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On the representations of an integer as a sum of two squares and a product of four factors

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§ 1. Introduction.

1.1. The purpose of the present paper is to establish the asymptotic formula for the number of representations of an integer as a sum of two integral squares and a product of four positive integral factors.

Our problem is obviously equivalent to the study of the asymptotical behaviour of the sum

$$\sum_{n<N} r(N-n)d_4(n) \qquad (\text{as } N\to\infty), \tag{1}$$

where $r(n)$ and $d_4(n)$ stand for the number of representations of n as a sum of two squares and as a product of four factors, respectively.

Our problem and the so-called additive divisor problem are similar in that each sum can be expressed as a combination of sums of iterated divisor functions over arithmetic progressions with variable modulus, whose size depends on the parameter N. But our problem has much greater difficulty caused mainly by the inner structure of $r(n)$. The same fact has been already noticed by Hooley [1] between the divisor problem of Titchmarsh and a conjecture of Hardy and Littlewood. Hence our proof depends on various devices of Hooley, and also the large sieve method plays an important role in this paper.

1.2. Notation: To avoid the unnecessary complications we assume that throughout this paper the parameter N is a sufficiently large *odd* integer.

ε is assumed to be positive and sufficiently small, and the constants in the symbols "O" and "$\ll$" depend on ε at most.

(m, n) stands for the greatest common divisor of m and n. A prime number is denoted by p, and $p^\alpha \| n$ means that p^α is the highest power of p which divides n. The symbol $m \subset n$ indicates that all prime divisors of m divide n.

$\omega(n)$ and $\Omega(n)$ are respectively the numbers of different prime factors of n and the total number of prime factors of n. $d(n)$ is the number of divisors of n, and $d_k(n)$ is the number of representations of n as a product of k factors.

$\mu(n)$ and $\varphi(n)$ denote the Möbius and the Euler functions, respectively.

χ denotes generally a Dirichlet character and χ_g is the principal character mod g. $s=\sigma+it$ is a complex variable, and $\zeta(s)$ and $L(s,\chi)$ are the Riemann zeta-function and the Dirichlet's L-function attached to χ, respectively.

§ 2. Fundamental decomposition of the sum.

2.1. As is well known, $r(n)$ has the expression

$$r(n)=4\sum_{q\mid n}\rho(q),$$

where ρ is the non-principal character mod 4.

Using this we decompose our sum (1) into three parts:

$$\begin{aligned}(2)\qquad &\frac{1}{4}\sum_{n<N}r(N-n)d_4(n)\\ &=\sum_{n<N}d_4(n)\Big\{\sum_{\substack{q\mid N-n\\ q\leqq N^{1/2}\log^{-B}N}}\rho(q)+\sum_{\substack{q\mid N-n\\ q\geqq N^{1/2}\log^{B}N}}\rho(q)+\sum_{\substack{q\mid N-n\\ N^{1/2}\log^{-B}N<q<N^{1/2}\log^{B}N}}\rho(q)\Big\}\\ &=\Sigma_{\mathrm{I}}+\Sigma_{\mathrm{II}}+\Sigma_{\mathrm{III}},\qquad \text{say}.\end{aligned}$$

The sums Σ_{I} and Σ_{II} contribute to the main-term of the asymptotic formula, while the sum Σ_{III} is of a lower order of magnitude and its estimation presents considerable difficulty.

To Σ_{I} and Σ_{II} we apply the large sieve method, and for this sake we need to deform them.

2.2. We have

$$\begin{aligned}(3)\qquad \Sigma_{\mathrm{I}}&=\sum_{q\leqq N^{1/2}\log^{-B}N}\rho(q)\sum_{\substack{n\equiv N\ (\mathrm{mod}\ q)\\ n<N}}d_4(n)\\ &=\sum_{q\leqq N^{1/2}\log^{-B}N}\rho(q)\sum_{\substack{n\equiv N/(N,q)\ (\mathrm{mod}\ q/(N,q))\\ n<N/(N,q)}}d_4((N,q)n)\\ &=\sum_{\substack{u\mid N\\ u\leqq N^{1/2}\log^{-B}N}}\rho(u)\sum_{\substack{(q,N/u)=1\\ q\leqq (N^{1/2}/u)\log^{-B}N}}\rho(q)\sum_{\substack{n\equiv N/u\ (\mathrm{mod}\ q)\\ n<N/u}}d_4(un)\,.\end{aligned}$$

For the sum Σ_{II} we need a little careful treatment. We have

$$\begin{aligned}\Sigma_{\mathrm{II}}&=\sum_{n<N}d_4(n)\sum_{\substack{qt=N-n\\ q\geqq N^{1/2}\log^{B}N}}\rho(q)\\ &=\sum_{t<N^{1/2}\log^{-B}N}\ \sum_{tN^{1/2}\log^{B}N\leqq qt=N-n}\rho(q)d_4(n)\end{aligned}$$

and, classifying q by mod 4, we see that

$$\sum_{\mathrm{II}} = \sum_{t<N^{1/2}\log^{-B}N} \Big\{ \sum_{\substack{n\equiv N-t \pmod{4t} \\ n\leqq N-tN^{1/2}\log^{B}N}} d_4(n) - \sum_{\substack{n\equiv N+t \pmod{4t} \\ n\leqq N-tN^{1/2}\log^{B}N}} d_4(n) \Big\}.$$

We decompose this right side into three parts according to t mod 4 as follows:

$$\sum_{\mathrm{II}} = \sum_{2|t} + \sum_{t\equiv N \pmod 4} + \sum_{t\equiv -N \pmod 4}$$

$$= \sum_{\mathrm{II}_1} + \sum_{\mathrm{II}_2} + \sum_{\mathrm{II}_3}, \qquad \text{say}.$$

2.3. Now, since N is odd, we have for even t

$$(N-t, 4t) = (N+t, 4t) = (N, t),$$

and thus, as the decomposition (3) of $\sum_{\mathrm{I}}$, we have

$$(4) \qquad \sum_{\mathrm{II}_1} = \sum_{\substack{u|N \\ u<N^{1/2}\log^{-B}N}} \sum_{\substack{2|t \\ (t,N/u)=1 \\ t<(N^{1/2}/u)\log^{-B}N}} \Big\{ \sum_{\substack{n\equiv N/u-t \pmod{4t} \\ n\leqq N/u-tN^{1/2}\log^{B}N}} d_4(un) - \sum_{\substack{n\equiv N/u+t \pmod{4t} \\ n\leqq N/u-tN^{1/2}\log^{B}N}} d_4(un) \Big\}.$$

In the sum $\sum_{\mathrm{II}_2}$ we have $t\equiv N \pmod 4$, and then we have

$$(N-t, 4t) = 4(N, t), \qquad (N+t, 4t) = 2(N, t),$$

which give

$$\sum_{\mathrm{II}_2} = \sum_{\substack{u|N \\ u<N^{1/2}\log^{-B}N}} \sum_{\substack{t\equiv N/u \pmod 4 \\ (t,N/u)=1 \\ t<(N^{1/2}/u)\log^{-B}N}} \Big\{ \sum_{\substack{n\equiv (1/4)(N/u-t) \pmod t \\ n\leqq (1/4)(N/u-tN^{1/2}\log^{B}N)}} d_4(4un) - \sum_{\substack{n\equiv (1/2)(N/u+t) \pmod{2t} \\ n\leqq (1/2)(N/u-tN^{1/2}\log^{B}N)}} d_4(2un) \Big\}.$$

In the same way we have

$$\sum_{\mathrm{II}_3} = \sum_{\substack{u|N \\ u<N^{1/2}\log^{-B}N}} \sum_{\substack{t\equiv -N/u \pmod 4 \\ (t,N/u)=1 \\ t<(N^{1/2}/u)\log^{-B}N}} \Big\{ \sum_{\substack{n\equiv (1/2)(N/u-t) \pmod{2t} \\ n\leqq (1/2)(N/u-tN^{1/2}\log^{B}N)}} d_4(2un) - \sum_{\substack{n\equiv (1/4)(N/u+t) \pmod t \\ n\leqq (1/4)(N/u-tN^{1/2}\log^{B}N)}} d_4(4un) \Big\}.$$

Comparing these expressions of $\sum_{\mathrm{II}_2}$ and $\sum_{\mathrm{II}_3}$ we find easily that

$$(5) \qquad \sum_{\mathrm{II}_2} + \sum_{\mathrm{II}_3} = \rho(N) \sum_{\substack{u|N \\ u<N^{1/2}\log^{-B}N}} \rho(u) \sum_{\substack{(t,N/u)=1 \\ t<(N^{1/2}/u)\log^{-B}N}} \rho(t)$$

$$\times \Big\{ \sum_{\substack{n\equiv (1/4)(N/u-\rho(tN/u)t) \pmod t \\ n\leqq (1/4)(N/u-tN^{1/2}\log^{B}N)}} d_4(4un) - \sum_{\substack{n\equiv (1/2)(N/u+\rho(tN/u)t) \pmod{2t} \\ n\leqq (1/2)(N/u-tN^{1/2}\log^{B}N)}} d_4(2un) \Big\}.$$

This completes the decomposition of $\sum_{\mathrm{I}}$ and $\sum_{\mathrm{II}}$.

2.4. Thus the estimation of $\sum_{\mathrm{I}}$ and $\sum_{\mathrm{II}}$ is reduced to the study of the asymptotical behaviour of the sum

$$\sum_{\substack{n\equiv l \pmod q \\ n\leqq y}} d_4(hn),$$

where $(q, l) = 1$ and h is an arbitrary positive integer. Namely we need a very uniform estimation of the rest-term in the asymptotic formula for the above sum, since in the above expressions of $\sum_{\mathrm{I}}$ and $\sum_{\mathrm{II}}$ the corresponding parameters to y, h, q and l depend heavily each other. This is the problem to be solved in the next paragraph.

§ 3. A mean value theorem.

3.1. The aim of this paragraph is to prove an analogue to Bombieri's mean value theorem for the rest-term in the prime number theorem for arithmetic progressions: we are going to estimate the expression

$$\sum_{q\leqq Q}\max_{y\leqq x}\max_{\substack{l\\(q,l)=1}}\Bigg|\sum_{\substack{n\equiv l\ (\mathrm{mod}\ q)\\ n\leqq y}} d_4(hn)-\frac{1}{\varphi(q)}\sum_{\substack{(n,q)=1\\ n\leqq y}} d_4(hn)\Bigg|$$

with Q as large as possible.

