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On meromorphic maps into the complex projective space

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§ 1. Introduction.

In [10], the big Picard theorem was generalized by P. Montel to the case of a meromorphic function $\varphi(z)$ ($\not\equiv 0$) which satisfies the condition that the multiplicities of any zeros of $\varphi(z)$, $\frac{1}{\varphi(z)}$ and $\varphi(z)-1$ are always multiples of p, q and r, respectively, where p, q and r are arbitrarily fixed positive integers with

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1.$$

The main purpose of this paper is to give analogous generalizations of the extension theorems and degeneracy theorems of holomorphic maps into the N-dimensional complex projective space $P_N(C)$ omitting some hyperplanes given in the previous papers [4] and [5].

Let $\{H_i;\ 1\leqq i\leqq q\}$ $(q\geqq N+2)$ be hyperplanes in $P_N(C)$ located in general position. Associate with each H_i a positive integer m_i $(\leqq +\infty)$ such that

$$\sum_{i=1}^{N+1}\frac{1}{m_i}+\frac{1}{m_q}<\frac{1}{N} \tag{1.1}$$

when they are arranged as $m_1\geqq m_2\geqq\cdots\geqq m_q$ by a suitable change of indices. We consider in this paper a meromorphic map f of a domain D in C^n into $P_N(C)$ with the property that $f(D)\not\subset H_i$ $(1\leqq i\leqq q)$ and the intersection multiplicity of the image of f with each H_i at a point w is always a common multiple of all m_j's for j with $w\in H_j$. If the image of f omits any H_i $(1\leqq i\leqq q)$, then we can take $m_i=\infty$ or $\frac{1}{m_i}=0$ in the above and so (1.1) is necessarily valid. Holomorphic maps studied in [4] and [5] are thus a special case of what is treated here.

The first main result in this paper is the following generalization of Theorem A in [4].

Let f be a meromorphic map of a domain D excluding a nowhere dense analytic subset S into $P_N(C)$ with the above property. Then f has a meromor-

phic extension to the totality of D or $f(D-S)$ is included in some linear subvariety of dimension $2N+1-q$ (Corollary 5.7).

Theorem B in [4] will be generalized as follows:

If f is a meromorphic map of C^n into $P_N(C)$ with the above property, then $f(C^n)$ is included in a linear subvariety of dimension $\left[\frac{N}{q-N}\right]$, *where* $[a]$ *denotes the largest integer which does not exceed a number a.*

By using this, it will be possible to determine completely types of meromorphic maps of C^n into $P_{N+1}(C)$ with images in the hypersurface

$$V^d:\ w_0^d+w_1^d+\cdots+w_{N+1}^d=0$$

in the case $d>N(N+2)$ (Corollary 6.4)[1]. Here, the author does not know if the number $N(N+2)$ is best possible for analogous conclusions. Further studies in this direction are expected.

We shall prove the above main theorems under slightly weaker conditions (Theorem 5.4 and Theorem 6.2). The main tool to be used for the proof is a defect relation given by H. Cartan in [2] which is a generalization of Nevanlinna's second fundamental theorem to the case of a system of holomorphic functions. With the aid of his defect relation, we shall give a generalization of a classical theorem of E. Borel to the case of holomorphic functions of several variables with zeros of sufficiently large multiplicities and, using this, prove the above main theorems by a similar argument as in [4].

§2. Multiplicities of zeros of holomorphic functions of several variables.

Let $f(z)$ be a not identically zero holomorphic function on a domain D in C^n. Take a point z^0 in the analytic set $N_f:=\{z\in D\,;\,f(z)=0\}$. We denote by $\mathcal{O}_{z^0}$ the local ring consisting of all germs at z^0 of holomorphic functions in a neighborhood of z^0 and by $\mathcal{I}_{z^0}(N_f)$ the ideal of all elements in $\mathcal{O}_{z^0}$ which vanish identically on N_f. As is well-known, $\mathcal{I}_{z^0}(N_f)$ is a principal ideal of $\mathcal{O}_{z^0}$ and so has a generator g.

DEFINITION 2.1. We shall call f to have *a zero of multiplicity m* at z^0 if there is some $h\in\mathcal{O}_{z^0}$ with $h\notin\mathcal{I}_{z^0}(N_f)$ such that $f=g^m h$, where m is obviously determined independently of any choice of a generator g.

Let $N_f=\bigcup_\iota N_f^\iota$ be the irreducible decomposition of N_f. We have easily

(2.2) *For each N_f^ι, f has a zero of the same multiplicity m_ι at any point in $N_f^\iota-\bigcup_{\iota\neq\iota'} N_f^{\iota'}$. And, any other z in N_f is of multiplicity* $m:=\min\{m_\iota\,;\,z\in N_f^\iota\}$.

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1) Recently, the author received the information that M. L. Green obtained also the similar result on holomorphic maps into V^d (cf., [7]).

Now, let us consider a holomorphic function f ($\not\equiv 0$) on a domain $D:=G\times\tilde{D}$ in C^n, where G is an arbitrary domain in C and $\tilde{D}:=\{|z_2|<r_2, \cdots, |z_n|<r_n\}$ $(r_2>0, \cdots, r_n>0)$. For any fixed $\tilde{z}=(z_2, \cdots, z_n)\in\tilde{D}$ we define a holomorphic function $f^*_{\tilde{z}}(z_1):=f(z_1, \tilde{z})$ of z_1 on G.

(2.3) *If $f(z)$ has a zero of multiplicity m at $z^0:=(z_1^0, \tilde{z}^0)$ and $f^*_{\tilde{z}^0}(z_1)\not\equiv 0$, then $f^*_{\tilde{z}^0}(z_1)$ has a zero of multiplicity $\geqq m$ at z_1^0.*

In fact, a generator g of $\mathcal{I}_{z^0}(N_f)$ can be written as $g^*_{\tilde{z}^0}(z_1)=(z_1-z_1^0)\tilde{g}(z_1)$ with some holomorphic function $\tilde{g}(z_1)$ in a neighborhood of z_1^0. Therefore, if f can be written $f=g^m h$ $(h\in\mathcal{O}_{z^0})$, we get

$$f^*_{\tilde{z}^0}(z_1)=(z_1-z_1^0)^m\tilde{g}(z_1)^m h^*_{\tilde{z}^0}(z_1),$$

which implies (2.3).

