)$  we have

$$(3.6) \quad S(x; q, l) \leq S_1(x; q, l),$$

where the right side is defined by

$$(3.7) \quad S_1(x; q, l) = \sum_{\substack{n \equiv l \pmod{q} \\ n \leq x}} \left( \sum_{d|n} \lambda_d \right)^2.$$

#### § 4. Analytic expression of $S_1(x; q, l)$ .

Since we have  $(q, l) = 1$ , we can write (3.7) as follows:

$$(4.1) \quad S_1(x; q, l) = \frac{1}{\varphi(q)} \sum_{x \pmod{q}} \bar{\chi}(l) \sum_{n \leq x} \chi(n) \left( \sum_{d|n} \lambda_d \right)^2$$

$$= \frac{1}{\varphi(q)} \sum_{x \pmod q} \bar{\chi}(l) S(x, \chi), \quad \text{say.}$$

And we consider the generating series of  $S(x, \chi)$ . This is

$$(4.2) \quad \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \left( \sum_{d|n} \lambda_d \right)^2 = L(s, \chi) \sum_{d_1, d_2 \leq z} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]^s} \chi([d_1, d_2]) \\ = L(s, \chi) K(s, \chi), \quad \text{say.}$$

We note here that

$$(4.3) \quad K(s, \chi) = \sum_{d \leq z^2} \frac{\chi(d)}{d^s} \rho_d,$$

where

$$\rho_d = \sum_{[d_1, d_2]=d} \lambda_{d_1} \lambda_{d_2}$$

and so

$$(4.4) \quad |\rho_d| \leq \tau^2(d)$$

since (3.5).

For the proof of Theorem 2 we shall need a different expression of  $K(s, \chi)$ , originally due to Selberg: Defining  $G_s(d, \chi)$  by

$$(4.5) \quad G_s(d, \chi) = \sum_{u|d} \mu\left(\frac{d}{u}\right) \bar{\chi}(u) u^s,$$

we have

$$(4.6) \quad K(s, \chi) = \sum_{d \leq z} \frac{\chi^2(d)}{d^{2s}} G_s(d, \chi) \left( \sum_{d_1 \leq z/d} \frac{\chi(d_1)}{d_1^s} \lambda_{dd_1} \right)^2 \\ = \sum_{d \leq z} \frac{\chi^2(d)}{d^{2s}} G_s(d, \chi) H^2\left(s, \chi, \frac{z}{d}\right), \quad \text{say,}$$

which is the consequence of

$$\sum_{d|n} G_s(d, \chi) = \bar{\chi}(n) n^s.$$

Now let

$$(4.7) \quad T = x^{10}$$

and we express  $S(x, \chi)$  by the integral

$$S(x, \chi) = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} L(s, \chi) K(s, \chi) \frac{x^s}{s} ds + O\left(\frac{x^2}{T}\right).$$

From (4.5) we see that

$$|G_s(d, \chi)| \leq d^\sigma \tau(d),$$

and so from (4.6) we have

$$K(s, \chi) \ll z$$

uniformly for  $\frac{1}{2} \leq \sigma \leq 2$ . Also we have

$$|L(s, \chi)| \ll (q|s|)^{\frac{1}{2}} \quad (|s| \geq 1)$$

uniformly for  $\frac{1}{2} \leq \sigma \leq 2$ . Thus shifting the line of integration to the line  $\sigma = \frac{1}{2}$ , we get

$$(4.8) \quad S(x, \chi) = \frac{\varphi(q)}{q} K(1, \chi) x E(\chi) + O\left(\frac{1}{x}\right) \\ + \frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} L(s, \chi) K(s, \chi) \frac{x^s}{s} ds,$$

where  $E(\chi) = 1$  if  $\chi$  is principal, and  $= 0$  otherwise.