3.2. For $(q, l)=1$ we have

$$\sum_{\substack{n\equiv l\ (\mathrm{mod}\ q)\\ n\leqq y}} d_4(hn)\log\frac{y}{n}-\frac{1}{\varphi(q)}\sum_{\substack{(n,q)=1\\ n\leqq y}} d_4(hn)\log\frac{y}{n} \tag{6}$$

$$=\frac{1}{2\pi i\varphi(q)}\sum_{\chi\neq\chi_q\ (\mathrm{mod}\ q)}\bar{\chi}(l)\int_{(2)}\sum_{n=1}^{\infty}\frac{\chi(n)d_4(hn)}{n^s}\frac{y^s}{s^2}ds\,.$$

Here we have, for $\sigma>1$,

$$\sum_{n=1}^{\infty}\frac{\chi(n)d_4(hn)}{n^s}=\Big\{\sum_{n\subset h}\frac{\chi(n)d_4(hn)}{n^s}\Big\}\Big\{\sum_{(n,h)=1}\frac{\chi(n)d_4(n)}{n^s}\Big\}$$

$$=\Big\{\sum_{n\subset h}\frac{\chi(n)d_4(hn)}{n^s}\Big\}\prod_{p|h}\Big(1-\frac{\chi(p)}{p^s}\Big)^4 L^4(s,\chi)\,.$$

Obviously the last product represents the analytic continuation for $\sigma>0$, and moreover we have, uniformly in the region $\sigma\geqq 1/2$, the inequality

$$\Big|\Big\{\sum_{n\subset h}\frac{\chi(n)d_4(hn)}{n^s}\Big\}\prod_{p|h}\Big(1-\frac{\chi(p)}{p^s}\Big)^4\Big|\leqq d_4(h)\prod_{p|h}\Big(1+\frac{1}{\sqrt{p}}\Big)^4\sum_{n\subset h}\frac{d_4(n)}{\sqrt{n}}$$

$$\leqq d^3(h)\prod_{p|h}\Big(1+\frac{1}{\sqrt{p}}\Big)^4\prod_{p|h}\Big(1-\frac{1}{\sqrt{p}}\Big)^{-4}$$

$$\leqq d^{15}(h)\,.$$

Thus, shifting the line of integration from $\sigma=2$ to $\sigma=1/2$ in the expession (6), we get

$$\max_{y\leqq x}\max_{\substack{l\\(q,l)=1}}\Bigg|\sum_{\substack{n\equiv l\ (\mathrm{mod}\ q)\\ n\leqq y}} d_4(hn)\log\frac{y}{n}-\frac{1}{\varphi(q)}\sum_{\substack{(n,q)=1\\ n\leqq y}} d_4(hn)\log\frac{y}{n}\Bigg| \tag{7}$$

$$\ll\frac{x^{1/2}d^{15}(h)}{\varphi(q)}\sum_{\chi\neq\chi_q\ (\mathrm{mod}\ q)}\int_{(1/2)}|L(s,\chi)|^4\frac{|ds|}{|s|^2}\,.$$

3.3. Now let χ^* denote a primitive character mod q^* which induces χ mod q, then we have

$$L(s, \chi)=\prod_{p\mid\frac{q}{q^*}}\left(1-\frac{\chi^*(p)}{p^s}\right)L(s, \chi^*) .$$

Here we quote the following result of Montgomery [2]: for $T\geqq 2$ we have the inequality

$$\sum_{\chi \bmod q}^* \int_{1/2-iT}^{1/2+iT}|L(s, \chi)|^4|ds|\ll\varphi(q)T(\log qT)^4 ,$$

where $\sum^*$ denotes a sum over all primitive characters mod q.

Thus we have, for any $Q\geqq 2$,

$$\text{(8)}\qquad \sum_{q\leqq Q}\frac{1}{\varphi(q)}\sum_{\chi\neq\chi_q \pmod q}\int_{(1/2)}|L(s, \chi)|^4\frac{|ds|}{|s|^2}$$

$$\leqq\sum_{q^*\leqq Q}\sum_{\chi \bmod q^*}^*\int_{(1/2)}|L(s, \chi)|^4\frac{|ds|}{|s|^2}\sum_{\substack{q^*|q\\ q\leqq Q}}\frac{1}{\varphi(q)}\prod_{p\mid\frac{q}{q^*}}\left(1+\frac{1}{\sqrt{p}}\right)^4$$

$$\ll\log Q\sum_{q\leqq Q}\frac{1}{\varphi(q)}\sum_{\chi \bmod q}^*\int_{(1/2)}|L(s, \chi)|^4\frac{|ds|}{|s|^2}$$

$$\ll Q(\log Q)^5 .$$

3.4. Hence, combining (7) with (8), we get the inequality

$$\text{(9)}\qquad \sum_{q\leqq Q}\max_{y\leqq x}\max_{\substack{l\\ (q,l)=1}}\left|\sum_{\substack{n\equiv l \pmod q\\ n\leqq y}}d_4(hn)\log\frac{y}{n}-\frac{1}{\varphi(q)}\sum_{\substack{(n,q)=1\\ n\leqq y}}d_4(hn)\log\frac{y}{n}\right|$$

$$\ll Qx^{1/2}d^{15}(h)(\log Q)^5 .$$

Now, by the standard way of smoothening, we have for any $0<\lambda\leqq 1$

$$\text{(10)}\qquad \lambda\left|\sum_{\substack{n\equiv l \pmod q\\ n\leqq y}}d_4(hn)-\frac{1}{\varphi(q)}\sum_{\substack{(n,q)=1\\ n\leqq y}}d_4(hn)\right|$$

$$\ll\frac{\lambda}{\varphi(q)}\sum_{e^{-\lambda}y\leqq n\leqq e^{\lambda}y}d_4(hn)$$

$$+\max_{\xi\leqq e^{\lambda}y}\left|\sum_{\substack{n\equiv l \pmod q\\ n\leqq\xi}}d_4(hn)\log\frac{\xi}{n}-\frac{1}{\varphi(q)}\sum_{\substack{(n,q)=1\\ n\leqq\xi}}d_4(hn)\log\frac{\xi}{n}\right| .$$

To estimate the first sum of the right side we quote the following well-known result: if $z\geqq x^{\varepsilon}$, we have

$$\sum_{x\leqq n\leqq x+z}d_4(n)\ll z\log^3 x .$$

Thus if we restrict the value of λ by

$$x^{-1/4}\leqq\lambda\leqq 1 ,$$

then we have

$$(11)\qquad \max_{y\le x}\sum_{e^{-\lambda}y\le n\le e^{\lambda}y} d_4(hn)\le \max_{y\le x^{1/2}}\sum+\max_{x^{1/2}<y\le x}\sum$$
$$\le d_4(h)\{x^{1/2+\varepsilon}+\max_{x^{1/2}<y\le x}\sum\}$$
$$\ll d_4(h)(x^{1/2}+\lambda x\log^3 x)$$
$$\ll \lambda d^3(h)x\log^3 x\,.$$

Finally, collecting (9), (10), (11) and further putting

$$\lambda=(\log x)^{-(1/2)(B-1)}\,,$$
$$Q=x^{1/2}\log^{-B} x\,,$$

we obtain the crucial result:

$$(12)\qquad \sum_{q\le x^{1/2}\log^{-B}x}\max_{y\le x}\max_{\substack{l\\(q,l)=1}}\left|\sum_{\substack{n\equiv l\ (\mathrm{mod}\ q)\\ n\le y}} d_4(hn)-\frac{1}{\varphi(q)}\sum_{\substack{(n,q)=1\\ n\le y}} d_4(hn)\right|$$
$$\ll d^{15}(h)x\log^{-A} x\,,$$

where $B=2A+9$ and the constant in the symbol "$\ll$" depends only on A.

3.5. It seems very probable that the inequality of the same type holds also for $d_k(n)$ with $k\geqq 5$. And we would like to remark that, if we have such inequality for $d_k(n)$, then our method employed in this paper can be used without any alterations to prove the asymptotic formula for the number of representations of an integer as a sum of two squares and a product of k factors. Moreover it may be easy to see that even in this general case the sum corresponding to $\sum_{\mathrm{III}}$ of our problem can be treated completely analogously and the proof is free from any hypothesis.

§ 4. Estimation of $\sum_{\mathrm{I}}$.

4.1. Having obtained the inequality (12), we are now ready to start the evaluation of the sums $\sum_{\mathrm{I}}$ and $\sum_{\mathrm{II}}$. First in this paragraph we treat only $\sum_{\mathrm{I}}$, since $\sum_{\mathrm{II}}$ has a little difference and we have to be more careful in its estimation.

4.2. Now from (3) we have

$$\sum\nolimits_{\mathrm{I}}=\sum_{\substack{u\mid N\\ u\le N^{1/2}\log^{-B}N}}\rho(u)\sum_{\substack{(q,N/u)=1\\ q\le(N^{1/2}/u)\log^{-B}N}}\frac{\rho(q)}{\varphi(q)}\sum_{\substack{(n,q)=1\\ n<N/u}} d_4(un)$$
$$+O\Big\{\sum_{\substack{u\mid N\\ u\le N^{1/2}\log^{-B}N}}\ \sum_{\substack{(q,N/u)=1\\ q\le(N^{1/2}/u)\log^{-B}N}}\left|\sum_{\substack{n\equiv N/u\ (\mathrm{mod}\ q)\\ n<N/u}} d_4(un)-\frac{1}{\varphi(q)}\sum_{\substack{(n,q)=1\\ n<N/u}} d_4(un)\right|\Big\}$$

$$= \sum_{\substack{u|N \\ u \leqq N^{1/2}\log^{-B}N}} \rho(u)\Sigma_1 + O\{\sum_{\substack{u|N \\ u \leqq N^{1/2}\log^{-B}N}} \Sigma_2\}, \quad \text{say}.$$

In the sum Σ_2 we have

$$q \leqq \frac{N^{1/2}}{u} \log^{-B} N \leqq \left(\frac{N}{u}\right)^{1/2} \log^{-B}\frac{N}{u}$$

and thus, putting $x = N/u$ and $h = u$ in the inequality (12), we get

$$\Sigma_2 \ll \frac{d^{15}(u)}{u} N \log^{-A}\frac{N}{u},$$

which gives

$$\sum_{\substack{u|N \\ u \leqq N^{1/2}\log^{-B}N}} \Sigma_2 \ll N\log^{-A}N \sum_{u|N} \frac{d^{15}(u)}{u}$$

$$\ll N\log^{-A}N \prod_{p|N}\left(1-\frac{1}{p}\right)^{-2^{15}}$$

$$\ll N\log^{-A}N(\log\log N)^{2^{15}}.$$

Hence we have

$$\Sigma_{\text{I}} = \sum_{\substack{u|N \\ u \leqq N^{1/2}\log^{-B}N}} \rho(u)\Sigma_1 + O(N\log^{1-A}N). \tag{13}$$

4.3. Now for Σ_1 we have

$$\Sigma_1 = \sum_{n<N/u} d_4(un) \sum_{\substack{(q,n(N/u))=1 \\ q \leqq (N^{1/2}/u)\log^{-B}N}} \frac{\rho(q)}{\varphi(q)}.$$

The inner sum can be estimated by applying Perron's formula to the function

$$\sum_{(q,n(N/u))=1} \frac{\rho(q)}{q^s\varphi(q)}$$

$$= L(s+1,\rho) \prod_{p|n(N/u)}\left(1-\frac{\rho(p)}{p^{s+1}}\right) \prod_{p\nmid n(N/u)}\left(1-\frac{\rho(p)}{p^{s+1}}+\frac{\rho(p)}{p^{s+1}}\left(1-\frac{1}{p}\right)^{-1}\right),$$

where the inner product converges absolutely for $\sigma > -1$. Thus we get easily

$$\sum_{\substack{(q,n(N/u))=1 \\ q \leqq x}} \frac{\rho(q)}{\varphi(q)} = \frac{\pi}{4}\mathfrak{S} \prod_{p|n(N/u)}\left(1-\frac{\rho(q)}{p}\right)\left(1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\left(1-\frac{1}{p}\right)^{-1}\right)^{-1} \tag{14}$$

$$+O\left(x^{-1/2}d\left(n\frac{N}{u}\right)\log x\right),$$

where

$$\mathfrak{S} = \prod_p\left(1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\left(1-\frac{1}{p}\right)^{-1}\right). \tag{15}$$

From this formula we have

$$(16)\qquad \Sigma_1=\frac{\pi}{4}\mathfrak{S}\sum_{n<N/u}d_4(un)\prod_{p\mid n(N/u)}\Big(1-\frac{\rho(p)}{p}\Big)\Big(1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\Big(1-\frac{1}{p}\Big)^{-1}\Big)^{-1}$$
$$+O\Big\{\log N\sum_{n<N/u}\Big(\frac{N^{1/2}}{u}\log^{-B}N\Big)^{-1/2}d_4(un)\,d\Big(n\frac{N}{u}\Big)\Big\}$$
$$=\frac{\pi}{4}\mathfrak{S}\Sigma_3+O(N^{3/4+\varepsilon}),\qquad\text{say}.$$

