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# On pseudoconvexity of complex abelian Lie groups

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## § 1. Introduction.

The purpose of this paper is to prove the following theorem.

THEOREM. *Let $G$ be a complex abelian Lie group of complex dimension $n$ and $K$ the maximal compact subgroup of the connected component of $G$ with Lie algebra $\mathfrak{k}$. Let $q$ be the complex dimension of $\mathfrak{k} \cap \sqrt{-1}\mathfrak{k}$. Then there exists a real-valued $C^\infty$ function $\varphi$ on $G$ satisfying the following conditions:*

(1) *The Levi form of $\varphi$:*

$$L(\varphi, x) = \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j$$

*is positive semi-definite and has $n-q$ positive eigenvalues at every point $x$ of $G$, where $(z_1, z_2, \cdots, z_n)$ denotes a system of coordinates in some neighborhood of $x$.*

(2) *The set*

$$G_c = \{g \in G : \varphi(g) < c\}$$

*is a relatively compact subset of $G$ for any $c \in \boldsymbol{R}$.*

By the above theorem any complex abelian Lie group is always pseudoconvex. In the last part we shall find a complex Lie group of arbitrary dimension, on which every holomorphic function is a constant and which is pseudoconvex and 1-complete.

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## § 2. Proof of Theorem.

Since all connected components of $G$ are biholomorphically isomorphic, we may assume that $G$ is connected. Let $\mathfrak{O}$ be the sheaf of all germs of holomorphic functions on $G$. We put

$$G^0 = \{g \in G : f(g) = f(e) \quad \text{for all } f \in H^0(G, \mathfrak{O})\}$$

where $e$ is the unit element of $G$. Then Morimoto [5] proved that $G^0$ is a complex abelian Lie subgroup of $G$ and that every holomorphic function on

$G^0$ is a constant. By Morimoto [6] we may assume that

$$G = G^0 \times \boldsymbol{C}^m \times \boldsymbol{C}^{*p}$$

for non-negative integers $m$ and $p$ where $\boldsymbol{C}^*$ is the multiplicative group of all non-zero complex numbers. Let $K$ be the maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k}$ and $K^0$ be the maximal compact subgroup of $G^0$ with Lie algebra $\mathfrak{k}^0$. Since there holds $\mathfrak{k} \cap \sqrt{-1}\,\mathfrak{k} = \mathfrak{k}^0 \cap \sqrt{-1}\,\mathfrak{k}^0$ and since $\boldsymbol{C}^m \times \boldsymbol{C}^{*p}$ is a Stein group, it suffices to prove the theorem in the case that $G = G^0$. Then every holomorphic function on $G$ is a constant and $G$ is isomorphic to $\boldsymbol{C}^n/\Gamma(d^1, d^2, \cdots, d^s)$, where $\Gamma(d^1, d^2, \cdots, d^s)$ is a discrete subgroup generated by linearly independent vectors $d^1, d^2, \cdots, d^s$ of $\boldsymbol{C}^n$ over $\boldsymbol{R}$. The complex linear subspace of $\boldsymbol{C}^n$ spanned by $\{d^i : 1 \leqq i \leqq s\}$ is expressed by $\langle d^1, d^2, \cdots, d^s \rangle_{\boldsymbol{C}}$. Then we have $\boldsymbol{C}^n = \langle d^1, d^2, \cdots, d^s \rangle_{\boldsymbol{C}}$. Actually, if $\dim_{\boldsymbol{C}} \langle d^1, d^2, \cdots, d^s \rangle_{\boldsymbol{C}} \leqq n-1$, there exists a complex linear subspace $V$ of $\boldsymbol{C}^n$ of positive dimension such that $\boldsymbol{C}^n = \langle d^1, \cdots, d^s \rangle_{\boldsymbol{C}} \oplus V$. Consequently we have $G = \langle d^1, d^2, \cdots, d^s \rangle_{\boldsymbol{C}} / \Gamma(d^1, d^2, \cdots, d^s) \oplus V$. Since every holomorphic function on $G$ is a constant, it is a contradiction. Therefore $s \geqq n$ and we may assume that $d^1, d^2, \cdots, d^{n-1}$ and $d^n$ are linearly independent over $\boldsymbol{C}$. There exists an $(n, n)$-matrix $M \in GL(n, \boldsymbol{C})$ such that $e^i = M(d^i)$ where $e^i$ is the $i$-th unit vectors of $\boldsymbol{C}^n$. We put $f^j = M(d^{j+n})$, $1 \leqq j \leqq s-n$. Then $G$ is isomorphic to the complex Lie group $\boldsymbol{C}^n/\Gamma(e^1, \cdots, e^n, f^1, \cdots, f^{s-n})$. From now, we put $\Gamma = \Gamma(e^1, \cdots, e^n, f^1, \cdots, f^{s-n})$ and regard $G$ as $\boldsymbol{C}^n/\Gamma$. Since $d^1, d^2, \cdots, d^{s-1}$ and $d^s$ are linearly independent over $\boldsymbol{R}$, $e^1, e^2, \cdots, e^n, f^1, f^2, \cdots, f^{s-n-1}$ and $f^{s-n}$ are linearly independent over $\boldsymbol{R}$. We put