We have from (3.3) for the principal character  $\chi_0$

$$K(1, \chi_0) = \sum_{d_1, d_2 \leq x} \frac{\lambda_{d_1} \lambda_{d_2}}{[d_1, d_2]} \\ = \frac{1}{Y}.$$

Hence we get from (4.1) and (4.8)

$$(4.9) \quad S_1(x; q, l) = \frac{x}{qY} + O\left(\frac{1}{x}\right) \\ + \frac{1}{2\pi i \varphi(q)} \sum_{\chi \bmod q} \bar{\chi}(l) \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} L(s, \chi) K(s, \chi) \frac{x^s}{s} ds.$$

### §5. Lemmas.

To the estimation of the third term of (4.9) we apply some of the recent results in the theory of numbers, which are embodied in the following three lemmas:

LEMMA A. *We have for any  $T_1 \geq 0$*

$$\sum_{\substack{\chi \bmod q \\ (\chi: \text{primitive})}} \int_{-T_1}^{T_1} \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \ll \varphi(q) T_1 \log^4 q (T_1 + 2).$$

This is essentially due to Lavrik [7], and for the proof see [9], where the restriction  $T_1 \geq 2$  is imposed but it is easy to see that we may write their result as in the above form. We can eliminate the restriction that  $\chi$  is primitive and we get

COROLLARY OF LEMMA A. *We have for any  $T_1 \geq 0$*

$$\sum_{\chi \bmod q} \int_{-T_1}^{T_1} \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \ll q T_1 \log^4 q (T_1 + 2).$$

PROOF. Let  $\chi^*$  be the primitive character mod  $q^*$  which induces  $\chi \bmod q$ . Then we have

$$L(s, \chi) = \prod_{p|q} \left(-1 \frac{\chi^*(p)}{p^s}\right) L(s, \chi^*),$$

where  $p$  denotes generally a prime number. Thus we have

$$\sum_{\chi \bmod q} \left|L\left(\frac{1}{2} + it, \chi\right)\right|^4 \leq \sum_{d|q} \prod_{p|\frac{q}{d}} \left(1 + \frac{1}{\sqrt{p}}\right)^4 \sum_{\substack{\chi \bmod d \\ (\chi: \text{primitive})}} \left|L\left(\frac{1}{2} + it, \chi\right)\right|^4,$$

which, with Lemma A, gives

$$\sum_{\chi \bmod q} \int_{-T_1}^{T_1} \left|L\left(\frac{1}{2} + it, \chi\right)\right|^4 dt \ll T_1 \log^4 q (T_1 + 2) \sum_{d|q} \varphi(d) \prod_{p|\frac{q}{d}} \left(1 + \frac{1}{\sqrt{p}}\right)^4.$$

Since

$$\varphi(q) \geq \varphi(d) \varphi\left(\frac{q}{d}\right),$$

we have

$$\begin{aligned} \sum_{d|q} \varphi(d) \prod_{p|\frac{q}{d}} \left(1 + \frac{1}{\sqrt{p}}\right)^4 &\leq \varphi(q) \sum_{d|q} \frac{1}{\varphi(d)} \prod_{p|\frac{q}{d}} \left(1 + \frac{1}{\sqrt{p}}\right)^4 \\ &\leq \varphi(q) \prod_{p|q} \left(1 + \frac{1}{p} \left(1 + \frac{1}{\sqrt{p}}\right)^4 \left(1 - \frac{1}{p}\right)^{-2}\right) \\ &= q \prod_{p|q} \left\{1 - \frac{1}{p} + \frac{1}{p} \left(1 - \frac{1}{p}\right)^{-1} \left(1 + \frac{1}{\sqrt{p}}\right)^4\right\} \\ &\ll q. \end{aligned}$$

This proves the corollary.

LEMMA B. If  $T_1 \gg 1$ , we have for any positive  $M, N$  and complex numbers  $a_n$

$$\begin{aligned} \sum_{\chi \bmod q} \int_{-T_1}^{T_1} \left| \sum_{M < n \leq M+N} a_n \chi(n) n^{it} \right|^2 dt \\ \ll (qT_1 + N) \sum_{M < n \leq M+N} |a_n|^2. \end{aligned}$$

Also we have

$$\sum_{\chi \bmod q} \left| \sum_{M < n \leq M+N} a_n \chi(n) \right|^2 \ll (q + N) \sum_{M < n \leq M+N} |a_n|^2.$$

This is due to Gallagher [4] and plays a fundamental role in this paper.