More precisely, we can prove

PROPOSITION 2.4. *There exists a subset E of $\tilde{D}$ which is almost analytically thin, i.e., included in the union of at most countably many nowhere dense locally analytic sets, such that for any $\tilde{z}\in\tilde{D}-E$ $f^*_{\tilde{z}}\not\equiv 0$ and the multiplicity of any zero z_1 of $f^*_{\tilde{z}}$ equals that of a zero $z:=(z_1, \tilde{z})$ of f.*

PROOF. Since D is a Cousin II domain, we can find a holomorphic function g on D which gives a generator of $\mathcal{I}_z(N_f)$ for any z in N_f. Consider the set

$$E_1:=\{\tilde{z}\in\tilde{D};\ g(z_1, \tilde{z})\equiv 0 \quad \text{as a function of } z_1\}$$

which is evidently a nowhere dense analytic subset of $\tilde{D}$. Then

$$A:=N_f\cap\left\{\frac{\partial g}{\partial z_1}=0\right\}\cap(G\times(\tilde{D}-E_1))$$

is an analytic set of codimension $\geqq 2$ in $G\times(\tilde{D}-E_1)$. In fact, otherwise, there is a point $z^0=(z_1^0, \tilde{z}^0)\in A$ such that $N_f\cap U\subset\left\{\frac{\partial g}{\partial z_1}=0\right\}\cap U$ for a neighborhood U of z^0. We may write

$$\frac{\partial g}{\partial z_1}=h\cdot g \qquad \text{or} \qquad \frac{dg^*_{\tilde{z}^0}}{dz_1}=h^*_{\tilde{z}^0}\cdot g^*_{\tilde{z}^0} \tag{2.5}$$

with some $h\in\mathcal{O}_{z^0}$. Since $g^*_{\tilde{z}^0}(z_1^0)=0$, by differentiating (2.5) repeatedly and observing their values at $z_1=z_1^0$ successively, we have easily $\frac{d^l g^*_{\tilde{z}^0}}{dz_1^l}=0$ at z_1^0 for any $l=1, 2, \cdots$. It then follows from the theorem of identity that $g^*_{\tilde{z}^0}\equiv 0$, i.e., $\tilde{z}^0\in E_1$, which is a contradiction. Now, by the projection map $\pi:(z_1, \tilde{z})\mapsto\tilde{z}$, we define $E_2:=\pi(A\cup(N_f)_{\mathrm{sing}})$, where $(N_f)_{\mathrm{sing}}$ denotes the set of all singularities of the analytic set N_f. Since $\dim(A\cup(N_f)_{\mathrm{sing}})\leqq n-2$, E_2 is an almost analytically thin subset of $\tilde{D}-E_1$. We shall show that $E:=E_1\cup E_2$ satisfies the desired condition of Proposition 2.4. Let f have a zero of multiplicity m at $z^*=(z_1^*, \tilde{z}^*)$ in $N_f\cap(G\times(D-E))$. Since $\frac{\partial g}{\partial z_1}\neq 0$ at z^*, $g^*_{\tilde{z}^*}(z_1)$ has a zero of multiplicity one at z_1^*. On the other hand, if $f=g^m h$ in a neighborhood of z^* for some $h\in\mathcal{O}_{z^*}$ with $h\notin\mathcal{I}_{z^*}(N_f)$, then $h(z^*)\neq 0$ because N_f is regular at z^*. Therefore, $f_{z^*}^*(z_1)$ has a zero of multiplicity m at $z_1=z_1^*$. This completes the proof.

Let us take next a holomorphic function $f(z_1,\cdots,z_n)$ on C^n, where we assume $f(0,\cdots,0)\neq 0$. For any arbitrarily fixed $z=(z_1,\cdots,z_n)$ $(\neq 0:=(0,\cdots,0))$, consider a holomorphic function $f_z^\#(u):=f(zu)$ of u, where $zu=(z_1u, z_2u,\cdots, z_nu)$.

Similarly to (2.3), we see easily

(2.6) *If f has a zero of multiplicity m at z^0 $(\neq 0)$ and $f_{z^0}^\#(u)\not\equiv 0$, then $f_{z^0}^\#(u)$ has a zero of multiplicity $\geqq m$ at $u=1$.*

And, we can prove

PROPOSITION 2.7. *Let E be the set of all points z $(\neq 0)$ in C^n such that $f_z^\#(u)\equiv 0$ or the multiplicity of some zero u^0 of $f_z^\#$ is larger than that of a zero zu^0 of f. Then, for the canonical map $\tilde{\pi}:(z_1,\cdots,z_n)\mapsto z_1:z_2:\cdots:z_n$ of $C^n-\{0\}$ into $P_{n-1}(C)$, the set $\tilde{\pi}(E)$ is almost analytically thin in $P_{n-1}(C)$.*

PROOF. As in the proof of Proposition 2.4, we take a holomorphic function g on C^n which gives a generator of $\mathcal{I}_z(N_f)$ for any $z\in N_f$. If we put

$$E_1:=\{z\in C^n-\{0\}\ ;\ g_z^\#(u)\equiv 0\}$$

in this case, $\tilde{\pi}(E_1)$ is obviously the union of at most countably many locally analytic sets in $P_{n-1}(C)$, each of which has no interior point by the assumption $f\not\equiv 0$. That is to say, $\tilde{\pi}(E_1)$ is almost analytically thin. Now, assume that the analytic set

$$A:=N_f\cap\left\{z_1\frac{\partial g}{\partial z_1}+\cdots+z_n\frac{\partial g}{\partial z_n}=0\right\}\cap(C^n-E_1)$$

is of codimension one. There is a point z^0 in A such that

$$N_f\cap U\subset\left\{z_1\frac{\partial g}{\partial z_1}+\cdots+z_n\frac{\partial g}{\partial z_n}=0\right\}\cap U$$

for a neighborhood U of z^0. Taking some $h\in\mathcal{O}_{z^0}$ and shrinking U if necessary, we may describe

$$z_1\frac{\partial g}{\partial z_1}+\cdots+z_n\frac{\partial g}{\partial z_n}=h\cdot g \tag{2.8}$$

on U. By substituting $z_1^0u,\cdots,z_n^0u$ into $z_1,\cdots,z_n$ in (2.8), we may rewrite (2.8) as

$$\frac{dg_{z^0}^\#}{du}=\frac{1}{u}h_{z^0}^\#(u)g_{z^0}^\#(u)$$

in a neighborhood of $u=1$. Therefore, since $\left[\frac{d}{du}(g_{z^0}^\#(u))\right]_{u=1}=h(z^0)g(z^0)=0$, as in the proof of Proposition 2.4 we can show

$$\left[\frac{d^l}{du^l}(g^{\#}_{z^0}(u))\right]_{u=1}=0$$

for any $l=1, 2, \cdots$ and so $z^0 \in E_1$. This is a contradiction. We have codim $A \geqq 2$. Then, $E^* := \tilde{\pi}(E_1 \cup A \cup (N_f)_{\mathrm{sing}})$ is almost analytically thin. To complete the proof, we have only to show $\tilde{\pi}(E) \subset E^*$. Take a point $z \in N_f$ with $\tilde{\pi}(z) \notin E^*$. Then

$$\left[\frac{d}{du}g^{\#}_z(u)\right]_{u=u^0}=\frac{1}{u^0}\left(z_1u^0\frac{\partial g}{\partial z_1}+\cdots+z_nu^0\frac{\partial g}{\partial z_n}\right)(zu^0)\neq 0$$

for any u^0 $(\neq 0)$ with $zu^0 \in N_f$ because $zu^0 \notin A \cup E_1$. So, $g^{\#}_z(u)$ has no zeros of multiplicity $\geqq 2$. By the same reason as in the proof of Proposition 2.4, we can easily conclude that the multiplicity of any zero zu^0 of f equals that of a zero u^0 of $f^{\#}_z(u)$. Thus we have Proposition 2.7.