4.4. Next we have to estimate the sum Σ_3. For this sake we introduce the function

$$(17)\qquad G(s)=\sum_{n=1}^{\infty}\frac{d_4(un)}{n^s}F\Big(n\frac{N}{u}\Big),$$

where

$$(18)\qquad F(m)=\prod_{p\mid m}\Big(1-\frac{\rho(p)}{p}\Big)\Big(1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\Big(1-\frac{1}{p}\Big)^{-1}\Big)^{-1}.$$

Then we have

$$(19)\qquad G(s)=\Big\{\sum_{n\subset N}\frac{d_4(un)}{n^s}F\Big(\frac{N}{u}n\Big)\Big\}\Big\{\sum_{(n,N)=1}\frac{d_4(n)}{n^s}F(n)\Big\}$$
$$=\Big\{\sum_{n\subset N}\frac{d_4(un)}{n^s}F\Big(\frac{N}{u}n\Big)\Big\}\frac{1}{\mathfrak{S}}\prod_{p\mid N}\Big(1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\Big(1-\frac{1}{p}\Big)^{-1}\Big)\Big(1-\frac{1}{p^s}\Big)^4$$
$$\times\prod_{p\nmid N}\Big(1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\Big(1-\frac{1}{p}\Big)^{-1}\Big(1-\frac{1}{p^s}\Big)^4\Big)\zeta^4(s)$$
$$=\frac{1}{\mathfrak{S}}H_u(s)\zeta^4(s),\qquad\text{say},$$

where $\mathfrak{S}$ is defined by (15) and it is easy to see that $H_u(s)$ converges absolutely for $\sigma>0$ and moreover we have uniformly for $\sigma\geqq 1/2$

$$(20)\qquad H_u(s)\ll\prod_{p\mid N}\Big(1+\frac{1}{\sqrt{p}}\Big)^4\sum_{n\subset N}\frac{d_4(un)}{\sqrt{n}}\Big|F\Big(\frac{N}{u}n\Big)\Big|$$
$$\ll N^{\varepsilon}.$$

Hence applying Perron's formula in conjunction with the mean value theorem of $|\zeta(s)|^4$ on the line $\sigma=1/2$ we get easily

$$(21)\qquad \sum_{n\leqq x}d_4(un)F\Big(\frac{N}{u}n\Big)=\frac{1}{\mathfrak{S}}\operatorname*{Res}_{s=1}H_u(s)\zeta^4(s)\frac{x^s}{s}+O(x^{1/2}N^{\varepsilon}).$$

4.5. Now we see that

$$(22)\qquad \operatorname*{Res}_{s=1}H_u(s)\zeta^4(s)\frac{x^s}{s}=\frac{1}{6}H_u(1)x\log^3 x+O(x\log^2 x\max_{0\leqq j\leqq 3}|H_u^{(j)}(1)|).$$

Thus we have to estimate $H_u^{(j)}(1)$ ($j\leqq 3$).

By Cauchy's theorem we have

$$H_u^{(j)}(1)=\frac{j\,!}{2\pi i}\int_C\frac{H(s)}{(s-1)^{j+1}}ds\,, \tag{23}$$

where C is the circle of radius $(\log\log N)^{-1}$ with the center $s=1$. On this circle we have, putting $\sigma_0=1-(\log\log N)^{-1}$,

$$|H_u(s)|\ll\prod_{p|N}\Big(1+\frac{1}{p^{\sigma_0}}\Big)^4\sum_{n\subset N}\frac{d_4(un)}{n^{\sigma_0}}\Big|F\Big(\frac{N}{u}n\Big)\Big|$$

$$\ll\prod_{p|N}\Big(1+\frac{1}{p^{\sigma_0}}\Big)^4\sum_{n\subset N}\frac{d_4(un)}{n^{\sigma_0}}\prod_{p|n(N/u)}\Big(1+\frac{1}{p}\Big)\,,$$

since by the definition (18) of $F(m)$ we have

$$|F(m)|\ll\prod_{p|m}\Big(1+\frac{1}{p}\Big)\,.$$

And so we have on C

$$|H_u(s)|\ll d_4(u)\prod_{p|N/u}\Big(1+\frac{1}{p}\Big)\prod_{p|N}\Big(1+\frac{1}{p^{\sigma_0}}\Big)^4\prod_{p|N}\Big(1-\frac{1}{p^{\sigma_0}}\Big)^{-4} \tag{24}$$

$$\ll d_4(u)\log\log N\prod_{p|N}\Big(1+\frac{1}{p^{\sigma_0}}\Big)^8\,.$$

Here we have

$$\prod_{p|N}\Big(1+\frac{1}{p^{\sigma_0}}\Big)^8=\exp\Big\{8\sum_{p|N}\frac{1}{p^{\sigma_0}}+O(1)\Big\}\,,$$

and further we have

$$\sum_{p|N}\frac{1}{p^{\sigma_0}}\leqq\sum_{p\leqq\log N}\frac{1}{p^{\sigma_0}}+\frac{1}{(\log N)^{\sigma_0}}\sum_{p|N}1$$

$$\leqq(\log N)^{1/\log\log N}\sum_{p\leqq\log N}\frac{1}{p}+O\Big(\frac{1}{\log N}\sum_{p|N}1\Big)$$

$$=(1+0(1))e\log\log\log N\,.$$

Hence we get

$$\prod_{p|N}\Big(1+\frac{1}{p^{\sigma_0}}\Big)^8\ll(\log\log N)^{24}\,,$$

which, with (24), gives

$$|H_u(s)|\ll d_4(u)(\log\log N)^{25}$$

uniformly on C.

Now from this inequality and (23) we get

$$\max_{j\leqq3}|H_u^{(j)}(1)|\ll d_4(u)(\log\log N)^{28}\,.$$

This gives, with (21) and (22),

$$\sum_{n\leqq x}d_4(un)F\Big(\frac{N}{u}n\Big)=\frac{1}{6\mathfrak{S}}H_u(1)x\log^3x+O(x\log^2x\,d_4(u)(\log\log N)^{28})\,.$$

Turning to the formula (16), we have

$$(25)\qquad \Sigma_3=\frac{1}{6\mathfrak{S}}\frac{H_u(1)}{u}N\log^3\frac{N}{u}+O\Big(\frac{d_4(u)}{u}N\log^2\frac{N}{u}(\log\log N)^{28}\Big)$$

$$=\frac{1}{6\mathfrak{S}}\frac{H_u(1)}{u}N\log^3 N+O\Big(\frac{d_4(u)}{u}\log u\, N\log^2 N(\log\log N)^{28}\Big).$$

By the way we notice here that

$$(26)\qquad |\Sigma_3|\ll\frac{d_4(u)}{u}N^{1+\varepsilon}.$$

4.6. Now from (13) and (16) we have

$$\Sigma_{\mathrm{I}}=\frac{\pi}{4}\mathfrak{S}\sum_{\substack{u|N\\ u\leqq N^{1/2}\log^{-B}N}}\rho(u)\Sigma_3+O(N(\log N)^{1-A})$$

$$=\frac{\pi}{4}\mathfrak{S}\sum_{u|N}\rho(u)\Sigma_3+O\Big(\sum_{\substack{u|N\\ u>N^{1/2}\log^{-B}N}}|\Sigma_3|\Big)+O(N(\log N)^{1-A})$$

$$=\frac{\pi}{4}\mathfrak{S}\sum_{u|N}\rho(u)\Sigma_3+O(N(\log N)^{1-A}),$$

since from (26) we get

$$\sum_{\substack{u|N\\ u>N^{1/2}\log^{-B}N}}|\Sigma_3|\ll N^{1/2+2\varepsilon}\sum_{u|N}1\ll N^{1/2+3\varepsilon}.$$

Inserting the result (25) into the above expression of Σ_{I}, we have

$$\Sigma_{\mathrm{I}}=\frac{\pi}{24}N\log^3 N\sum_{u|N}\frac{\rho(u)H_u(1)}{u}+O\Big(N\log^2 N(\log\log N)^{28}\sum_{u|N}\frac{d_4(u)}{u}\log u\Big).$$

Here we have

$$\sum_{u|N}\frac{d_4(u)}{u}\log u\ll(\log\log N)^{13},$$

which can be proved similarly as the estimation of $H_u^{(j)}(1)$ of the preceding section.

Thus we get

$$(27)\qquad \Sigma_{\mathrm{I}}=\frac{\pi}{24}N\log^3 N\sum_{u|N}\frac{\rho(u)H_u(1)}{u}+O(N\log^2 N(\log\log N)^{41})$$

$$=\frac{\pi}{24}N\log^3 N\prod_{p\nmid N}\Big(1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\Big(1-\frac{1}{p}\Big)^3\Big)$$

$$\times\prod_{p|N}\Big(1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\Big(1-\frac{1}{p}\Big)^{-1}\Big)\Big(1-\frac{1}{p}\Big)^4\sum_{u|N}\frac{\rho(u)}{u}\sum_{n\subset N}\frac{d_4(un)}{n}F\Big(\frac{N}{u}n\Big)$$

$$+O(N\log^2 N(\log\log N)^{41}),$$

since $H_u(s)$ is defined by (19).

4.7. Let $N=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ and $u=p_1^{\beta_1}p_2^{\beta_2}\cdots p_r^{\beta_r}$ be the prime-power decompositions of N and u, then we have

$$\sum_{n\subset N}\frac{d_4(un)}{n}F\Big(\frac{N}{u}n\Big)=\sum_{\delta_1,\cdots,\delta_r=0}^{\infty}\frac{d_4(p_1^{\beta_1+\delta_1}\cdots p_r^{\beta_r+\delta_r})}{p_1^{\delta_1}\cdots p_r^{\delta_r}}F(p_1^{\alpha_1-\beta_1+\delta_1}\cdots p_r^{\alpha_r-\beta_r+\delta_r})$$

$$=\prod_{j\leq r}\Big\{\sum_{\delta=0}^{\infty}\frac{d_4(p_j^{\delta+\beta_j})}{p_j^{\delta}}F(p_j^{\alpha_j-\beta_j+\delta})\Big\}.$$

Thus we have

$$\sum_{u|N}\frac{\rho(u)}{u}\sum_{n\subset N}\frac{d_4(un)}{n}F\Big(\frac{N}{u}n\Big) \tag{28}$$

$$=\sum_{\beta_1,\cdots,\beta_r=0}^{\alpha_1,\cdots,\alpha_r}\frac{\rho(p_1^{\beta_1})\cdots\rho(p_r^{\beta_r})}{p_1^{\beta_1}\cdots p_r^{\beta_r}}\prod_{j\leq r}\Big\{\prod_{\delta=0}^{\infty}\frac{d_4(p_j^{\delta+\beta_j})}{p_j^{\delta}}F(p_j^{\alpha_j-\beta_j+\delta})\Big\}$$

$$=\prod_{p^\alpha\,\|\,N}\Big\{\sum_{\beta=0}^{\alpha}\frac{\rho(p^\beta)}{p^\beta}\sum_{\delta=0}^{\infty}\frac{d_4(p^{\delta+\beta})}{p^\delta}F(p^{\alpha-\beta+\delta})\Big\}.$$

Here we notice that, only when $\alpha=\beta$ and $\delta=0$, we have

$$F(p^{\alpha-\beta+\delta})=1$$

and otherwise

$$F(p^{\alpha-\beta+\delta})=F(p).$$

And so we have

$$\sum_{\beta=0}^{\alpha}\frac{\rho(p^\beta)}{p^\beta}\sum_{\delta=0}^{\infty}\frac{d_4(p^{\delta+\beta})}{p^\delta}F(p^{\alpha-\beta+\delta}) \tag{29}$$

$$=(1-F(p))\frac{\rho(p^\alpha)}{p^\alpha}d_4(p^\alpha)$$

$$+F(p)\Big(1-\frac{1}{p}\Big)^{-4}+F(p)\sum_{\beta=1}^{\alpha}\frac{\rho(p^\beta)}{p^\beta}\sum_{\delta=0}^{\infty}\frac{d_4(p^{\delta+\beta})}{p^\delta}$$

$$=F(p)\Big(1-\frac{1}{p}\Big)^{-4}\Big\{1+\Big(1-\frac{1}{p}\Big)^4\sum_{\beta=1}^{\alpha}\frac{\rho(p^\beta)}{p^\beta}\sum_{\delta=0}^{\infty}\frac{d_4(p^{\delta+\beta})}{p^\delta}$$

$$+\frac{\rho(p^{\alpha+1})}{p^{\alpha+1}}d_4(p^\alpha)\Big(1-\frac{1}{p}\Big)^3\Big(1-\frac{\rho(p)}{p}\Big)^{-1}\Big\}$$

$$=F(p)\Big(1-\frac{1}{p}\Big)^{-4}\{p,\alpha\},\quad\text{say}.$$

Inserting this into the right side of (28) we get

$$\sum_{u|N}\frac{\rho(u)}{u}\sum_{n\subset N}\frac{d_4(un)}{n}F\Big(\frac{N}{u}n\Big) \tag{30}$$

$$=\prod_{p|N}\frac{\Big(1-\frac{\rho(p)}{p}\Big)\Big(1-\frac{1}{p}\Big)^{-4}}{1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\Big(1-\frac{1}{p}\Big)^{-1}}\prod_{p^\alpha\,\|\,N}\{p,\alpha\},$$

which, with (27), gives rise to the final result of this paragraph:

$$\Sigma_{\mathrm{I}}=\frac{\pi}{24}N\log^3 N\prod_{p|N}\Big(1-\frac{\rho(p)}{p}\Big)\prod_{p\nmid N}\Big(1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\Big(1-\frac{1}{p}\Big)^3\Big)\prod_{p^\alpha\|N}\{p,\alpha\} + O(N\log^2 N(\log\log N)^{41}). \tag{31}$$

§5. Estimation of Σ_{II}.