LEMMA C. We have for any character  $\chi \bmod q$

$$\left|L\left(\frac{1}{2} + it, \chi\right)\right| \ll q^{\frac{3}{16} + \epsilon} (|t| + 1).$$

This is a deep result of Burgess [3], where the factor  $|t| + 1$  is neglected but to make the estimation uniform we insert it.



## § 6. Proof of Theorem 1.

From (4.9) we have

$$\begin{aligned} & \sum_{\substack{l \bmod q \\ (q, l)=1}} \left\{ S_1(x; q, l) - \frac{x}{qY} \right\}^2 \\ &= \frac{1}{4\pi^2 \varphi(q)} \sum_{x \bmod q} \left| \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} L(s, \chi) K(s, \chi) \frac{x^s}{s} ds \right|^2 + O\left(\frac{q}{x^2}\right) \\ &\ll \frac{x \log x}{\varphi(q)} \left\{ \sum_{x \bmod q} \int_{-T}^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 \frac{dt}{|t|+1} \right\}^{\frac{1}{2}} \\ &\quad \times \left\{ \sum_{x \bmod q} \int_{-T}^T \left| K\left(\frac{1}{2} + it, \chi\right) \right|^4 \frac{dt}{|t|+1} \right\}^{\frac{1}{2}} + O\left(\frac{1}{x}\right), \end{aligned}$$

since we have (4.7).

From Corollary to Lemma A and by the partial integration we get

$$(6.1) \quad \sum_{x \bmod q} \int_{-T}^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 \frac{dt}{|t|+1} \ll q \log^5 x,$$

and from Lemma B we get

$$\begin{aligned} & \sum_{x \bmod q} \int_{-T}^T \left| K\left(\frac{1}{2} + it, \chi\right) \right|^4 \frac{dt}{|t|+1} \\ &\ll (q \log T + z^4) \sum_{\substack{d \leq z^4 \\ d \equiv 2 \pmod{4}}} \frac{1}{d} \left( \sum_{d_1 d_2 = d} \rho_{d_1} \rho_{d_2} \right)^2 \\ &\ll q \left(1 + \frac{z^4}{q}\right) (\log x)^{2^{10}+1}, \end{aligned}$$

since we have (4.4).

Thus we have

$$\begin{aligned} & \sum_{\substack{l \bmod q \\ (q, l)=1}} \left\{ S_1(x; q, l) - \frac{x}{qY} \right\}^2 \\ &\ll x (\log x)^{2^9+5} \left(1 + \frac{z^2}{\sqrt{q}}\right). \end{aligned}$$

Now we put

$$z = \left(\frac{x}{\sqrt{q}}\right)^{\frac{1}{2}-4\epsilon}$$

and

$$(6.2) \quad q \leq x^{1-5\epsilon}.$$

Then we have

$$\frac{z^2}{\sqrt{q}} \geq 1.$$

Thus if we denote by  $E_q$  the number of  $l \pmod q$  with  $(q, l) = 1$  such that

$$\left| S_1(x; q, l) - \frac{x}{qY} \right| \geq \frac{x^{1-\varepsilon}}{qY},$$

then we have

$$\begin{aligned} E_q &\ll q^{\frac{3}{2}} x^{-1+2\varepsilon} Y^2 (\log x)^{29+5\varepsilon} z^2 \\ &\leq qx^{-\varepsilon} \leq q^{1-\varepsilon} \end{aligned}$$

for sufficiently large  $x \geq x_0(\varepsilon)$ .

Hence we get from (3.4)

$$\begin{aligned} S_1(x; q, l) &= \frac{x}{qY} (1 + O(x^{-\varepsilon})) \\ &\leq \frac{2}{1-8\varepsilon} \cdot \frac{x}{\varphi(q) \log \frac{x}{\sqrt{q}}} (1 + O(x^{-\varepsilon})), \end{aligned}$$

save for at most  $q^{1-\varepsilon}$  residue classes  $l \pmod q$  under the condition (6.2). Collecting this result and (3.1), (3.6), we obtain Theorem 1.