§ 3. Generalizations of a theorem of E. Borel I.

To generalize a classical theorem of E. Borel (cf., [**12**], Proposition 5.15), we recall some results given by H. Cartan ([**2**]). Let D be a domain in the complex plane C which includes $\{r_0 \leqq |z| < +\infty\}$ $(r_0 \geqq 0)$ and $f=(f_1, f_2, \cdots, f_p)$ a system of p holomorphic functions on D with no common zeros, where we mean $D=C$ in the case $r_0=0$. The characteristic function of f is defined as

$$T(r,f):=\frac{1}{2\pi}\int_0^{2\pi}u(re^{i\theta})d\theta-\frac{1}{2\pi}\int_0^{2\pi}u(r_0e^{i\theta})d\theta\,,$$

by the function

$$u(z):=\max_{1\leqq j\leqq p}\log|f_j(z)|.$$

As is easily shown (cf., [**2**], p. 10),

(3.1) *If f is transcendental, i.e., $\dfrac{f_i}{f_j}$ has an essential singularity at ∞ for some i and j $(\neq)$, then*

$$\lim_{r\to+\infty}\frac{T(r,f)}{\log r}=+\infty\,.$$

DEFINITION 3.2. Let $f(z)$ be a holomorphic function on D such that $f(z)\neq 0$ on the set $\gamma_{r_0} := \{|z|=r_0\}$. For a positive integer p, we define

$$N_p(r,f)=\sum_{a_\mu}\min(m_\mu, p)\log\frac{r}{|a_\mu|}$$

where we sum up over all zeros a_μ of multiplicity m_μ of $f(z)$ in $\{r_0 < |z| \leqq r\}$.

H. Cartan gave the following fundamental inequality for the case $r_0=0$, which can be proved by the same argument as in [**2**] for the case $r_0>0$ too (cf., [**1**]).

THEOREM 3.3. *Let $f=(f_1, f_2, \cdots, f_p)$ be a system of holomorphic functions on D with no common zeros whose Wronskian*

$$W^f(z):=\begin{vmatrix} f_1 & f_2 & \cdots & f_p \\ f_1' & f_2' & \cdots & f_p' \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(p-1)} & f_2^{(p-1)} & \cdots & f_p^{(p-1)} \end{vmatrix} \neq 0$$

on γ_{r_0} and consider q $(>p)$ linear combinations

$$F_j:=a_j^1 f_1+a_j^2 f_2+\cdots+a_j^p f_p \qquad (1\leqq j\leqq q)$$

such that any minor of order p of the matrix $((a_j^i))$ does not vanish and $F_j(z) \neq 0$ on γ_{r_0} for any j. Then

$$(q-p)T(r, f)\leqq \sum_{j=1}^{q} N_{p-1}(r, F_j)+S(r),$$

where

$$S(r)=O(\log r)+O(\log T(r, f))$$

as $r\to+\infty$ outside a set of finite linear measure.

REMARK. In Theorem 3.3, if $r_0=0$ and f is not transcendental, we have

$$S(r)=O(1).$$

In fact, in the original proof of Theorem 3.3, the evaluation of $S(r)$ is essentially reduced to the evaluations of $m\left(r, \frac{F'}{F}\right)$ for some meromorphic functions F. As is easily seen, we have always

$$m\left(r, \frac{F'}{F}\right)=O(1)$$

for any rational function F $(\not\equiv 0)$, which concludes $S(r)=O(1)$ in our case.

As an immediate consequence of Theorem 3.3, putting

$$\delta_{p-1}(f, F_i):=1-\varliminf_{r\to+\infty} \frac{N_{p-1}(r, F_i)}{T(r, f)}$$

we have the following defect relation (cf., [2], p. 20).

COROLLARY 3.4. *Let f and F_j satisfy the same conditions as in Theorem 3.3 and, for the particular case $r_0>0$, suppose furthermore that f is transcendental. Then,*

$$\sum_{i=1}^{q} \delta_{p-1}(f, F_i)\leqq p.$$

With the help of these H. Cartan's results, we can prove the following generalization of the theorem of E. Borel.

THEOREM 3.5. *Let $f_0, f_1, \cdots, f_p$ $(p\geqq 2)$ be not identically zero holomorphic* *functions on* C^n *satisfying the following conditions;*

a) *each* f_i *has no zeros of multiplicity* $< m_i$ *for a fixed positive integer* m_i,

b) $\sum_i \frac{1}{m_i} < \frac{1}{p-1}$,

c) *if* $f_{i_0}, \cdots, f_{i_k}$ $(1 \leqq k \leqq p)$ *have a common zero* z_0 *of multiplicities* $n_{i_0}, n_{i_1}, \cdots, n_{i_k}$ *respectively, then*

$$n_l' := n_{i_l} - \min(n_{i_0}, n_{i_1}, \cdots, n_{i_k}) \geqq m_{i_l}$$

for any l *with* $n_l' > 0$ *and*

d) $\frac{f_i}{f_j} \not\equiv$ const. *for any* i *and* j $(\neq)$. *Then* $f_0, f_1, \cdots, f_p$ *are linearly independent over* C.