5.1. This sum is divided into three parts as (3). The estimation of Σ_{II_1} has no difficulty: we have

$$\Sigma_{\mathrm{II}_1}=\sum_{\substack{u|N\\ u<N^{1/2}\log^{-B}N}}\ \sum_{\substack{2|t\\ (t,N/u)=1\\ t<(N^{1/2}/u)\log^{-B}N}}\Big\{\Big(\sum_{\substack{n\equiv N/u-t\ (\mathrm{mod}\ 4t)\\ n\le N/u-tN^{1/2}\log^B N}}d_4(un)-\frac{1}{\varphi(4t)}\sum_{\substack{(n,4t)=1\\ n\le N/u-tN^{1/2}\log^B N}}d_4(un)\Big)$$
$$-\Big(\sum_{\substack{n\equiv N/u+t\ (\mathrm{mod}\ 4t)\\ n\le N/u-tN^{1/2}\log^B N}}d_4(un)-\frac{1}{\varphi(4t)}\sum_{\substack{(n,4t)=1\\ n\le N/u-tN^{1/2}\log^B N}}d_4(un)\Big)\Big\}.$$

And since

$$\frac{N}{u}-tN^{1/2}\log^B N<\frac{N}{u},$$

we have obviously

$$|\Sigma_{\mathrm{II}_1}| \ll \sum_{\substack{u|N\\ u\le N^{1/2}\log^{-B}N}}\ \sum_{t<(N^{1/2}/u)\log^{-B}N}\ \max_{y\le N/u}\ \max_{\substack{l\\ (4t,l)=1}}\Big|\sum_{\substack{n\equiv l\ (\mathrm{mod}\ 4t)\\ n\le y}}d_4(un)-\frac{1}{\varphi(4t)}\sum_{\substack{(n,4t)=1\\ n\le y}}d_4(un)\Big|,$$

which, with (12), gives

$$|\Sigma_{\mathrm{II}_1}|\ll N\log^{-A}N\sum_{u|N}\frac{d^{15}(u)}{u}\ll N(\log N)^{1-A}. \tag{32}$$

5.2. In the same way, applying (12) to the right side of (5), we get

$$\Sigma_{\mathrm{II}_2}+\Sigma_{\mathrm{II}_3} \tag{33}$$
$$=\rho(N)\sum_{\substack{u|N\\ u<N^{1/2}\log^{-B}N}}\rho(u)\sum_{\substack{(t,N/u)=1\\ t<(N^{1/2}/u)\log^{-B}N}}\rho(t)\Big\{\frac{1}{\varphi(t)}\sum_{\substack{(n,t)=1\\ n\le(1/4)(N/u-tN^{1/2}\log^B N)}}d_4(4un)$$
$$-\frac{1}{\varphi(2t)}\sum_{\substack{(n,2t)=1\\ n\le(1/2)(N/u-tN^{1/2}\log^B N)}}d_4(2un)\Big\}+O(N(\log N)^{1-A})$$
$$=\rho(N)\sum_{\substack{u|N\\ u<N^{1/2}\log^{-B}N}}\rho(u)\Big\{\sum_{\substack{(t,N/u)=1\\ t<(N^{1/2}/u)\log^{-B}N}}\frac{\rho(t)}{\varphi(t)}\sum_{\substack{(n,t)=1\\ n\le(1/4)(N/u-tN^{1/2}\log^B N)}}d_4(4un)$$

$$-\sum_{\substack{(t,N/u)=1\\ t<(N^{1/2}/u)\log^{-B}N}}\frac{\rho(t)}{\varphi(2t)}\sum_{\substack{(n,2t)=1\\ n\leqq(1/2)(N/u-tN^{1/2}\log^{B}N)}}d_4(2un)\Big\}+O(N(\log N)^{1-A})$$

$$=\rho(N)\sum_{\substack{u|N\\ u<N^{1/2}\log^{-B}N}}\rho(u)\{\Sigma_4-\Sigma_5\}+O(N(\log N)^{1-A}),\qquad \text{say}.$$

5.3. In the sum Σ_4 changing the order of summation, we have

$$\Sigma_4=\sum_{n\leqq(1/4)(N/u-N^{1/2}\log^{B}N)}d_4(4un)\sum_{\substack{(t,(N/u)n)=1\\ t\leqq(N/4u-n)(4/N^{1/2}\log^{B}N)}}\frac{\rho(t)}{\varphi(t)}.$$

To the inner sum we apply the result (14), and we get, using the notation (18),

$$\Sigma_4=\frac{\pi}{4}\mathfrak{S}\sum_{n\leqq(1/4)(N/u-N^{1/2}\log^{B}N)}d_4(4un)F\Big(\frac{N}{u}n\Big)\tag{34}$$

$$+O\Big(N^{1/4+\varepsilon}\sum_{n\leqq(1/4)(N/u-N^{1/2}\log^{B}N)}\Big(\frac{N}{4u}-n\Big)^{-1/2}\Big)$$

$$=\frac{\pi}{4}\mathfrak{S}\Sigma_6+O(N^{3/4+\varepsilon}),\qquad \text{say},$$

since we have

$$\sum_{n\leqq(1/4)(N/u-N^{1/2}\log^{B}N)}\Big(\frac{N}{4u}-n\Big)^{-1/2}\ll\int_0^{(1/4)(N/u-N^{1/2}\log^{B}N)}\Big(\frac{N}{4u}-x\Big)^{-1/2}dx$$

$$\ll\Big(\frac{N}{u}\Big)^{1/2}+N^{1/4}\log^{B/2}N\ll N^{1/2}.$$

Thus we have to estimate Σ_6, and to do this we need to investigate the function

$$\sum_{n=1}^{\infty}\frac{d_4(4un)}{n^s}F\Big(\frac{N}{u}n\Big),$$

which obviously can be treated similarly as $G(s)$ defined by (17). And we get

$$\sum_{n\leqq x}d_4(4un)F\Big(\frac{N}{u}n\Big)=\frac{1}{96\mathfrak{S}}\Big\{\sum_{n\subset 2N}\frac{d_4(4un)}{n}F\Big(\frac{N}{u}n\Big)\Big\}\tag{35}$$

$$\times\prod_{p|N}\Big(1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\Big(1-\frac{1}{p}\Big)^{-1}\Big)\Big(1-\frac{1}{p}\Big)^4$$

$$\times\prod_{p\nmid N}\Big(1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\Big(1-\frac{1}{p}\Big)^3\Big)x\log^3 x$$

$$+O(x\log^2 x\, d_4(u)(\log\log N)^{28})$$

$$=\frac{1}{96\mathfrak{S}}T_u\, x\log^3 x+O(x\log^2 x\, d_4(u)(\log\log N)^{28}),\qquad \text{say}.$$

Hence we have, from (34),

$$\Sigma_4 = \frac{\pi}{1536} T_u \Big(\frac{N}{u} - N^{1/2}\log^B N\Big) \log^3 \Big(\frac{N}{u} - N^{1/2}\log^B N\Big)$$
$$+ O\Big(\frac{d_4(u)}{u} N \log^2 N (\log\log N)^{28}\Big)$$
$$= \frac{\pi}{1536} T_u \frac{N}{u} \log^3 \Big(\frac{N}{u} - N^{1/2}\log^B N\Big)$$
$$+ O\Big(\frac{d_4(u)}{u} N \log^2 N (\log\log N)^{28}\Big).$$

5.4. Now turning to the expression (33), we have

$$(36) \qquad \sum_{\substack{u|N \\ u<N^{1/2}\log^{-B}N}} \rho(u)\Sigma_4 = \frac{\pi}{1536} N \sum_{\substack{u|N \\ u<N^{1/2}\log^{-B}N}} \frac{\rho(u)T_u}{u} \log^3 \Big(\frac{N}{u} - N^{1/2}\log^B N\Big)$$
$$+ O(N \log^2 N (\log\log N)^{41})$$
$$= \frac{\pi}{1536} N \Sigma_7 + O(N \log^2 N (\log\log N)^{41}), \qquad \text{say}.$$

We divide Σ_7 into two parts as follows

$$(37) \qquad \Sigma_7 = \sum_{\substack{u|N \\ u\leq N^{1/4}}} + \sum_{\substack{u|N \\ N^{1/4}<u<N^{1/2}\log^{-B}N}} = \Sigma_8 + \Sigma_9, \qquad \text{say}.$$

In the sum Σ_8 we have obviously

$$\log^3 \Big(\frac{N}{u} - N^{1/2}\log^B N\Big) = \log^3 \frac{N}{u} + O((u/N^{1/2}) \log^{B+2} N)$$
$$= \log^3 N + O(\log^2 N \log u),$$

and thus

$$\Sigma_8 = \log^3 N \sum_{\substack{u|N \\ u\leq N^{1/4}}} \frac{\rho(u)T_u}{u} + O\Big(\log^2 N \sum_{u|N} \frac{\log u}{u} |T_u|\Big).$$

Here we notice that we can prove easily

$$(38) \qquad |T_u| \ll d_4(u)(\log\log N)^9,$$

which gives

$$(39) \qquad \Sigma_8 = \log^3 N \sum_{u|N} \frac{\rho(u)T_u}{u} + O(\log^2 N (\log\log N)^{22}).$$

The inequality (38) yields also

$$(40) \qquad |\Sigma_9| \ll N^{-1/4+\varepsilon}.$$

Collecting (37), (39) and (40) we get

$$\Sigma_7 = \log^3 N \sum_{u|N} \frac{\rho(u)T_u}{u} + O(\log^2 N (\log\log N)^{22}),$$

which, with (36), gives

$$\sum_{\substack{u|N \\ u<N^{1/2}\log^{-B}N}} \rho(u)\Sigma_4$$

$$=\frac{\pi}{1536}N\log^3 N\prod_{p\nmid N}\Big(1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\Big(1-\frac{1}{p}\Big)^3\Big)$$

$$\times\prod_{p|N}\Big(1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\Big(1-\frac{1}{p}\Big)^{-1}\Big)\Big(1-\frac{1}{p}\Big)^4$$

$$\times\sum_{u|N}\frac{\rho(u)}{u}\sum_{n\subset 2N}\frac{d_4(4un)}{n}F\Big(\frac{N}{u}n\Big)+O(N\log^2 N(\log\log N)^{41}),$$

since T_u is defined by (35).