**§ 7. Proof of Theorem 2, Part I.**

We denote by  $I(x, z)$  the third term of the right side of (4.9), and so

$$(7.1) \quad S_1(x; q, l) = \frac{x}{qY} + O\left(\frac{1}{x}\right) + I(x, z).$$

Noticing that

$$G_s(d, \chi) = \sum_{uv=d} \mu(v) \bar{\chi}(u) u^s,$$

we divide  $K(s, \chi)$  into two parts as follows:

$$\begin{aligned} (7.2) \quad K(s, \chi) &= \sum_{u \leq z} \frac{\chi(u)}{u^s} \sum_{\substack{v \leq \frac{z}{u} \\ v \leq \frac{z}{u}}} \frac{\mu(v)}{v^{2s}} \chi(v^2) H^2\left(s, \chi, \frac{z}{uv}\right) \\ &= \sum_{u \leq z_1} + \sum_{z_1 < u \leq z} \\ &= K_1(s, \chi) + K_2(s, \chi), \end{aligned}$$

where  $z_1 \geq 1$  is to be determined explicitly. Thus we have

$$(7.3) \quad I(x, z) = I_1(x, z) + I_2(x, z),$$

where

$$I_j(x, z) = \frac{1}{2\pi i \varphi(q)} \sum_{x \pmod q} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} L(s, \chi) K_j(s, \chi) \frac{x^s}{s} ds \quad (j = 1, 2).$$

We have

$$\begin{aligned}
(7.4) \quad & |I_1(x, z)| \\
& \leq \frac{x^{\frac{1}{2}}}{2\pi\varphi(q)} \sum_{u \leq z_1} \frac{1}{\sqrt{u}} \sum_{v \leq \frac{z}{u}} \frac{1}{v} \sum_{\chi \bmod q} \int_{-T}^T \left| L\left(\frac{1}{2} + it, \chi\right) \right| \left| H\left(\frac{1}{2} + it, \chi, \frac{z}{uv}\right) \right|^2 \frac{dt}{|t|+1} \\
& = \frac{x^{\frac{1}{2}}}{2\pi\varphi(q)} \sum_{u \leq z_1} \frac{1}{\sqrt{u}} \sum_{v \leq \frac{z}{u}} \frac{1}{v} Q(u, v), \quad \text{say.}
\end{aligned}$$

Since

$$\left| H\left(\frac{1}{2} + it, \chi, \frac{z}{uv}\right) \right| \ll \left(\frac{z}{uv}\right)^{\frac{1}{2}},$$

we have

$$\begin{aligned}
Q(u, v) & \ll \left(\frac{z}{uv}\right)^{\frac{1}{4}} \sum_{\chi \bmod q} \int_{-T}^T \left| L\left(\frac{1}{2} + it, \chi\right) \right| \left| H\left(\frac{1}{2} + it, \chi, \frac{z}{uv}\right) \right|^{\frac{3}{2}} \frac{dt}{|t|+1} \\
& \ll \left(\frac{z}{uv}\right)^{\frac{1}{4}} \left\{ \sum_{\chi \bmod q} \int_{-T}^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 \frac{dt}{|t|+1} \right\}^{\frac{1}{4}} \\
& \quad \times \left\{ \sum_{\chi \bmod q} \int_{-T}^T \left| H\left(\frac{1}{2} + it, \chi, \frac{z}{uv}\right) \right|^2 \frac{dt}{|t|+1} \right\}^{\frac{3}{4}},
\end{aligned}$$

where we used the Hölder inequality twice.

From Lemma B we have

$$\begin{aligned}
& \sum_{\chi \bmod q} \int_{-T}^T \left| H\left(\frac{1}{2} + it, \chi, \frac{z}{uv}\right) \right|^2 \frac{dt}{|t|+1} \\
& \ll q \left( \log T + \frac{z}{quv} \right) \sum_{d_1 \leq \frac{z}{uv}} \frac{1}{d_1} \\
& \ll q \left( 1 + \frac{z}{quv} \right) \log^2 x,
\end{aligned}$$

since we have (3.5).