PROOF. It suffices to show that for any given relation

$$c^0 f_0 + \cdots + c^p f_p = 0 \tag{3.6}$$

$(c^i \in C,\ 0 \leqq i \leqq p)$ at least one c^i vanishes, under the assumption that Theorem 3.5 is true for the case $\leqq p-1$. In fact, from this we can easily conclude Theorem 3.5 by the induction on p, because any f_i and f_j $(i \neq j)$ are trivially linearly independent and for any $f_{i_0}, f_{i_1}, \cdots, f_{i_k}$ $(0 \leqq i_0 < i_1 < \cdots < i_k \leqq p)$ the system $(f_{i_0}, f_{i_1}, \cdots, f_{i_k})$ satisfies also the conditions a)$\sim$d). On the other hand, by Proposition 2.7 we can find a point $z \in C^n - \{0\}$ such that for any i $(f_i)_z^\# \not\equiv 0$ and the multiplicity of any zero u^0 of $(f_i)_z^\#(u)$ equals that of a zero zu^0 of f_i, where we may assume $\frac{(f_i)_z^\#}{(f_j)_z^\#} \not\equiv$ const. for any i and j $(\neq)$ because the $\tilde{\pi}$-image of the set

$$\left\{z;\ \frac{f_i(z_1 u, \cdots, z_n u)}{f_j(z_1 u, \cdots, z_n u)} \equiv \frac{f_i(0, \cdots, 0)}{f_j(0, \cdots, 0)} \text{ as a function of } u\right\}$$

is almost analytically thin in $P_{n-1}(C)$. Then, $(f_0)_z^\#, (f_1)_z^\#, \cdots, (f_p)_z^\#$ satisfy all assumptions of Theorem 3.5. This shows that there is no harm in assuming $n=1$ for our purpose. The variable z_1 is replaced by z in the following. Assume that $c^0 \neq 0, c^1 \neq 0, \cdots, c^p \neq 0$. We put $n_f(z) = m$ if z is a zero of multiplicity m of a holomorphic function f and $n_f(z) = 0$ if $f(z) \neq 0$. Take a holomorphic function g on C such that $n_g(z) = \min(n_{f_0}(z), \cdots, n_{f_{p-1}}(z))$ for any $z \in C$. By (3.6) $n_{f_p}(z) \geqq n_g(z)$ for any $z \in C$. Each $g_i = \frac{c^i f_i}{g}$ $(0 \leqq i \leqq p)$ is a well-defined holomorphic function on C. Consider the system $g = (g_0, g_1, \cdots, g_{p-1})$ which has no common zeros. Then, the Wronskian W^g of g does not vanish identically, because $f_0, f_1, \cdots, f_{p-1}$ are linearly independent over C by the induction hypothesis. Let $F_i := g_i$ $(0 \leqq i \leqq p-1)$ and $F_p := g_0 + g_1 + \cdots + g_{p-1}$. Each F_i $(0 \leqq i \leqq p)$ has no zeros of mulitiplicity $< m_i$ by the assumptions a) and c) and, moreover, may be assumed not to vanish at the origin. Therefore,

$$N_{p-1}(r, F_i) \leqq (p-1)N_1(r, F_i) = \frac{p-1}{m_i} m_i N_1(r, F_i)$$
$$\leqq \frac{p-1}{m_i} N(r, F_i)\,.$$

On the other hand, we know

$$N(r, F_i) \leqq T(r, f) + O(1)$$

(cf., [2], p. 11). From these facts, we can easily conclude

$$\delta_{p-1}(f, F_i) \geqq 1 - \frac{p-1}{m_i} \qquad (0 \leqq i \leqq p)\,.$$

By Corollary 3.4, we have

$$\sum_{i=0}^{p}\left(1-\frac{p-1}{m_i}\right) \leqq \sum_{i=0}^{p} \delta_{p-1}(f, F_i) \leqq p\,,$$

which contradicts the assumption b). The proof is complete.

REMARK 3.7. (i) In Theorem 3.5, if f_i has no zeros, we can take $m_i=\infty$ and then the assumptions a), b) and c) are satisfied. So, Theorem 3.5 deduces Proposition 5.15 in [12] as a special case.

(ii) If for any zero z of f_i $n_{f_i}(z)$ is a common multiple of all m_j's for j with $f_j(z)=0$, then $\{f_i\}$ satisfies the conditions a) and c) of Theorem 3.5.

The author does not know if Theorem 3.5 remains valid without the assumption c). In this connection, we can show only the following theorem, which is essentially the same as Theorem 1 in [11].

THEOREM 3.8. *Let $f_0, f_1, \cdots, f_{p-1}$ $(p \geqq 2)$ be holomorphic functions on C^n satisfying the conditions* a) *and* b) *of Theorem* 3.5, *where we put $m_p=\infty$. If*

$$f_0+f_1+\cdots+f_{p-1}=1\,, \tag{3.9}$$

then at least one f_i is of constant.

PROOF. As in the proof of Theorem 3.5, there is a point $z \in C^n-\{0\}$ such that $(f_i)_z^{\#}(u):=f_i(zu)$ is not of constant and satisfies the condition a) for any f_i with $f_i \not\equiv \text{const.}$ Therefore, we may assume $n=1$ in Theorem 3.8 from the beginning. Consider the system $f=(f_0, f_1, \cdots, f_{p-1})$, which has obviously no common zeros. By Corollary 3.4 and assumptions a) and b), $f_0, f_1, \cdots, f_{p-1}$ are necessarily linearly dependent. So,

$$c^0 f_0 + c^1 f_1 + \cdots + c^{p-1} f_{p-1} = 0 \tag{3.10}$$

with some $(c^0, c^1, \cdots, c^{p-1}) \neq (0, 0, \cdots, 0)$, where we assume $c^{p-1}=1$. By subtracting (3.10) from the both sides of (3.9), we obtain

$$d^0 f_0 + d^1 f_1 + \cdots + d^{p-2} f_{p-2} = 1\,.$$

Thus, the proof is reduced to the case $\leqq p-1$. By the induction on p, we have easily Theorem 3.8.

§ 4. Generalizations of a theorem of E. Borel II.

The purpose of this section is to prove

THEOREM 4.1. *Consider the domains* $D:=\{|z_1|<1, \cdots, |z_n|<1\}$ *and* $D^*:=\{0<|z_1|<1, |z_2|<1, \cdots, |z_n|<1\}$ *in* C^n *and not identically zero holomorphic functions* $f_0, f_1, \cdots, f_p$ *on* D^* *satisfying the conditions* a), b), c) *of Theorem* 3.5 *and*

d′) *each* $\dfrac{f_i}{f_j}$ $(i \neq j)$ *can not be extended to the totality of* D *as a meromorphic function.*

Then, if

$$\alpha^0 f_0 + \alpha^1 f_1 + \cdots + \alpha^p f_p \equiv 0$$

for meromorphic functions $\alpha^0, \alpha^1, \cdots, \alpha^p$ *on* D, *then*

$$\alpha^0 \equiv \alpha^1 \equiv \cdots \equiv \alpha^p \equiv 0 .$$

For the proof, we need

LEMMA 4.2. *Let* D *and* D^* *be domains as in Theorem* 4.1 *and* $f(z_1, \cdots, z_n)$ *a meromorphic function on* D^*. *Suppose that there is a subset* P *of* $\tilde{D}:=\{|z_2|<1, \cdots, |z_n|<1\}$ *of positive capacity in the sense of* [**3**], *p.* 3 *such that for any fixed* $\tilde{z}$ *in* P *a meromorphic function* $f_{\tilde{z}}^*(z_1):=f(z_1, \tilde{z})$ *of* z_1 *is well-defined and has a removable singularity at* $z_1=0$. *Then,* f *is extended to the totality of* D *as a meromorphic function.*