5.5. Now we have, since N is odd,

$$\sum_{n\subset 2N}\frac{d_4(4un)}{n}F\Big(\frac{N}{u}n\Big)=\sum_{\delta=0}^{\infty}\frac{d_4(2^{\delta+2})}{2^\delta}\sum_{n\subset N}\frac{d_4(un)}{n}F\Big(\frac{N}{u}n\Big)$$

$$=52\sum_{n\subset N}\frac{d_4(un)}{n}F\Big(\frac{N}{u}n\Big)$$

and thus, by the result (30) of the preceding paragraph, we have

$$\sum_{u|N}\frac{\rho(u)}{u}\sum_{n\subset 2N}\frac{d_4(4un)}{n}F\Big(\frac{N}{u}n\Big)$$

$$=52\prod_{p|N}\frac{\Big(1-\frac{\rho(p)}{p}\Big)\Big(1-\frac{1}{p}\Big)^{-4}}{1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\Big(1-\frac{1}{p}\Big)^{-1}}\prod_{p^\alpha\| N}\{p,\alpha\},$$

where $\{p,\alpha\}$ is defined by (29).

Inserting this into the right side of (41) we get

$$\sum_{\substack{u|N \\ u<N^{1/2}\log^{-B}N}} \rho(u)\Sigma_4=\frac{13}{384}\pi N\log^3 N\prod_{p|N}\Big(1-\frac{\rho(p)}{p}\Big)$$

$$\times\prod_{p\nmid N}\Big(1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\Big(1-\frac{1}{p}\Big)^3\Big)\prod_{p^\alpha\| N}\{p,\alpha\}$$

$$+O(N\log^2 N(\log\log N)^{41}).$$

Comparing this with the expression (31) of $\Sigma_{\rm I}$ we find

(42) $$\sum_{\substack{u|N \\ u<N^{1/2}\log^{-B}N}} \rho(u)\Sigma_4=\frac{13}{16}\Sigma_{\rm I}+O(N\log^2 N(\log\log N)^{41}).$$

Completely analogously we can prove

(43) $$\sum_{\substack{u|N \\ u<N^{1/2}\log^{-B}N}} \rho(u)\Sigma_5=\frac{1}{8}\Sigma_{\rm I}+O(N\log^2 N(\log\log N)^{41}).$$

Hence collecting (32), (33), (42) and (43) we get the final result of this paragraph:

$$\Sigma_{\text{II}}=\frac{11}{16}\rho(N)\Sigma_{\text{I}}+O(N\log^2 N(\log\log N)^{41}). \tag{44}$$

§ 6. A decomposition of $d_4(n)$.

6.1. The estimation of Σ_{III}, which is defined by (2), is our main object hereafter. Our proof depends on a kind of analogue to the so-called quasi-prime numbers of Hooley [1].

6.2. We introduce the quantity

$$\bar{N}=N^{(\log\log N)^{-3}} \tag{45}$$

and we define the following decomposition of any integer n:

$$n^{(1)}=\prod_{\substack{p^\alpha \| n\\ p\nmid 2N,\ p\leqq \bar{N}}} p^\alpha, \qquad n^{(2)}=\prod_{\substack{p^\alpha \| n\\ p|2N \text{ or } p>\bar{N}}} p^\alpha . \tag{46}$$

Then we decompose $d_4(n)$ into two parts:

$$f(n)=d_4(n^{(1)}), \qquad g(n)=d_4(n^{(2)}). \tag{47}$$

Here we should call one's attention to the fact that from the expression of $n^{(1)}$ the prime numbers which divide $2N$ are excluded. This will turn out to be useful at the last stage of our estimation of Σ_{III}.

To simplify the notations in what follows we introduce the symbols Δ_N and Γ_N which are the sets of all positive integers composed entirely of prime numbers that does not exceed $\bar{N}$ nor divide $2N$, and that exceed $\bar{N}$ or divide $2N$, respectively. And we assume that both Δ_N and Γ_N contain the integer 1.

6.3. We now investigate the summation property of $f(n)$ over arithmetic progressions, or the sum

$$\sum_{\substack{n\equiv l \pmod q\\ n\leqq y}} f(n),$$

where $(q, l)=1$ and the size of q and y is arbitrary.

First if

$$q>y^{1-\varepsilon/2} \tag{48}$$

then we see easily that

$$\begin{aligned}\sum_{\substack{n\equiv l \pmod q\\ n\leqq y}} f(n) &\ll y^{\varepsilon/2}\sum_{\substack{n\equiv l \pmod q\\ n\leqq y}} 1\\ &\ll y^{\varepsilon/2}\Big(\frac{y}{q}+1\Big)\\ &\ll y^{\varepsilon},\end{aligned} \tag{49}$$

since we have obviously

$$f(n) \ll y^{\varepsilon/2}.$$

Thus next we assume that

(50) $$q \leqq y^{1-\varepsilon/2}.$$

We have

$$f(n) = \sum_{\substack{u \mid N \\ u \in \Delta_N}} d_3(u),$$

and we decompose the sum as follows:

(51) $$\sum_{\substack{n \equiv l \ (\mathrm{mod}\ q) \\ n \leqq y}} f(n) = \sum_{\substack{n \equiv l \ (\mathrm{mod}\ q) \\ n \leqq y}} \{ \sum_{\substack{u \mid N \\ u \in \Delta_N \\ \Omega(u) \leqq (A_1 \log\log N)^2}} d_3(u) + \sum_{\substack{u \mid N \\ u \in \Delta_N \\ \Omega(u) > (A_1 \log\log N)^2}} d_3(u) \}$$

$$= \Sigma_{10} + \Sigma_{11}, \quad \text{say},$$

where A_1 is a constant to be determined later.

In the sum Σ_{10} we have, by (45),

$$u \leqq N^{A_1^2/\log\log N},$$

and thus we get

(52) $$\Sigma_{10} = \sum_{\substack{(u,q)=1 \\ u \in \Delta_N \\ \Omega(u) \leqq (A_1 \log\log N)^2}} d_3(u) \sum_{\substack{n \equiv l \ (\mathrm{mod}\ q) \\ n \equiv 0 \ (\mathrm{mod}\ u) \\ n \leqq y}} 1$$

$$= \frac{y}{q} \sum_{\substack{(u,q)=1 \\ u \in \Delta_N \\ \Omega(u) \leqq (A_1 \log\log N)^2}} \frac{d_3(u)}{u} + O\Big(\sum_{u \leqq N^{A_1^2/\log\log N}} d_3(u) \Big)$$

$$= \frac{y}{q} \Big\{ \sum_{\substack{(u,q)=1 \\ u \in \Delta_N}} \frac{d_3(u)}{u} - \sum_{\substack{(u,q)=1 \\ u \in \Delta_N \\ \Omega(u) > (A_1 \log\log N)^2}} \frac{d_3(u)}{u} \Big\} + O(N^{\varepsilon}).$$

We have for the first term

(53) $$\sum_{\substack{(u,q)=1 \\ u \in \Delta_N}} \frac{d_3(u)}{u} = \prod_{\substack{p \nmid 2qN \\ p \leqq \bar{N}}} \Big(1 - \frac{1}{p}\Big)^{-3},$$

and the second term is estimated as follows:

(54) $$\sum_{\substack{(u,q)=1 \\ u \in \Delta_N \\ \Omega(u) > (A_1 \log\log N)^2}} \frac{d_3(u)}{u} \leqq 2^{-(A_1 \log\log N)^2} \sum_{u \in \Delta_N} \frac{2^{\Omega(u)} d_3(u)}{u}$$

$$\leqq (\log N)^{-A_1^2 \log 2 \log\log N} \prod_{2 < p \leqq \bar{N}} \Big(1 - \frac{2}{p}\Big)^{-3}$$

$$\ll (\log N)^{6 - A_1^2}.$$

From (52), (53) and (54) we get

$$\Sigma_{10}=\frac{y}{q}\prod_{\substack{p\nmid 2qN\\ p\leqq N}}\left(1-\frac{1}{p}\right)^{-3}+O\left(\frac{y}{q}(\log N)^{6-A_1^2}+N^{\varepsilon}\right). \tag{55}$$

We turn to the sum Σ_{11}. Obviously we have

$$\Sigma_{11}\ll\sum_{\substack{n\equiv l(\mathrm{mod}\ q)\\ n\leqq y\\ \Omega(n^{(1)})>(A_1\log\log N)^2}} d_4(n), \tag{56}$$

which is divided into two parts according to the number of different prime factors of n:

$$\sum_{\substack{n\equiv l(\mathrm{mod}\ q)\\ n\leqq y\\ \Omega(n^{(1)})>(A_1\log\log N)^2\\ \omega(n)\leqq A_1\log\log N}} d_4(n)+\sum_{\substack{n\equiv l(\mathrm{mod}\ q)\\ n\leqq y\\ \Omega(n^{(1)})>(A_1\log\log N)^2\\ \omega(n)>A_1\log\log N}} d_4(n)=\Sigma_{12}+\Sigma_{13}, \quad \text{say}. \tag{57}$$

In the sum Σ_{13} we have

$$d(n)>2^{A_1\log\log N}$$

and thus we have

$$\begin{aligned}\Sigma_{13}&\leqq 2^{-A_1\log\log N}\sum_{\substack{n\equiv l(\mathrm{mod}\ q)\\ n\leqq y}} d(n)d_4(n)\\ &\leqq(\log N)^{-A_1\log 2}\sum_{\substack{n\equiv l(\mathrm{mod}\ q)\\ n\leqq y}} d^4(n).\end{aligned} \tag{58}$$

Here we quote the following well-known result: under the condition (50) we have, for any j,

$$\sum_{\substack{n\equiv l(\mathrm{mod}\ q)\\ n\leqq y}} d^j(n)\ll\frac{y}{q}(\log y)^{2^j-1}. \tag{59}$$

Applying this to the right side of (58) we get, under (50),

$$\Sigma_{13}\ll\frac{y}{q}(\log N)^{-A_1\log 2}(\log y)^{15}. \tag{60}$$

To estimate Σ_{12} we apart two cases: first if

$$y\leqq N^{2A_1/\varepsilon(\log\log N)^2}$$

we have obviously

$$\Sigma_{12}\ll N^{\varepsilon} \tag{61}$$

for sufficiently large N.

Second we assume that

$$y>N^{2A_1/\varepsilon(\log\log N)^2}. \tag{62}$$

We apply the Cauchy-Schwarz inequality to Σ_{12}:

$$(\Sigma_{12})^2 \leqq \sum_{\substack{n\equiv l(\mathrm{mod}\ q)\\ n\leqq y}} d_4^2(n) \sum_{\substack{n\equiv l(\mathrm{mod}\ q)\\ n\leqq y\\ \Omega(n^{(1)})>(A_1 \log\log N)^2\\ \omega(n)\leqq(A_1\log\log N)}} 1 \tag{63}$$

$$\ll \frac{y}{q}(\log y)^{63}\Sigma_{14}, \qquad \text{say,}$$

since we have (50) and (59).

In the sum Σ_{14} the condition on the number of prime factors of n and $n^{(1)}$ implies that there is at least one prime power $p^{\alpha'}$ such that

$$p^{\alpha'} | n$$

with $\alpha' > A_1 \log\log N$ and $2 < p \leqq \bar{N}$.