Thus we have from this and (6.1)

$$Q(u, v) \ll q \left(\frac{z}{uv}\right)^{\frac{1}{4}} \left( 1 + \left(\frac{z}{quv}\right)^{\frac{3}{4}} \right) \log^3 x,$$

which, with (7.4), gives

$$\begin{aligned}
(7.5) \quad & |I_1(x, z)| \ll \frac{q}{\varphi(q)} x^{\frac{1}{2}} \log^3 x \left\{ z^{\frac{1}{4}} \sum_{u \leq z_1} \frac{1}{u^{\frac{3}{4}}} \sum_{v \leq \frac{z}{u}} \frac{1}{v^{\frac{5}{4}}} + \frac{z}{q^{\frac{3}{4}}} \sum_{u \leq z_1} \frac{1}{u^{\frac{3}{2}}} \sum_{v \leq \frac{z}{u}} \frac{1}{v^2} \right\} \\
& \ll x^{\frac{1}{2}} \log^4 x \left( (zz_1)^{\frac{1}{4}} + \frac{z}{q^{\frac{3}{4}}} \right).
\end{aligned}$$

We now turn to  $I_2(x, z)$ . In this case  $K_2(s, \chi)$  has the following expression:

$$\begin{aligned}
 (7.6) \quad K_2(s, \chi) &= \sum_{\substack{z_1 < u \leq z \\ uv \leq z \\ d_1, d_2 \leq \frac{z}{uv}}} \frac{\chi(uv^2d_1d_2)}{(uv^2d_1d_2)^s} \mu(v)\lambda_{uvd_1}\lambda_{uvd_2} \\
 &= \sum_{\substack{n \leq \frac{z^2}{z_1}}} \frac{\chi(n)}{n^s} f(n),
 \end{aligned}$$

where

$$f(n) = \sum_{\substack{n=uv^2d_1d_2 \\ z_1 < u \leq z \\ v \leq \frac{z}{u} \\ d_1, d_2 \leq \frac{z}{uv}}} \mu(v)\lambda_{uvd_1}\lambda_{uvd_2}.$$

Here we have from (3.5)

$$(7.7) \quad |f(n)| \leq \tau_4(n),$$

where  $\tau_4(n)$  is the number of representations of  $n$  as a product of four factors.

We have

$$\begin{aligned}
 |I_2(x, z)| &\ll \frac{x^{\frac{1}{2}}}{\varphi(q)} \left\{ \sum_{x \bmod q} \int_{-T}^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^2 \frac{dt}{|t|+1} \right\}^{\frac{1}{2}} \\
 &\quad \times \left\{ \sum_{x \bmod q} \int_{-T}^T \left| K_2\left(\frac{1}{2} + it, \chi\right) \right|^2 \frac{dt}{|t|+1} \right\}^{\frac{1}{2}}.
 \end{aligned}$$

From (6.1) we get easily

$$\sum_{x \bmod q} \int_{-T}^T \left| L\left(\frac{1}{2} + it, \chi\right) \right|^2 \frac{dt}{|t|+1} \ll q(\log x)^3.$$

From (7.6), (7.7) and by Lemma B we get

$$\begin{aligned}
 &\sum_{x \bmod q} \int_{-T}^T \left| K_2\left(\frac{1}{2} + it, \chi\right) \right|^2 \frac{dt}{|t|+1} \\
 &\quad \ll q\left(\log T + \frac{z^2}{qz_1}\right) \sum_{\substack{n \leq \frac{z^2}{z_1}}} \frac{f^2(n)}{n} \\
 &\quad \ll q\left(1 + \frac{z^2}{qz_1}\right) (\log x)^{17}.
 \end{aligned}$$

Thus we have

$$|I_2(x, z)| \ll x^{\frac{1}{2}} (\log x)^{11} \left(1 + \frac{z}{\sqrt{qz_1}}\right).$$

From this and (7.3), (7.5), we get

$$|I(x, z)| \ll x^{\frac{1}{2}} (\log x)^{11} \left\{ (zz_1)^{\frac{1}{4}} + \frac{z}{\sqrt{qz_1}} + \frac{z}{q^{\frac{3}{4}}} \right\}.$$