PROOF. We may assume $f \not\equiv 0$. Let N^0 and N^∞ be the set of all zeros and all poles of f respectively. And, consider the set

$$E:=\{\tilde{z} \in \tilde{D}\,;\,(z_1, \tilde{z}) \in N^0 \cup N^\infty \quad \text{for any } z_1\ (0<|z_1|<1)\},$$

which is evidently of capacity zero. Moreover, $\pi(N^0 \cap N^\infty)$ is also of capacity zero because $\operatorname{codim}(N^0 \cap N^\infty) \geqq 2$, where π denotes the canonical projection map of D onto $\tilde{D}$. It may be assumed that $P \cap (E \cup \pi(N^0 \cap N^\infty)) = \emptyset$. For the proof of Lemma 4.2, it suffices to show that f is meromorphically extended to a neighborhood of at least one point $z^0=(0, \tilde{z}^0)$ by virtue of the well-known theorem of E. E. Levi. We can easily take a point $\tilde{z}^0$ in P such that $P \cap U$ is of positive capacity for any neighborhood U of $\tilde{z}^0$. Choosing such U and a real number r $(0<r<1)$ suitably, we have

$$(\{|z_1|=r\} \times U) \cap (N^0 \cup N^\infty) = \emptyset ,$$

where U is chosen as $U \cap E = \emptyset$. Then the analytic set $(N^0 \cup N^\infty) \cap (\{0<|z_1|<r\} \times U)$ can be extended to an analytic set in $\{|z_1|<r\} \times U$. In fact, in this situation, we can apply a result in [3]. In Theorem III of [3], put $D_1:=U$, $D_2:=\{|z_1|<r\}$, $D:=\{|z_1|<r\} \times U$, $S:=\{z_1=0\} \times U$, $A:=$ an irreducible component of $(N^0 \cup N^\infty) \cap (\{|z_1|<r\} \times U)$ and consider a plurisubharmonic

function $u(z_1, \tilde{z})=\log|z_1|$ on A. Since $A\cap\{\tilde{z}=\tilde{z}^*\}$ is a finite set for any $\tilde{z}^*$ in $P\cap U$, all assumptions of Theorem III of [3] are satisfied. We can conclude that $\bar{A}\cap(\{|z_1|<r\}\times U)$ is analytic. Moreover, it is not difficult to show that $(N^0\cup N^\infty)\cap(\{|z_1|<r\}\times U)$ has only finitely many irreducible components. Thus $(\bar{N}^0\cup\bar{N}^\infty)\cap(\{|z_1|<r\}\times U)$ itself is analytic. Then, if U is chosen as a polydisc, we can easily take a not identically zero meromorphic function h on $\{|z_1|\leqq r\}\times U$ such that $h\cdot f$ has no zeros and no poles on $\{0<|z_1|\leqq r\}\times U$. For our purpose, f itself may be assumed to be holomorphic and vanish nowhere on $\{0<|z_1|\leqq r\}\times U$ from the beginning.

Now, for the above r, U and an arbitrary $\tilde{z}\in U\cap P$,

$$n(\tilde{z}):=\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{\frac{\partial f}{\partial z_1}(\zeta, \tilde{z})}{f(\zeta, \tilde{z})}d\zeta$$

gives the order of zero at $z_1=0$ or $-\tilde{n}(z)$ gives the order of pole as a meromorphic function of z_1. Since $n(\tilde{z})$ is a continuous function of $\tilde{z}$ on U, it is bounded below by an integer m_0 not depending on each $\tilde{z}\in U$. This shows that $g(z_1, \tilde{z}):=\frac{1}{z_1^{m_0}}f(z_1, \tilde{z})$ has a holomorphic extension to $\{|z_1|<1\}$ for any fixed $\tilde{z}\in P\cap U$, which equals

$$\frac{1}{2\pi i}\int_{|\zeta|=r}\frac{g(\zeta, \tilde{z})}{\zeta-z_1}d\zeta\,. \tag{4.3}$$

On the other hand, (4.3) defines a well-defined holomorphic function $g(z_1, \tilde{z})$ on the totality of $\{|z_1|<r\}\times U$. Then $f(z_1, \tilde{z})=z_1^{m_0}g(z_1, \tilde{z})$ on $\{0<|z_1|<r\}\times U$ because it holds on a set $\{0<|z_1|<r\}\times(P\cap U)$ of positive capacity. This shows that $f(z_1, \tilde{z})$ has a meromorphic extension $z_1^{m_0}g(z_1, \tilde{z})$ to $\{|z_1|<r\}\times U$. The proof is complete.

We shall prove next the following

LEMMA 4.4. *Let $f_0, f_1, \cdots, f_p$ $(p\geqq 2)$ be not identically zero holomorphic functions on a domain $\{r_0<|z|<+\infty\}$ $(r_0\geqq 0)$ which satisfy the conditions* a), b), c) *of Theorem* 3.5 *and*

d$''$) $\frac{f_i}{f_j}$ *has an essential singularity at ∞ for any i, j $(\neq)$.*

If $\alpha^0 f_0+\alpha^1 f_1+\cdots+\alpha^p f_p\equiv 0$ for holomorphic functions $\alpha^0, \alpha^1, \cdots, \alpha^p$ on $\{r_0<|z|<+\infty\}$ with removable singularities at ∞, then $\alpha^0\equiv\alpha^1\equiv\cdots\equiv\alpha^p\equiv 0$.

PROOF. The proof is given by the same argument as in the proof for the case $n=1$ of Theorem 3.5. We state here only the outline of it. It suffices to prove that at least one α^i vanishes identically under the assumption that Lemma 4.4 is valid for the case $\leqq p-1$. Suppose that $\alpha^i\not\equiv 0$ for any i $(0\leqq i\leqq p)$. Replacing r_0 by a sufficiently large one, we may assume

$\alpha^i \neq 0$ on $\{r_0<|z|<+\infty\}$ for any i. Choose a holomorphic function g on $\{r_0<|z|<+\infty\}$ such that each $g_i := \frac{\alpha^i f_i}{g}$ $(0 \leqq i \leqq p)$ is holomorphic and the system $g=(g_0, g_1, \cdots, g_{p-1})$ satisfies all assumptions of Corollary 3.4 for $F_i := g_i$ and $F_p := g_0+g_1+\cdots+g_{p-1}$. By using the assumptions a) and c) we can obtain an absurd conclusion

$$\sum_{i=0}^{p}\left(1-\frac{p-1}{m_i}\right) \leqq \sum_{j=0}^{p} \delta_{p-1}(g, F_i) \leqq p$$

in this case too. We have thus Lemma 4.4.