Hence Σ_{14} does not exceed the sum, with $\alpha_0 = [A_1 \log\log N]$,

$$\sum_{\substack{2<p\leqq\bar{N}\\ p\nmid q}} \sum_{\substack{n\equiv l(\mathrm{mod}\ q)\\ n\equiv 0(\mathrm{mod}\ p^{\alpha_0})\\ n\leqq y}} 1. \tag{64}$$

Here by virtue of the definition of $\bar{N}$ we have

$$p^{\alpha_0} \leqq N^{A_1/(\log\log N)^2} < y^{\varepsilon/2},$$

since we have (62), and the condition (50) on q implies

$$qp^{\alpha_0} < y.$$

Thus we have, from (64),

$$\Sigma_{14} \ll \frac{y}{q} \sum_{p\geqq 3} \frac{1}{p^{\alpha_0}} \ll \frac{y}{q}(\log N)^{-A_1 \log 2},$$

which, with (63), gives

$$\Sigma_{12} \ll \frac{y}{q}(\log N)^{-(A_1/2)\log 2}(\log y)^{32}. \tag{65}$$

From (57), (60), (61) and (65) we get, under the condition (50),

$$\Sigma_{11} \ll \frac{y}{q}(\log N)^{-(A_1/2)\log 2}(\log y)^{32} + N^{\varepsilon}.$$

This, with (49), (51) and (55), gives rise to the following result: if $y \leqq N$ and $(q, l) = 1$, then we have

$$\sum_{\substack{n\equiv l(\mathrm{mod}\ q)\\ n\leqq y}} f(n) = \frac{y}{q} \prod_{\substack{p\nmid 2qN\\ p\leqq\bar{N}}} \left(1-\frac{1}{p}\right)^{-3} + O\left(\frac{y}{q}(\log N)^{-E} + N^{\varepsilon}\right), \tag{66}$$

where E is a constant which can be made as large as desired, and here we should remark that the size of y is restricted only by $y \leqq N$.

6.4. Now we turn our attention to the function $g(n)$, for which we shall later need the estimation of the sum

$$\sum_{\substack{n\equiv l(\mathrm{mod}\ q)\\ n\leqq y}} g^2(n),$$

where $y \leqq N$ and $(q, l)=1$ and $q \leqq y^{1-\varepsilon}$.

Our proof of the estimation of this sum depends on the following recent idea of Wolke [3]: let $0<\delta \leqq \frac{1}{2}$ then the inequality

$$d(n) \ll \{ \sum_{\substack{v \mid n \\ v \leqq n^{\delta}}} 1\}^{c(\delta \log \delta^{-1})^{-1}} \tag{67}$$

holds with an absolute constant c.

In this inequality we take

$$\delta^{-1}=\max\left\{\left[\exp\left(\frac{12c}{\varepsilon}\right)\right], \left[\frac{2}{\varepsilon}\right]\right\}+1, \tag{68}$$

then we have

$$6c(\delta \log \delta^{-1})^{-1} \leqq \frac{\varepsilon}{2\delta} \leqq \left[\frac{\varepsilon}{2\delta}\right]+1=\eta, \quad \text{say.} \tag{69}$$

And we have

$$\begin{aligned} \sum_{\substack{n \equiv l (\bmod q) \\ n \leqq y}} g^2(n) &\ll \sum_{\substack{n \equiv l (\bmod q) \\ n \leqq y}} d^6(n^{(2)}) \\ &\ll \sum_{\substack{n \equiv l (\bmod q) \\ n \leqq y}} \{ \sum_{\substack{v \mid n \\ v \in \Gamma_N \\ v \leqq n^{\delta}}} 1\}^{\eta}, \end{aligned}$$

where the inner sum is less than

$$\sum_{\substack{t \mid n \\ t \in \Gamma_N \\ t \leqq y^{\delta\eta}}} \sum_{t=[v_1, \cdots, v_\eta]} 1 \leqq \sum_{\substack{t \mid n \\ t \in \Gamma_N \\ t \leqq y^{\varepsilon}}} d^{\eta}(t),$$

since we have, from (68) and (69),

$$\delta\eta \leqq \frac{\varepsilon}{2}+\delta \leqq \varepsilon .$$

Hence we get the inequality, noticing that $q \leqq y^{1-\varepsilon}$,

$$\begin{aligned} \sum_{\substack{n \equiv l (\bmod q) \\ n \leqq y}} g^2(n) &\ll \sum_{\substack{(t, q)=1 \\ t \in \Gamma_N \\ t \leqq y^{\varepsilon}}} d^{\eta}(t) \sum_{\substack{n \equiv l (\bmod q) \\ n \equiv 0 (\bmod t) \\ n \leqq y}} 1 \\ &\ll \frac{y}{q} \sum_{\substack{t \in \Gamma_N \\ t \leqq N}} \frac{d^{\eta}(t)}{t} \\ &\ll \frac{y}{q} \prod_{\substack{p \mid 2N \\ \text{or} \\ \bar{N}<p \leqq N}} \left(1-\frac{1}{p}\right)^{-2^{\eta}}. \end{aligned}$$

Thus, by the definition of $\bar{N}$, we have proved the following result: if $y \leqq N$, $(q, l)=1$ and $q \leqq y^{1-\varepsilon}$ then we have

$$\sum_{\substack{n \equiv l (\bmod q) \\ n \leqq y}} g^2(n) \ll \frac{y}{q}(\log \log N)^{2^{\eta+2}}, \tag{70}$$

where η is defined by (69).

6.5. Further we estimate the sum

$$\sum_{\substack{n\equiv l \pmod q \\ n\leqq y}} f(n)g^2(n)$$

under the condition:

(71) $$N^{1/8}\leqq y\leqq N, \qquad (q, l)=1, \qquad q\leqq y^{1-\varepsilon}.$$

We divide the sum into two parts as follows

(72) $$\sum_{\substack{n\equiv l \pmod q \\ n\leqq y}} f(n)g^2(n)=\sum_{\substack{n\equiv l \pmod q \\ n\leqq y}} g^2(n)\Big\{\sum_{\substack{u\mid n \\ u\in \Delta_N \\ \Omega(u)\leqq (A_2\log\log N)^2}} d_3(u)+\sum_{\substack{u\mid n \\ u\in \Delta_N \\ \Omega(u)>(A_2\log\log N)^2}} d_3(u)\Big\}$$
$$=\Sigma_{15}+\Sigma_{16}, \qquad \text{say,}$$

where A_2 is a constant to be determined later. Obviously we have

$$\Sigma_{16}\leqq \sum_{\substack{n\equiv l \pmod q \\ n\leqq y \\ \Omega(n^{(1)})>(A_2\log\log N)^2}} d^9(n),$$

which can be treated just like the sum Σ_{11} of (56), and we find

(73) $$\Sigma_{16}\leqq \frac{y}{q}(\log N)^{-E_1}$$

with an arbitrarily large constant E_1.

Now for Σ_{15} we have

(74) $$\Sigma_{15}=\sum_{\substack{(u,q)=1 \\ u\in \Delta_N \\ \Omega(u)\leqq (A_2\log\log N)^2}} d_3(u)\sum_{\substack{n\equiv l \pmod q \\ n\equiv 0 \pmod u \\ n\leqq y}} g^2(n)$$
$$=\sum_{\substack{(u,q)=1 \\ u\in \Delta_N \\ \Omega(u)\leqq (A_2\log\log N)^2}} d_3(u)\sum_{\substack{uv\equiv l \pmod q \\ v\leqq y/u}} g^2(v)$$

since for $u\in \Delta_N$ we have

$$g(uv)=d_4(u^{(2)}v^{(2)})=d_4(v^{(2)})=g(v).$$

Now, by the definition of $\bar{N}$ and the condition (71) on y, we have in the sum

$$q\leqq y^{1-\varepsilon}\leqq \left(\frac{y}{u}\right)^{1-\varepsilon/2},$$

and so we can apply the result (70) to the inner sum of (74). Thus we get

(75) $$\Sigma_{15}\ll \frac{y}{q}(\log\log N)^K\sum_{u\in \Delta_N}\frac{d_3(u)}{u}$$
$$\ll \frac{y}{q}(\log N)^3(\log\log N)^K,$$

where K depends on ε at most.

From (72), (73) and (75) we get the following result: under the condition (71) we have

$$\sum_{\substack{n\equiv l \pmod q\\ n\leqq y}} f(n)g^2(n) \ll \frac{y}{q}(\log N)^3(\log\log N)^K. \tag{76}$$

6.6. Finally we estimate the sum

$$\sum_{n<N} d(N-n)f(n)g^2(n).$$

This sum does not exceed

$$2\sum_{n<N} f(n)g^2(n) \sum_{\substack{q\mid N-n\\ q\leqq N^{1/2}}} 1$$

$$\ll \sum_{\substack{u\mid N\\ u\leqq N^{1/2}}} f(u)g^2(u) \sum_{\substack{(q,N/u)=1\\ q\leqq N^{1/2}/u}} \sum_{\substack{n\equiv (N/u) \pmod q\\ n<N/u}} f(n)g^2(n).$$

Now noticing that

$$q\leqq \frac{N^{1/2}}{u} < \left(\frac{N}{u}\right)^{1/2},$$

we apply (76) to the inner most sum. Thus we find

$$\sum_{\substack{(q,N/u)=1\\ q\leqq N^{1/2}/u}} \sum_{\substack{n\equiv (N/u) \pmod q\\ n<N/u}} f(n)g^2(n) \ll \sum_{q\leqq N} \frac{N}{qu}(\log N)^3(\log\log N)^K$$

$$\ll \frac{N}{u}(\log N)^4(\log\log N)^K,$$

which gives

$$\sum_{n<N} d(N-n)f(n)g^2(n) \ll N(\log N)^4(\log\log N)^K \sum_{u\mid N} \frac{f(u)g^2(u)}{u} \tag{77}$$

$$\ll N(\log N)^4(\log\log N)^{K_1}$$

with an absolute constant K_1.

§ 7. Estimation of $\sum_{\mathrm{III}}$, (I).

7.1. We are now ready to start the estimation of $\sum_{\mathrm{III}}$ along the line of Hooley [1].

Let

$$H(m) = \sum_{\substack{q\mid m\\ N^{1/2}\log^{-B} N<q<N^{1/2}\log^B N}} 1, \tag{78}$$

then, using the fact that

$$d_4(n) = f(n)g(n),$$

we have the fundamental inequality

(79) $$(\Sigma_{\mathrm{III}})^2 \leqq \{ \sum_{\substack{n<N \\ H(N-n)\neq 0}} f(n)g^2(n)\} \{ \sum_{n<N} f(n) | \sum_{\substack{q|N-n \\ N^{1/2}\log^{-B}N<q<N^{1/2}\log^{B}N}} \rho(q)|^2\}$$
$$= \{\Sigma_{\mathrm{III}_1}\} \{\Sigma_{\mathrm{III}_2}\}\,, \quad \text{say.}$$

The aim of this paragraph is to estimate Σ_{III_1}, while in the next paragraph we treat Σ_{III_2}.

7.2. Now we divide Σ_{III_1} into two parts as follows

(80) $$\Sigma_{\mathrm{III}_1} = \sum_{\substack{n<N \\ H(N-n)\neq 0 \\ \omega(N-n)\leqq\beta\log\log N}} f(n)g^2(n) + \sum_{\substack{n<N \\ H(N-n)\neq 0 \\ \omega(N-n)>\beta\log\log N}} f(n)g^2(n)$$
$$= \Sigma_{17} + \Sigma_{18}\,, \quad \text{say,}$$

where β is in the interval
$$1 < \beta < 2$$
and to be determined explicitly later.

We treat Σ_{18} first. The condition on the number of different prime divisors of $N-n$ implies
$$d(N-n) \geqq 2^{\beta\log\log N}\,.$$
And thus we have

(81) $$\Sigma_{18} \leqq 2^{-\beta\log\log N} \sum_{n<N} d(N-n)f(n)g^2(n)$$
$$\ll N(\log N)^{4-\beta\log 2}(\log\log N)^{K_1}\,,$$

since we have the result (77).