The right side is minimized at  $z_1 = \frac{z}{q^{\frac{2}{3}}}$ , and hence we have proved

LEMMA D. *If*

$$z \geq q^{\frac{2}{3}},$$

*we have*

$$|I(x, z)| \ll x^{\frac{1}{2}}(\log x)^{11} \left\{ \frac{z^{\frac{1}{2}}}{q^{\frac{1}{6}}} + \frac{z}{q^{\frac{3}{4}}} \right\}.$$

### § 8. Proof of Theorem 2, Part II.

We now give another estimation of  $I(x, z)$  which is stronger than Lemma D when  $q$  is small.

We divide  $I(x, z)$  into two parts as follows:

$$(8.1) \quad I(x, z) = \frac{1}{2\pi i \varphi(q)} \left\{ \int_{T \geq |t| \geq T_0} + \int_{|t| \leq T_0} \right\} L(s, \chi) K(s, \chi) \frac{x^s}{s} ds \quad \left( s = \frac{1}{2} + it \right) \\ = I_3(x, z) + I_4(x, z), \quad \text{say,}$$

where  $T_0 \geq 1$  is to be determined explicitly.

We have from (4.6)

$$|I_3(x, z)| \ll \frac{x^{\frac{1}{2}}}{\varphi(q)} \sum_{d \leq z} \frac{\tau(d)}{\sqrt{d}} \sum_{x \bmod q} \int_{T_0}^T \left| L\left(\frac{1}{2} + it, \chi\right) \right| \left| H\left(\frac{1}{2} + it, \chi, \frac{z}{d}\right) \right|^2 \frac{dt}{|t|} \\ = \frac{x^{\frac{1}{2}}}{\varphi(q)} \sum_{d \leq z} \frac{\tau(d)}{\sqrt{d}} Q_1(d), \quad \text{say.}$$

As for  $Q_1(d)$  we have

$$Q_1(d) \ll \sum_{T_0 \leq 2^j \leq T} \frac{1}{2^j} \sum_{x \bmod q} \int_{2^j}^{2^{j+1}} \left| L\left(\frac{1}{2} + it, \chi\right) \right| \left| H\left(\frac{1}{2} + it, \chi, \frac{z}{d}\right) \right|^2 dt \\ \ll \left(\frac{z}{d}\right)^{\frac{1}{4}} \sum_{T_0 \leq 2^j \leq T} \frac{1}{2^j} \left\{ \sum_{x \bmod q} \int_{2^j}^{2^{j+1}} \left| L\left(\frac{1}{2} + it, \chi\right) \right|^4 dt \right\}^{\frac{1}{4}} \\ \times \left\{ \sum_{x \bmod q} \int_{2^j}^{2^{j+1}} \left| H\left(\frac{1}{2} + it, \chi, \frac{z}{d}\right) \right|^2 dt \right\}^{\frac{3}{4}} \\ \ll \left(\frac{z}{d}\right)^{\frac{1}{4}} \sum_{T_0 \leq 2^j \leq T} \frac{1}{2^j} (2^j q \log^4 x)^{\frac{1}{4}} \left\{ \left(2^j q + \frac{z}{d}\right) \sum_{d_1 \leq \frac{z}{d}} \frac{1}{d_1} \right\}^{\frac{3}{4}} \\ \ll \left(\frac{z}{d}\right)^{\frac{1}{4}} q (\log x)^{\frac{11}{4}} \left(1 + \left(\frac{z}{q T_0 d}\right)^{\frac{3}{4}}\right).$$

Thus we get

$$(8.2) \quad |I_3(x, z)| \ll x^{\frac{1}{2}} (\log x)^{\frac{15}{4}} \left\{ z^{\frac{1}{4}} \sum_{d \leq z} \frac{\tau(d)}{d^{\frac{3}{4}}} + \frac{z}{(q T_0)^{\frac{3}{4}}} \sum_{d \leq z} \frac{\tau(d)}{d^{\frac{3}{2}}} \right\}$$

$$\ll x^{\frac{1}{2}}(\log x)^5 \left( z^{\frac{1}{2}} + \frac{z}{(qT_0)^{\frac{3}{4}}} \right).$$