PROOF OF THEOREM 4.1. By multiplied by the denominator of each α^i if necessary, α^i $(0 \leqq i \leqq p)$ may be assumed to be holomorphic. By Proposition 2.4, we can take an almost analytically thin set E with the property that for any f_i $(0 \leqq i \leqq p)$ $(f_i)^*_{\tilde{z}} \not\equiv 0$ and the multiplicity of any zero $z:=(z_1, \tilde{z})$ of f_i equals that of a zero z_1 of $(f_i)^*_{\tilde{z}}$ whenever $\tilde{z} \notin E$. Then, for any $\tilde{z} \notin E$, $(f_0)^*_{\tilde{z}}, \cdots, (f_p)^*_{\tilde{z}}$ satisfy the conditions a), b) and c) of Theorem 3.5 as functions on $\{0<|z_1|<1\}$. On the other hand, if for any i, j $(\neq)$ we denote by E_{ij} the set of all $\tilde{z}$ in $\tilde{D} := \{|z_2|<1, \cdots, |z_n|<1\}$ with $\tilde{z} \notin E$ such that $\frac{(f_i)^*_{\tilde{z}}(z_1)}{(f_j)^*_{\tilde{z}}(z_1)}$ has a removable singularity at $z_1=0$, then E_{ij} is not of positive capacity by the assumption d′) and Lemma 4.2. The union $E^* := E \cup (\bigcup_{i<j} E_{ij})$ is of capacity zero and for any $\tilde{z} \in \tilde{D}-E^*$ each $\frac{(f_i)^*_{\tilde{z}}}{(f_j)^*_{\tilde{z}}}$ is well-defined and has an essential singularity at $z_1=0$. Then, we can apply Lemma 4.4 after a change of variable $z := \frac{1}{z_1}$. It follows that $\alpha^0 \equiv \cdots \equiv \alpha^p \equiv 0$ on $\{|z_1|<1\} \times (\tilde{D}-E^*)$. Since $\tilde{D}-E^*$ is of positive capacity, we have the desired conclusion $\alpha^0 \equiv \cdots \equiv \alpha^p \equiv 0$ on the totality of D.

§ 5. Extensions of meromorphic maps.

Let f be a meromorphic map of a domain D $(\subset C^n)$ into $P_N(C)$. By definition, outside an analytic set A of codimension $\geqq 2$, f is a well-defined holomorphic map into $P_N(C)$ and, for any point z^0 in D, it can be written as

$$f(z)=f_0(z) : f_1(z) : \cdots : f_N(z)$$

with holomorphic functions $f_i(z)$ on a neighborhood U of z^0 such that

$$\{z \in U ; f_0(z)=f_1(z)=\cdots=f_N(z)=0\} \quad (\subseteqq A \cap U)$$

is of codimension $\geqq 2$. In the following, such a representation of f is referred to as an admissible representation on U. If D is a Cousin II domain, f has an admissible representation on the totality of D.

DEFINITION 5.1. Let H be a hyperplane in $P_N(C)$ defined as

$$H: a^0w_0+a^1w_1+\cdots+a^Nw_N=0$$

such that $f(D)\subsetneqq H$. For a point z^0 with $f(z^0)\in H$ the image of f is said to *intersect* H *with multiplicity* m at $f(z^0)$ if it has an admissible representation $f=f_0:f_1:\cdots:f_N$ on a neighborhood of z^0 such that $F:=a^0f_0+a^1f_1+\cdots+a^Nf_N$ has a zero of multiplicity m at z^0.

As is easily seen, the intersection multiplicity of the image of f with a hyperplane is independent of any choice of an admissible representation.

Now, in $P_N(C)$, let us take q $(\geqq N+2)$ hyperplanes $\{H_i\}$ in general position. For convenience' sake we label them as $H_0, H_1, \cdots, H_{N+t}$, where $t:=q-N-1$ $(\geqq 1)$. With a suitable choice of a system of homogeneous coodinates, we may write

$$\text{(5.2)}\qquad \begin{aligned} &H_i: w_i=0, \qquad 0\leqq i\leqq N\\ &H_{N+s}: a_s^0w_0+a_s^1w_1+\cdots+a_s^Nw_N=0, \qquad 1\leqq s\leqq t. \end{aligned}$$

We wish to generalize some results in the previous papers [4] and [5]. As in §5 of [5], we consider a partition $J=(J_1, J_2, \cdots, J_p)$ of the set of indices $I:=\{0, 1, \cdots, N\}$, i.e., $I=J_1\cup\cdots\cup J_p$, $J_l\neq\emptyset$ and $J_l\cap J_m=\emptyset$ $(l\neq m)$ and a map $\chi:\{1,,\cdots,t\}\to\{1,,\cdots,p\}$, where we assume $p\geqq 2$. By $E_{J,\chi}$ we denote the set of all points $w_0:w_1:\cdots:w_N$ in $P_N(C)$ such that

$$\sum_{i\in J_l} a_s^iw_i=0$$

for any l and s with $1\leqq l\leqq p$, $l\neq\chi(s)$ and $1\leqq s\leqq t$.

By the same argument as in the proof of Lemma 2.2 in [5], we can prove easily

(5.3) *$E_{J,\chi}$ is a linear subvariety in $P_N(C)$. If $E_{J,\chi}\subsetneqq\{w_i=0\}$ for any i $(0\leqq i\leqq N)$, then $E_{J,\chi}$ is of dimension $\leqq N-(p-1)t$ and so of dimension $\leqq N-t$ by the assumption $p\geqq 2$.*

As a generalization of Theorem A in [4], we give

THEOREM 5.4. *Let f be a meromorphic map into $P_N(C)$ defined on a domain D excluding a thin analytic subset S. Suppose that in $P_N(C)$ there are hyperplanes $\{H_i; 0\leqq i\leqq N+t\}$ $(t\geqq 1)$ in general position satisfying the following conditions;*

a) *$f(D-S)\subsetneqq H_i$ $(0\leqq i\leqq N+t)$ and the image of f intersects each H_i with multiplicity $<m_i$ nowhere,*

b) $$\sum_{i=0}^{N}\frac{1}{m_i}+\frac{1}{m_{N+t}}<\frac{1}{N},$$

c) *if the image of f intersects $H_{i_0}, H_{i_1}, \cdots, H_{i_k}$ $(0\leqq i_0<i_1<\cdots<i_k\leqq N+t$, $1\leqq k\leqq p)$ with multiplicity $n_{i_0}, n_{i_1}, \cdots, n_{i_k}$ respectively at a point $f(z)$ for $z\in D-S$, then*

$$n_l' := n_{i_l} - \min\{n_{i_0}, n_{i_1}, \cdots, n_{i_k}\} \geqq m_{i_l}$$

for any l *with* $n_l' > 0$, *where* m_i $(0 \leqq i \leqq N+t)$ *are arbitrarily fixed positive integers and assume that* $m_0 \geqq m_1 \geqq \cdots \geqq m_{N+t}$. *Then* (i) f *has a meromorphic extension to totality of* D *or* (ii) *the image of* f *is included in some* $E_{J,\chi}$.