7.3. The sum Σ_{17} requires much careful treatment than Σ_{18}. And, since hereafter we have to treat sums over complicated ranges of the variables, we adopt abbreviated notations for lengthy conditions of summation: following Hooley [1] we introduce the symbols

(82) $$\left.\begin{aligned} (L'_j) &\equiv \{N^{1/2}\log^{-B}N < l'_j < N^{1/2}\log^{B}N\} \\ (L_j) &\equiv \left\{\frac{N^{1/2}}{d}\log^{-B}N < l_j < \frac{N^{1/2}}{d}\log^{B}N\right\} \end{aligned}\right\} \quad (j=1,2)$$
$$(L^*) \equiv \{(l_1, l_2) = 1\}\,.$$

Then we have

(83) $$\Sigma_{17} \leqq \sum_{\substack{n<N \\ \omega(N-n)\leqq\beta\log\log N}} f(n)g^2(n)H(N-n)$$
$$= \sum_{n<N} f(n)g^2(n) \sum_{\substack{l'_1 l'_2 = N-n \\ \omega(l'_1 l'_2)\leqq\beta\log\log N \\ (L'_1),\,(L'_2)}} 1$$

$$\leqq \sum_{n<N} f(n)g^2(n) \sum_{\substack{l_1l_2d^2=N-n \\ \omega(l_1l_2)\leqq\beta\log\log N \\ (L_1),(L_2),(L^*)}} 1$$

$$= \sum_{d\leqq N^{1/8}} + \sum_{N^{1/8}<d<N^{1/2}} = \sum\nolimits_{19} + \sum\nolimits_{20}, \qquad \text{say.}$$

It is easy to see that

$$\sum\nolimits_{20} \ll N^{\varepsilon} \sum_{N^{1/8}<d<N^{1/2}} \sum_{\substack{n\equiv N(\text{mod } d^2) \\ n<N}} 1 \ll N^{7/8+\varepsilon}. \tag{84}$$

Now in the sum $\sum_{19}$ the conditions (L^*) and $\omega(l_1l_2)\leqq\beta\log\log N$ imply that at least one of $\omega(l_1)$ and $\omega(l_2)$ does not exceed

$$\frac{1}{2}\beta\log\log N,$$

and hence we have

$$\sum\nolimits_{19} \ll \sum_{d\leqq N^{1/8}} \sum_{\substack{\omega(l_1)\leqq(1/2)\beta\log\log N \\ (L_1)}} \sum_{\substack{n\equiv N(\text{mod } l_1d^2) \\ n<N}} f(n)g^2(n) \tag{85}$$

$$\ll \sum_{\substack{u\mid N \\ u\leqq N^{5/8}\log^B N}} f(u)g^2(u) \sum_{\substack{(d^2l_1,N)=u \\ d\leqq N^{1/8} \\ (L_1) \\ \omega(l_1)\leqq(1/2)\beta\log\log N}} \sum_{\substack{n\equiv(N/u)\left(\text{mod } \frac{d^2l_1}{u}\right) \\ n<N/u}} f(n)g^2(n).$$

Thus, noticing that

$$\frac{d^2l_1}{u} \leqq \frac{N^{3/4}}{u} \leqq \left(\frac{N}{u}\right)^{3/4},$$

we apply the result (76) to the inner-most sum of (85), and we get

$$\sum\nolimits_{19} \ll N(\log N)^3(\log\log N)^K \sum_{u\mid N} f(u)g^2(u) \sum_{\substack{d^2l_1\equiv0(\text{mod } u) \\ d\leqq N^{1/8} \\ (L_1) \\ \omega(l_1)\leqq(1/2)\beta\log\log N}} \frac{1}{d^2l_1} \tag{86}$$

$$= N(\log N)^3(\log\log N)^K\Big\{ \sum_{u\leqq N^{1/8}} + \sum_{u>N^{1/8}} \Big\}$$

$$= N(\log N)^3(\log\log N)^K\{\textstyle\sum_{21}+\sum_{22}\}, \qquad \text{say.}$$

We have easily

$$\sum\nolimits_{22} \ll \sum_{\substack{u\mid N \\ u>N^{1/8}}} f(u)g^2(u) \sum_{d=1}^{\infty} \frac{1}{d^2} \sum_{\substack{l\equiv0(\text{mod } u/(d^2,u)) \\ l\leqq N}} \frac{1}{l} \tag{87}$$

$$\ll N^{\varepsilon} \sum_{\substack{u\mid N \\ u>N^{1/8}}} \frac{1}{u} \sum_{d=1}^{\infty} \frac{(d^2,u)}{d^2}.$$

Here we have

$$\sum_{d=1}^{\infty} \frac{(d^2,u)}{d^2} \ll \sum_{d\subset u} \frac{(d^2,u)}{d^2} = \prod_{p^\alpha\parallel u}\Big\{\sum_{2j\leqq\alpha} 1 + p^\alpha \sum_{2j>\alpha} \frac{1}{p^{2j}}\Big\} \tag{88}$$

$$\leqq \prod_{p^\alpha \| u} \left\{ \left[\frac{\alpha}{2} \right] + 1 + \frac{1}{1-1/p^2} \right\}$$

$$\leqq \prod_{p^\alpha \| u} \left(\frac{\alpha}{2} + \frac{7}{3} \right) \ll d^3(u),$$

which gives

(89) $$\sum\nolimits_{22} \ll N^{2\varepsilon} \sum_{\substack{u|N \\ u>N^{1/8}}} \frac{1}{u} \ll N^{-1/8+3\varepsilon}.$$

Now for $\sum_{21}$ we have analogously as (87)

(90) $$\sum\nolimits_{21} \ll \sum_{\substack{u|N \\ u\leqq N^{1/8}}} \frac{f(u)g^2(u)}{u} \sum_{d\leqq N^{1/8}} \frac{(d^2, u)}{d^2} \sum_{\substack{\omega(l)\leqq(1/2)\beta \log\log N \\ ((d^2,u)/u)(N^{1/2}/d)\log^{-B} N<l<((d^2,u)/u)(N^{1/2}/d)\log^B N}} \frac{1}{l}.$$

Here we quote the following result: if $1/2 \leqq \xi < 1$, then we have, uniformly for any $t \leqq N^{1/4}$,

(91) $$\sum_{\substack{\omega(m)\leqq\xi \log\log N \\ (N^{1/2}/t)\log^{-B} N<m<(N^{1/2}/t)\log^B N}} \frac{1}{m} \ll (\log N)^{\gamma_\xi - 1} \log\log N$$

where $\gamma_\xi = \xi - \xi \log \xi$.

This can be proved just like the corresponding result of Hooley [**1**], in which $\Omega(m)$ appears instead of $\omega(m)$ but this does not make any essential differences.

Since in the right side of (90) we have obviously

$$\frac{ud}{(u, d^2)} \leqq N^{1/4},$$

we can apply (91), and we find

$$\sum\nolimits_{21} \ll (\log N)^{\gamma_{\beta/2}-1} (\log\log N) \sum_{u|N} \frac{f(u)g^2(u)}{u} \sum_{d=1}^{\infty} \frac{(d^2, u)}{d^2}$$

$$\ll (\log N)^{\gamma_{\beta/2}-1} (\log\log N) \sum_{u|N} \frac{d^{12}(u)}{u},$$

since we have (88) and $f(u)g^2(u) \leqq d^9(u)$.

Thus we get

(92) $$\sum\nolimits_{21} \ll (\log N)^{\gamma_{\beta/2}-1} (\log\log N)^{2^{12}+1}.$$

Now from (86), (89) and (92) we have

$$\sum\nolimits_{19} \ll N (\log N)^{2+\gamma_{\beta/2}} (\log\log N)^{K_2}$$

with an absolute constant K_2. Collecting (84), (83) and this result we see that the same inequality holds for $\sum_{17}$.

7.4. Thus, from (80), (81) and the above conclusion, we get

$$\Sigma_{\mathrm{III}_1} \ll N(\log\log N)^{K_3}(\log N)^3\{(\log N)^{1-\beta\log 2}+(\log N)^{\gamma_{\beta/2}-1}\}$$

with an absolute constant K_3.

Now the equation

$$\gamma_{\beta/2}-1=1-\beta\log 2$$

or

$$\log\frac{8e}{\beta}=\frac{4}{\beta}$$

has the root $\beta_0=1.494\cdots$.

Hence we get

$$\Sigma_{\mathrm{III}_1} \ll N(\log N)^{3-\gamma}(\log\log N)^{K_3}, \tag{93}$$

where

$$\gamma=\beta_0\log 2-1=0.0358\cdots.$$

§ 8. Estimation of Σ_{III}, (II).

8.1. We now pass on to the estimation of Σ_{III_2}, which is defined by (79) Our principal tool is the result (66).

We have

$$\Sigma_{\mathrm{III}_2}=\sum_{\substack{l_1'm_1=l_2'm_2=N-n\\ n<N\\ (L_1'),\,(L_2')}}\rho(l_1')\rho(l_2')f(n),$$

where (L_j') is defined by (82) of the previous paragraph.

Then we have

$$\Sigma_{\mathrm{III}_2}=\sum_{\substack{l_1l_2dm=N-n\\ n<N\\ (L_1),(L_2),(L^*)}}\rho^2(d)\rho(l_1)\rho(l_2)f(n) \tag{94}$$

$$=\sum_{d<N^{1/8}}+\sum_{N^{1/8}\leqq d\leqq N^{1/2}\log^B N}=\Sigma_{23}+\Sigma_{24}, \qquad \text{say.}$$

8.2. First we treat Σ_{24}. We have

$$\Sigma_{24}=\sum_{\substack{(L_1),(L_2),(L^*)\\ N^{1/8}\leqq d\leqq N^{1/2}\log^B N}}\rho^2(d)\rho(l_1)\rho(l_2)\sum_{\substack{n\equiv N/(N,dl_1l_2)(\mathrm{mod}\ dl_1l_2/(N,dl_1l_2))\\ n<N/(N,dl_1l_2)}}f((N,dl_1l_2)n) \tag{95}$$

$$=\sum_{\substack{(L_1),(L_2),(L^*)\\ N^{1/8}\leqq d\leqq N^{1/2}\log^B N}}\rho^2(d)\rho(l_1)\rho(l_2)\sum_{\substack{n\equiv N/(N,dl_1l_2)(\mathrm{mod}\ dl_1l_2/(N,dl_1l_2))\\ n<N/(N,dl_1l_2)}}f(n)$$

since, by virtue of the definition (46) of $n^{(1)}$, we have

$$(N,dl_1l_2)^{(1)}=1.$$

Here we notice that from the condition on d, l_1 and l_2 we have

$$dl_1l_2\leqq N^{7/8}\log^{2B}N.$$

Thus, applying (66) to the inner-sum of (95), we get

$$\Sigma_{24}=N\sum_{\substack{(L_1),(L_2),(L^*)\\ N^{1/8}\leqq d\leqq N^{1/2}\log^B N}}\frac{\rho^2(d)\rho(l_1)\rho(l_2)}{dl_1l_2}\prod_{\substack{p\nmid 2(Ndl_1l_2/(N,dl_1l_2))\\ p\leqq N}}\left(1-\frac{1}{p}\right)^{-3} \tag{96}$$

$$+O\Big(N(\log N)^{-E}\sum_{dl_1l_2\leqq N}\frac{1}{dl_1l_2}\Big)$$

$$+O(N^{\varepsilon}\sum_{dl_1l_2\leqq N^{7/8}\log^{2B}N}1)$$

$$=N\Sigma_{25}+O(N(\log N)^{3-E}),\qquad \text{say.}$$

Now, setting

$$P_N=\prod_{p\leqq N}\left(1-\frac{1}{p}\right)^{-3}$$

we have

$$\Sigma_{25}=\frac{1}{8}P_N\sum_{\substack{(L_1),(L_2),(L^*)\\ N^{1/8}\leqq d\leqq N^{1/2}\log^B N}}\frac{\rho^2(d)\rho(l_1)\rho(l_2)}{dl_1l_2}\prod_{\substack{p\mid Ndl_1l_2\\ p\leqq N}}\left(1-\frac{1}{p}\right)^3,$$

since $\dfrac{Ndl_1l_2}{(N,\,dl_1l_2)}$ is the least common multiple of N and so

$$\prod_{p\mid Ndl_1l_2/(N,dl_1l_2)}\left(1-\frac{1}{p}\right)^3=\prod_{p\mid Ndl_1l_2}\left(1-\frac{1}{p}\right)^3.$$