On the other hand we have from Lemma C

$$|I_4(x, z)| \ll \frac{x^{\frac{1}{2}}q^\alpha T_0}{\varphi(q)} \sum_{d \leq z} \frac{\tau(d)}{\sqrt{d}} \int_0^{T_0} \left| H\left(\frac{1}{2} + it, \chi, \frac{z}{d}\right) \right|^2 \frac{dt}{|t|+1},$$

where we put

$$\alpha = \frac{3}{16} + \varepsilon.$$

So we have

$$\begin{aligned} |I_4(x, z)| &\ll \frac{x^{\frac{1}{2}}q^\alpha T_0}{\varphi(q)} \sum_{d \leq z} \frac{\tau(d)}{\sqrt{d}} \left( q \log T_0 + \frac{z}{d} \right) \sum_{d_1 \leq \frac{z}{d}} \frac{1}{d_1} \\ &\ll x^{\frac{1}{2}}q^\alpha T_0 \log^4 x \left( z^{\frac{1}{2}} + \frac{z}{q} \right). \end{aligned}$$

Hence from this and (8.1), (8.2) we get

$$(8.3) \quad |I(x, z)| \ll x^{\frac{1}{2}} \log^5 x \left\{ T_0 q^\alpha \left( z^{\frac{1}{2}} + \frac{z}{q} \right) + \frac{z}{(qT_0)^{\frac{3}{4}}} \right\}.$$

Now if  $z \leq q^2$ , then we have  $z^{\frac{1}{2}} \geq \frac{z}{q}$ , and the right side of (8.3) is minimized at

$$T_0 = \left( \frac{z^{\frac{1}{2}}}{q^{\frac{3}{4} + \alpha}} \right)^{\frac{4}{7}}$$

which is  $\geq 1$  when  $z \geq q^{\frac{3}{2} + 2\alpha}$ . Thus we have

LEMMA E. *If*

$$q^2 \geq z \geq q^{\frac{3}{2} + 2\alpha}$$

*we have*

$$|I(x, z)| \ll x^{\frac{1}{2}}(\log x)^5 \frac{z^{\frac{11}{14}}}{q^{\frac{3}{7}(1-\alpha)}}.$$

And if  $z \geq q^2$ , we have  $z^{\frac{1}{2}} \leq \frac{z}{q}$ , and the right side of (8.3) is minimized at

$$T_0 = q^{\frac{1}{7}(1-4\alpha)} \geq 1,$$

which gives

LEMMA F. *If*

$$z \geq q^2,$$

*we have*

$$|I(x, z)| \ll x^{\frac{1}{2}} (\log x)^5 \frac{z}{q^{\frac{6}{7} - \frac{3}{7}\alpha}}.$$

**§ 9. Proof of Theorem 2, Conclusion.**

Now from Lemma D we have

$$|I(x, z)| \leq x^{\frac{1}{2}} (\log x)^{11} \frac{z^{\frac{1}{2}}}{q^{\frac{1}{6}}},$$

when

$$q^{\frac{7}{6}} \geq z \geq q^{\frac{2}{3}}.$$

In this case we put

$$z = \frac{x}{q^{\frac{5}{3}} (\log x)^{26}},$$

which gives

$$|I(x, z)| \leq \frac{x}{q (\log x)^2}.$$

Then from (3.4) and (7.1) we get

$$S_1(x; q, l) \leq \frac{x}{\varphi(q) \log \frac{x}{q^{\frac{5}{3}}}} \left( 1 + O\left( \frac{\log \log x}{\log x} \right) \right).$$