PROOF. We may assume that S is regular, because codim $S_{\text{sing}} \geqq 2$. Since the properties of (i) and (ii) are of local nature, it may be assumed that $D = \{|z_1| < 1, \cdots, |z_n| < 1\}$ and $S = \{z_1 = 0\} \cap D$. In this case, f has an admissible representation $f = f_0 : f_1 : \cdots : f_N$ on the totality of $D - S$ because $D - S$ is a Cousin II domain. On the other hand, $\{H_i; 0 \leqq i \leqq N+t\}$ can be written as (5.2). Put

$$F_i := f_i \ (\not\equiv 0), \qquad 0 \leqq i \leqq N, \tag{5.5}$$
$$F_{N+s} := a_s^0 f_0 + a_s^1 f_1 + \cdots + a_s^N f_N \ (\not\equiv 0), \qquad 1 \leqq s \leqq t.$$

By definition, the intersection multiplicity of the image of f with H_i $(0 \leqq i \leqq N+t)$ at a point $f(z^0)$ is the multiplicity of a zero z^0 of F_i. Now, assume that f cannot be extended to a meromorphic map of D into $P_N(C)$. Then, there are some indices i and j $(\neq)$ such that $\dfrac{f_i}{f_j}$ can not be extended to a meromorphic function on D. Now, consider the uniquely determined partition $J = (J_1, J_2, \cdots, J_p)$ of $I = \{0, 1, \cdots, N\}$ such that i and j are in the same class if and only if $\dfrac{f_i}{f_j}$ can be meromorphically extended to D, where $p \geqq 2$ by the above assumption. We may assume $J_1 = \{N_1 := 0, 1, \cdots, N_2 - 1\}$, $J_2 = \{N_2, N_2+1, \cdots, N_3-1\}, \cdots, J_p = \{N_p, N_p+1, \cdots, N\}$. Put $\beta_i := \dfrac{f_i}{f_{N_l}}$ for any i with $N_l \leqq i \leqq N_{l+1} - 1$, where $N_{p+1} := N+1$. And, define

$$\alpha_s^l := \sum_{i=N_l}^{i=N_{l+1}-1} a_s^i \beta_i,$$

which may be considered as meromorphic functions on D. Then, (5.5) can be rewritten as

$$F_{N+s} = \alpha_s^1 f_{N_1} + \alpha_s^2 f_{N_2} + \cdots + \alpha_s^p f_{N_p} \qquad (1 \leqq s \leqq t). \tag{5.6}$$

Now, for each s, we apply Theorem 4.1 to the functions f_{N_l} and F_{N+s}. Since $\dfrac{f_{N_l}}{f_{N_m}}$ $(l \neq m)$ can not be meromorphically extended to D, $\dfrac{F_{N+s}}{f_{N_{\chi(s)}}}$ is extended to a meromorphic function γ on D for some $\chi(s)$ $(1 \leqq \chi(s) \leqq p)$. Rewriting (5.6) as

$$\alpha_s^1 f_{N_1} + \cdots + (\alpha_s^{\chi(s)} - \gamma) f_{N_{\chi(s)}} + \cdots + \alpha_s^p f_{N_p} = 0$$

and using Theorem 4.1 again, we conclude that $\alpha_s^l \equiv 0$ for any l but $\chi(s)$. This shows that, for the map $\chi : s \mapsto \chi(s)$, $f(D-S) \subseteqq E_{J,\chi}$. The proof is complete.

COROLLARY 5.7. *Let f be a meromorphic map of a domain D excluding a nowhere dense analytic set S into $P_N(C)$, $\{H_i; 0 \leqq i \leqq N+t\}$ $(t \geqq 1)$ be hyperplanes in general position and m_i $(0 \leqq i \leqq N+t)$ be a positive integer associated with each H_i satisfying the condition* b) *of Theorem* 5.4. *If the intersection multiplicity of the image of f with each H_i at a point w is always a common multiple of all m_j's for j with $w \in H_j$, then* (i) *f is meromorphically extended to D or* (ii) *$f(D-S)$ is included in some linear subvariety of dimension $N-t$.*

This is a direct result of Theorem 5.4 by virtue of Remark 3.7, (ii) and (5.3).

COROLLARY 5.8. *Let f be a meromorphic map of $D-S$ into $P_{N+1}(C)$ whose image is included in a special hypersurface*

$$V^d: w_0^d + w_0^d + \cdots + w_{N+1}^d = 0$$

and assume that $d > N(N+2)$. Then (i) *f is meromorphically extended to D or* (ii) *the image of f is included in a proper subvariety of V^d.*

PROOF. Without loss of generality, we may assume $D = \{|z_1| < 1, \cdots, |z_n| < 1\}$, $S = \{z_1 = 0\} \cap D$. If we take an admissible representation $f = f_0 : f_1 : \cdots : f_{N+1}$, then

$$f_0^d + f_1^d + \cdots + f_{N+1}^d = 0$$

and $\operatorname{codim}\{z; f_0(z) = f_1(z) = \cdots = f_{N+1}(z) = 0\} \geqq 2$, where we may assume $f_i \not\equiv 0$ $(0 \leqq i \leqq N)$. We define $g := f_0^d : f_1^d : \cdots : f_N^d$ of $D-S$ into $P_N(C)$. Since the multiplicity of any zero of f_i^d $(0 \leqq i \leqq N+1)$ is a multiple of d, all assumptions of Corollary 5.7 are satisfied for a meromorphic map g, hyperplanes

$$H_i: w_i = 0, \qquad 0 \leqq i \leqq N$$

and

$$H_{N+1}: w_0 + w_1 + \cdots + w_N = 0$$

in $P_N(C)$ and $m_0 = m_1 = \cdots = m_{N+1} = d$. As a result of Corollary 5.7, we have easily Corollary 5.8.

§ 6. Degeneracy theorems of meromorphic maps.

Using Theorem 3.5, we can give some degeneracy theorems of meromorphic maps of C^n into $P_N(C)$. We shall show first the following generalization of a result of M. L. Green in [**6**].