Further we have, restricting the value of d,

$$\Sigma_{25}=\frac{1}{8}P_N\sum_{\substack{(L_1),(L_2),(L^*)\\ N^{1/8}\leqq d\leqq N^{1/2}\log^{-B} N}}\frac{\rho^2(d)\rho(l_1)\rho(l_2)}{dl_1l_2}\prod_{\substack{p\mid Ndl_1l_2\\ p\leqq N}}\left(1-\frac{1}{p}\right)^3 \tag{97}$$

$$+O\Big(P_N\sum_{\substack{(L_1),(L_2)\\ N^{1/2}\log^{-B}N<d\leqq N^{1/2}\log^B N}}\frac{1}{dl_1l_2}\Big)$$

$$=\frac{1}{8}P_N\sum_{\substack{(L_2)\\ N^{1/8}\leqq d\leqq N^{1/2}\log^{-B}N}}\frac{\rho^2(d)\rho(l_2)}{dl_2}\sum_{(L_1),(L^*)}\frac{\rho(l_1)}{l_1}\prod_{\substack{p\mid Ndl_1l_2\\ p\leqq N}}\left(1-\frac{1}{p}\right)^3$$

$$+O(\log^3 N).$$

To estimate the inner sum of (97), we consider the function

$$\sum_{(n,l_2)=1}\frac{\rho(n)}{n^s}\prod_{\substack{p\mid Ndl_2n\\ p\leqq N}}\left(1-\frac{1}{p}\right)^3,$$

which converges absolutely for $\sigma>1$, and by the standard way of decomposing the series into Euler product, we find easily that this is equal to

$$L(s,\rho)\prod_{\substack{p\mid Ndl_2\\ p\leqq N}}\left(1-\frac{1}{p}\right)^3\prod_{\substack{p\nmid Ndl_2\\ p\leqq N}}\left(1-\frac{\rho(p)}{p^s}+\frac{\rho(p)}{p^s}\left(1-\frac{1}{p}\right)^3\right)\prod_{p\mid l_2}\left(1-\frac{\rho(p)}{p^s}\right).$$

Applying the formula of Perron to this function we get uniformly for all parameters

$$\sum_{\substack{(n,l_2)=1\\ x\leqq n}}\frac{\rho(n)}{n}\prod_{\substack{p\mid Ndl_2n\\ p\leqq\bar{N}}}\Big(1-\frac{1}{p}\Big)^3\ll x^{-1/4}\prod_{p\mid l_2}\Big(1+\frac{1}{\sqrt{p}}\Big).$$

Thus we have

$$\sum_{(L_1),(L^*)}\frac{\rho(l_1)}{l_1}\prod_{\substack{p\mid Ndl_2l_1\\ p\leqq\bar{N}}}\Big(1-\frac{1}{p}\Big)\ll\Big(\frac{N^{1/2}}{d}\log^{-B}N\Big)^{-1/4}\prod_{p\mid l_2}\Big(1+\frac{1}{\sqrt{p}}\Big).$$

Inserting this into the right side of (97), we get

$$\sum\nolimits_{25}\ll P_N(N^{1/2}\log^{-B}N)^{-1/4}\sum_{d\leqq N^{1/2}\log^{-B}N}\frac{1}{d^{3/4}}\sum_{(L_2)}\frac{1}{l_2}\prod_{p\mid l_2}\Big(1+\frac{1}{\sqrt{p}}\Big)$$
$$\ll(\log N)^3(\log\log N)^{-2},$$

which, with (96), gives

(98) $$\sum\nolimits_{24}\ll N(\log N)^3(\log\log N)^{-2}.$$

8.3. We now turn to the sum $\sum_{23}$. Before starting the estimation we introduce further abbreviations:

$$(R_j)\equiv\Big\{\frac{N^{1/2}}{dt}\log^{-B}N<r_j<\frac{N^{1/2}}{dt}\log^{B}N\Big\}\qquad(j=1,2).$$

Then, using the fact that

$$\sum_{\substack{r_1t=l_1\\ r_2t=l_2}}\mu(t)=\begin{cases}1 & \text{if } (l_1,l_2)=1\\ 0 & \text{if } (l_1,l_2)>1,\end{cases}$$

we have

(99) $$\sum\nolimits_{23}=\sum_{\substack{N-n=r_1r_2t^2dm\\ n<N\\ (R_1),(R_2),d<N^{1/8}}}\mu(t)\rho^2(t)\rho^2(d)\rho(r_1)\rho(r_2)f(n)$$
$$=\sum_{t<N^{1/8}}+\sum_{t\geqq N^{1/8}}=\sum\nolimits_{26}+\sum\nolimits_{27},\qquad\text{say.}$$

It is easy to see that

(100) $$\sum\nolimits_{27}\ll N^{7/8+\varepsilon}.$$

Now in the sum $\sum_{26}$ we have

(101) $$r_1t^2dm<N^{3/4}\log^{B}N,$$

and so we have

(102) $$\Big|\sum\nolimits_{26}\Big|\leqq\sum_{\substack{r_1t^2dm<N^{3/4}\log^{B}N\\ (R_1)\\ d<N^{1/8},\,t<N^{1/8}}}\Big|\sum_{\substack{r_1r_2t^2dm=N-n\\ n<N\\ (R_2)}}\rho(r_2)f(n)\Big|.$$

We put $w=r_1t^2dm$ and denote the inner sum of the above inequality by $T(w)$. Then we have

$$T(w)=\sum_{\substack{r_2w=N-n\\ y_1<n<y_2}}\rho(r_2)f(n)\,,$$

where

$$y_1=N-\frac{w}{dt}N^{1/2}\log^B N\,,\quad y_2=N-\frac{w}{dt}N^{1/2}\log^{-B}N\,.$$

Thus we have

$$T(w)=\sum_{\substack{n\equiv N-w(\bmod 4w)\\ y_1<n<y_2}}f(n)-\sum_{\substack{n\equiv N+w(\bmod 4w)\\ y_1<n<y_2}}f(n)\,.$$

We now apart three cases:

(i) If w is even, then we have

$$(N-w,\,4w)=(N+w,\,4w)=(N,\,w)\,.$$

Thus we have

$$T(w)=\sum_{\substack{n\equiv(N-w)/(N,w)(\bmod 4w/(N,w))\\ y_1/(N,w)<n<y_2/(N,w)}}f(n)-\sum_{\substack{n\equiv(N+w)/(N,w)(\bmod 4w/(N,w))\\ y_1/(N,w)<n<y_2/(N,w)}}f(n)\,,$$

since by virtue of the definition (47) of $f(n)$ we have

$$f((N,\,w)n)=f(n)\,.$$

Applying the result (66) to the right side we get

$$\begin{aligned}T(w)&=\Big(\frac{y_2}{(N,w)}-\frac{y_1}{(N,w)}\Big)\frac{(N,w)}{4w}\prod_{\substack{p\nmid 2(Nw/(N,w))\\ p\leq N}}\Big(1-\frac{1}{p}\Big)^{-3}\\ &\quad-\Big(\frac{y_2}{(N,w)}-\frac{y_1}{(N,w)}\Big)\frac{(N,w)}{4w}\prod_{\substack{p\nmid 2(Nw/(N,w))\\ p\leq N}}\Big(1-\frac{1}{p}\Big)^{-3}\\ &\quad+O\Big(\frac{y_2}{w}(\log N)^{-E}+N^{\varepsilon}\Big)\\ &=O\Big(\frac{N}{w}(\log N)^{-E}\Big)\,,\end{aligned}$$

since we have (101).

(ii) If $w\equiv N\ (\bmod 4)$, then we have

$$(N-w,\,4w)=4(N,\,w)\,,\qquad (N+w,\,4w)=2(N,\,w)\,.$$

Thus we have

$$T(w)=\sum_{\substack{n\equiv(N-w)/4(N,w)(\bmod w/(N,w))\\ y_1/4(N,w)<n<y_2/4(N,w)}}f(n)-\sum_{\substack{n\equiv(N-w)/2(N,w)(\bmod 2w/(N,w))\\ y_1/2(N,w)<n<y_2/2(N,w)}}f(n)\,,$$

since we have $f(2(N,\,w)n)=f(n)$.

Applying (66) to the right side, we get

$$T(w)=\left(\frac{y_2}{4(N, w)}-\frac{y_1}{4(N, w)}\right)\frac{(N, w)}{w}\prod_{\substack{p\nmid 2(Nw/(N,w))\\ p\leq N}}\left(1-\frac{1}{p}\right)^{-3}$$
$$-\left(\frac{y_2}{2(N, w)}-\frac{y_1}{2(N, w)}\right)\frac{(N, w)}{2w}\prod_{\substack{p\nmid 2(Nw/(N,w))\\ p\leq N}}\left(1-\frac{1}{p}\right)^{-3}$$
$$+O\left(\frac{y_2}{w}(\log N)^{-E}+N^{\varepsilon}\right).$$

And we have

$$T(w)=O\left(\frac{N}{w}(\log N)^{-E}\right).$$

(iii) In the same way we get

$$T(w)=O\left(\frac{N}{w}(\log N)^{-E}\right),$$

when $w\equiv -N \pmod 4$.

Collecting these results we see that in any case

$$T(w)=O\left(\frac{N}{w}(\log N)^{-E}\right).$$

Inserting this into the right side of (102) we get

$$|\Sigma_{26}|\ll N(\log N)^{-E}\sum_{r_1t^2dm\leq N}\frac{1}{r_1t^2dm} \tag{103}$$
$$\ll N(\log N)^{3-E}.$$

From (99), (100) and (103) we have

$$\Sigma_{23}\ll N(\log N)^{3-E},$$

which, with (94) and (98), gives

$$\Sigma_{\mathrm{III}_2}\ll N(\log N)^{3}(\log\log N)^{-2}. \tag{104}$$

8.4. Now from (79), (93) and (104) we get

$$\Sigma_{\mathrm{III}}\ll N(\log N)^{3-\gamma/2}(\log\log N)^{(1/2)K_3}. \tag{105}$$

§ 9. Asymptotic formula.

Finally collecting (2), (31), (44) and (105) we now complete the proof of our main result:

THEOREM. *Let N be a sufficiently large odd integer, then we have the asymptotic formula*

$$\sum_{n<N}r(N-n)d_4(n)=\frac{\pi}{6}\left(1+\frac{11}{16}\rho(N)\right)\mathfrak{S}(N)N\log^3 N$$
$$+O(N(\log N)^{3-\theta}(\log\log N)^{L}),$$

where L is an absolute constant and $\theta=0.0179\cdots$.

$\mathfrak{S}(N)$ has the expression

$$\mathfrak{S}(N)=\prod_{p}\left(1-\frac{\rho(p)}{p}+\frac{\rho(p)}{p}\left(1-\frac{1}{p}\right)^{3}\right)\prod_{p\mid N}\frac{1-\rho(p)/p}{1-\rho(p)/p+(\rho(p)/p)(1-(1/p))^{3}}$$

$$\times\prod_{p^{\alpha}\parallel N}\left\{1+\frac{\rho(p^{\alpha+1})}{p^{\alpha+1}}d_{4}(p^{\alpha})\left(1-\frac{1}{p}\right)^{3}\left(1-\frac{\rho(p)}{p}\right)^{-1}\right.$$

$$\left.+\left(1-\frac{1}{p}\right)^{4}\sum_{\beta=1}^{\alpha}\frac{\rho(p^{\beta})}{p^{\beta}}\sum_{\delta=0}^{\infty}\frac{d_{4}(p^{\delta+\beta})}{p^{\delta}}\right\}.$$

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