Thus from (3.1) and (3.6) we get uniformly for all  $l \pmod q$  with  $(q, l) = 1$

$$(9.1) \quad \pi(x; q, l) \leq \frac{x}{\varphi(q) \log \frac{x}{q^{\frac{5}{3}}}} \left( 1 + O\left( \frac{\log \log x}{\log x} \right) \right),$$

when

$$q^{\frac{17}{6}} \geq \frac{x}{(\log x)^{26}} \geq q^{\frac{7}{3}}.$$

If

$$q^{\frac{3}{2} + 2\alpha} \geq z \geq q^{\frac{7}{6}},$$

then from Lemmas D and E we have

$$|I(x, z)| \leq x^{\frac{1}{2}} (\log x)^{11} \frac{z}{q^{\frac{3}{4}}}.$$

So we put

$$z = \frac{x^{\frac{1}{2}}}{q^{\frac{1}{4}} (\log x)^{13}},$$

which, with (3.1), (3.4), (3.6) and (7.1), gives uniformly for all  $l \pmod q$  with  $(q, l) = 1$

$$(9.2) \quad \pi(x; q, l) \leq 2 \frac{x}{\varphi(q) \log \frac{x}{\sqrt{q}}} \left( 1 + O\left( \frac{\log \log x}{\log x} \right) \right),$$

$$q^{\frac{7}{2} + 4\alpha} \geq \frac{x}{(\log x)^{26}} \geq q^{\frac{17}{6}}.$$

Further if

$$q^2 \geq z \geq q^{\frac{3}{2} + 2\alpha},$$

then we have

$$\frac{z^{\frac{11}{14}}}{q^{\frac{7}{3}(1-\alpha)}} \leq \frac{z^{\frac{3}{4}}}{q^{\frac{3}{4}}},$$

and so from Lemmas D and E we have

$$|I(x, z)| \leq x^{\frac{1}{2}} (\log x)^{11} \frac{z^{\frac{11}{14}}}{q^{\frac{3}{7}(1-\alpha)}}.$$

Thus we put

$$z = \left\{ \frac{x^7}{q^{8+6\alpha} (\log x)^{182}} \right\}^{\frac{1}{11}},$$

which gives as above uniformly for all  $l \pmod q$  with  $(q, l) = 1$

$$(9.3) \quad \pi(x; q, l) \leq 11 \frac{x}{\varphi(q) \log \frac{x^7}{q^{8+6\alpha}}} \left( 1 + O\left( \frac{\log \log x}{\log x} \right) \right),$$

when

$$q^{\frac{30}{7} + \frac{6}{7}\alpha} \geq \frac{x}{(\log x)^{26}} \geq q^{\frac{7}{2} + 4\alpha}.$$

Finally if

$$z \geq q^2,$$

we put

$$z = \frac{x^{\frac{1}{2}}}{q^{\frac{1}{7}(1+3\alpha)} (\log x)^{13}}$$

which, with Lemmas D and F, gives

$$|I(x, z)| \leq \frac{x}{q(\log x)^2}.$$

Hence we get uniformly for all  $l \pmod q$  with  $(q, l) = 1$



$$(9.4) \quad \pi(x; q, l) \leq 2 \frac{x}{\varphi(q) \log \frac{x}{q^{\frac{2}{7}(1+3\alpha)}}} \left(1 + O\left(\frac{\log \log x}{\log x}\right)\right),$$

when

$$\frac{x}{(\log x)^{26}} \geq q^{\frac{30}{7} + \frac{6}{7}\alpha}.$$

With a little further calculation all assertions of Theorem 2 follow from the results (9.1) - (9.4).

Concluding Remark: In the estimations of  $I(x, z)$  we did not make use of the inner structure of  $\lambda_d$ , (3.2), and one may ask what will happen if (3.2) is taken into account in a more detailed calculation of  $I(x, z)$ , especially when  $q$  is in the range

$$q \leq \exp((\log x)^{\frac{1}{2}}).$$

In this case we can give a fairly nice estimation of  $H\left(s, \chi, \frac{z}{d}\right)$  unless  $d$  is too large and  $\chi$  is the exceptional one, and so the summands of  $I(x, z)$  which corresponds to non-exceptional characters do not give any trouble. But it turns out that as long as we use (3.2), which means that  $\lambda_d$  is something near to

$$\mu(d) \left(1 - \frac{\log d}{\log z}\right),$$

the term corresponding to the exceptional character is connected essentially with the exceptional zero. Thus we run against a rock.

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