THEOREM 6.1. *Let f be a meromorphic map of C^n into $P_N(C)$ and $\{H_i; 0 \leqq i \leqq N+1\}$ be $N+2$ distinct hyperplanes in $P_N(C)$ satisfying the conditions* a), b) *and* c) *of Theorem* 5.4. *Then f is degenerate, i.e., the image of f is included in some hyperplane in $P_N(C)$.*

PROOF. Since C^n is a Cousin II domain, we can take an admissible representation $f = f_0 : f_1 : \cdots : f_N$ on the totality of C^n. Let each H_i $(0 \leqq i \leqq N+1)$ be given as

$$H_i : a_i^0 w_0 + a_i^1 w_1 + \cdots + a_i^N w_N = 0$$

and put

$$F_i : = a_i^0 f_0 + a_i^1 f_1 + \cdots + a_i^N f_N .$$

Evidently, $F_0, F_1, \cdots, F_{N+1}$ are linearly dependent over C and satisfy the conditions a), b) and c) of Theorem 3.5. Therefore, some $\frac{F_i}{F_j}$ $(i \neq j)$ is of constant, say c. Then $f(C^n)$ is included in a hyperplane

$$H : (a_i^0 - c a_j^0) w_0 + \cdots + (a_i^N - c a_j^N) w_N = 0 .$$

The proof is complete.

Now, we give the following generalization of Theorem B in [4].

THEOREM 6.2. *Let f be a meromorphic map of C^n into $P_N(C)$. Suppose that there exist hyperplanes $\{H_i ; 0 \leqq i \leqq N+t\}$ $(t \geqq 1)$ in general position satisfying the conditions* a), b) *and* c) *of Theorem* 5.4. *Then* (i) *f is of constant or* (ii) *the image of f is included in some $E_{J,\chi}$ and, more precisely, included in a linear subvariety of dimension $\leqq p-1$, where p denotes the number of classes of J and always $\leqq \left[\frac{t+N+1}{t+1}\right]$.*

PROOF. The proof is similar to that of Theorem 5.4. Let $\{H_i ; 0 \leqq i \leqq N+t\}$ be given as (5.2) and define holomorphic functions F_i $(0 \leqq i \leqq N)$ by (5.5) for an admissible representation $f = f_0 : f_1 : \cdots : f_N$. Consider a partition $J = (J_1, \cdots, J_p)$ of $I = \{0, 1, \cdots, N\}$ such that $\frac{f_i}{f_j} \equiv$ const. if i and j are in the same class and $\frac{f_i}{f_j} \equiv$ const. otherwise. Here, it may be assumed $k \geqq 2$ because, if not, $f \equiv$ const. As in the proof of Theorem 5.2, by setting $J_l = \{N_l, N_l + 1, \cdots, N_{l+1} - 1\}$ $(1 \leqq l \leqq p)$ and $c_s^l : = \sum_{i=N_l}^{i=N_{l+1}-1} a_s^i \frac{f_i}{f_{N_l}}$, we have

$$F_{N+s} = c_s^1 f_{N_1} + \cdots + c_s^p f_{N_p}$$

for each s $(1 \leqq s \leqq t)$. In this situation, according to Theorem 3.5 and by the same argument as in the proof of Theorem 5.4, we can conclude $c_s^l = 0$ for any l except exactly one index, say $\chi(s)$. It then follows easily that $f(C^n) \subseteqq E_{J,\chi}$ for the above given J and map $\chi : s \mapsto \chi(s)$. Moreover, since $\frac{f_i}{f_j} \equiv$ const. for any $i, j \in J_l$ $(1 \leqq l \leqq p)$, $f(C^n)$ is included in a linear subvariety of dimension $\leqq p-1$. The last assertion $p \leqq \left[\frac{N+t+1}{t+1}\right]$ is shown by the same argument as in the proof of Theorem B in [4].

By the same argument as in the proof of Corollary 5.7 and Corollary 5.8, Theorem 6.2 implies

COROLLARY 6.3. *Let f be a meromorphic map of C^n into $P_N(C)$ which satisfies the same conditions as in Corollary* 5.7 *for hyperplanes $\{H_i ; 0 \leqq i \leqq N+t\}$ in general position and positive integers $\{m_i\}$. Then $f(C^n)$ is included in* *some linear subvariety of dimension* $\left[\frac{N}{t+1}\right]$.

COROLLARY 6.4. *Take a hypersurface*

$$V^d : w_0^d + w_1^d + \cdots + w_{N+1}^d = 0$$

in $P_{N+1}(C)$ *and assume* $d > N(N+2)$. *Then any meromorphic map of* C^n *into* $P_{N+1}(C)$ *with values in* V^d *is necessarily of the following type after a suitable change of indices* $0, 1, \cdots, N+1$;

$$f = a_0 f_1 : a_1 f_1 : \cdots : a_{N_1} f_1 : a_{N_1+1} f_2 : \cdots : a_{N_2} f_2 : \cdots : a_{N_p} f_p ,$$

where $0 < N_1 < N_2 < \cdots < N_p := N+1$ $(p \geqq 1)$, a_i $(0 \leqq i \leqq N+1)$ *are constants with* $\sum_{i=N_{l-1}+1}^{i=N_l} a_i^d = 0$ $((a_0, a_1, \cdots, a_{N+1}) \neq (0, 0, \cdots, 0))$ *and* f_i $(1 \leqq i \leqq p)$ *are not identically zero holomorphic functions.*

We give lastly another degeneracy theorem as follows.

THEOREM 6.5. *In the same situation as in Theorem* 6.2, *suppose that* f *satisfies the conditions* a) *and, instead of* b) *and* c),

b′) $\sum_{l=0}^{N+1} \frac{1}{m_{i_l}} < \frac{1}{N}$ *for any* $i_0, i_1, \cdots, i_{N+1}$ $(0 \leqq i_0 \leqq i_1 \cdots \leqq i_{N+1} \leqq N+t)$,

c′) *there is some* i_0 $(0 \leqq i_0 \leqq N+t)$ *such that* $m_{i_0} = \infty$, *i.e.*, $f(C^n) \cap H_{i_0} = \emptyset$. *Then the image of* f *is included in an* $(N-t)$*-dimensional linear subspace of* $P_N(C)$.

PROOF. Let $f = f_0 : f_1 : \cdots : f_N$ be an admissible representation. We use the same notations as in the proof of Theorem 6.2. It may be assumed that $f(C^n) \cap H_0 = \emptyset$ and so $f_0 \equiv 1$. Changing indices if necessary, we may assume that $F_i \equiv$ const. $(0 \leqq i \leqq k)$ and $F_j \not\equiv$ const. $(k+1 \leqq j \leqq N+t)$. We have only to prove $k \geqq t$. Assume that $k < t$. Since $H_0, H_t, \cdots, H_{N+t}$ are located in general position, we can write

$$1 \equiv F_0 = c^0 F_t + c^1 F_{t+1} + \cdots + c^N F_{N+t} ,$$

where $c^i \neq 0$, $0 \leqq i \leqq N$. By Theorem 3.8, some F_i $(t \leqq i \leqq t+N)$ is of constant, which is a contradiction. We have Theorem 6.5.

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