$$K = \langle e^1, \cdots, e^n, f^1, \cdots, f^{s-n} \rangle_{\boldsymbol{R}} / \Gamma$$

where $\langle e^1, \cdots, e^n, f^1, \cdots, f^{s-n} \rangle_{\boldsymbol{R}}$ denotes the real linear subspace of $C^n$ spanned by $\{e^i, f^j : 1 \leqq i \leqq n,\ 1 \leqq j \leqq s-n\}$. Then $K$ is the maximal compact subgroup of $G$ with Lie algebra $\mathfrak{k} = \langle e^1, \cdots, e^n, f^1, \cdots, f^{s-n} \rangle_{\boldsymbol{R}}$. We put $f^j = \mathrm{Re}\, f^j + \sqrt{-1}\, \mathrm{Im}\, f^j$, where $\mathrm{Re}\, f^j$ and $\mathrm{Im}\, f^j$ are vectors of $\boldsymbol{R}^n$ for $1 \leqq j \leqq s-n$. Then we have

$$\mathfrak{k} = \langle e^1, \cdots, e^n, \sqrt{-1}\,\mathrm{Im}\, f^1, \cdots, \sqrt{-1}\,\mathrm{Im}\, f^{s-n} \rangle_{\boldsymbol{R}}$$

and

$$\mathfrak{k} \cap \sqrt{-1}\,\mathfrak{k} = \langle \mathrm{Im}\, f^1, \cdots, \mathrm{Im}\, f^{s-n} \rangle_{\boldsymbol{C}}\,.$$

Since $\mathrm{Im}\, f^1, \mathrm{Im}\, f^2, \cdots, \mathrm{Im}\, f^{s-n-1}$ and $\mathrm{Im}\, f^{s-n}$ are linearly independent over $\boldsymbol{C}$, $\dim_{\boldsymbol{C}} \mathfrak{k} \cap \sqrt{-1}\,\mathfrak{k}$ coincides with $s-n$ and we have $q = s-n$. Take a system $\{h^k \in \boldsymbol{R}^n : q+1 \leqq k \leqq n\}$ of $n-q$ vectors such that $\mathrm{Im}\, f^1, \mathrm{Im}\, f^2, \cdots, \mathrm{Im}\, f^q, h^{q+1}, h^{q+2}, \cdots, h^{n-1}$ and $h^n$ are linearly independent over $\boldsymbol{R}$. Then $e^1, e^2, \cdots, e^n$, $\sqrt{-1}\,\mathrm{Im}\, f^1, \cdots, \sqrt{-1}\,\mathrm{Im}\, f^q, \sqrt{-1}\, h^{q+1}, \cdots, \sqrt{-1}\, h^{n-1}$ and $\sqrt{-1}\, h^n$ are linearly independent over $\boldsymbol{R}$ and we have

$$C^n = \langle e^1, \cdots, e^n, \sqrt{-1}\,\mathrm{Im}\, f^1, \cdots, \sqrt{-1}\,\mathrm{Im}\, f^q, \sqrt{-1}\, h^{q+1}, \cdots, \sqrt{-1}\, h^n \rangle_{\boldsymbol{R}}.$$

There exists an $(n, n)$-matrix $A=(a^i_j) \in GL(n, \boldsymbol{R})$ such that

$$\begin{bmatrix} e^1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ e^n \end{bmatrix} = A \begin{bmatrix} \mathrm{Im}\, f^1 \\ \vdots \\ \mathrm{Im}\, f^q \\ h^{q+1} \\ \vdots \\ h^n \end{bmatrix}.$$

For every $(z_1, z_2, \cdots, z_n) \in C^n$ where $z_i = x_i + \sqrt{-1}\, y_i$, $1 \leqq i \leqq n$, we have

$$\begin{aligned}(z_1, z_2, \cdots, z_n) &= \sum_{i=1}^n x_i e^i + \sqrt{-1} \sum_{i=1}^n y_i e^i \\ &= \sum_{i=1}^n x_i e^i + \sum_{j=1}^q \Big(\sum_{i=1}^n y_i a^i_j\Big) \sqrt{-1}\,\mathrm{Im}\, f^j + \sum_{k=q+1}^n \Big(\sum_{i=1}^n y_i a^i_k\Big) \sqrt{-1}\, h^k.\end{aligned}$$

We consider the function $\varphi$ on $C^n$ defined by $\varphi(z_1, z_2, \cdots, z_n) = \sum_{k=q+1}^n \big(\sum_{i=1}^n y_i a^i_k\big)^2$ via the above equation. We define the real-valued $C^\infty$ function $\bar{\varphi}$ on $G = \boldsymbol{C}^n/\Gamma$ by putting $\bar{\varphi}(z+\Gamma) = \varphi(z)$. By the definition of $\bar{\varphi}$, the set $\{g \in G : \bar{\varphi}(g) < c\}$ is relatively compact for any $c \in \boldsymbol{R}$. We consider the $(n, n)$-matrix

$$B = \begin{bmatrix} 0 \cdots\cdots 0 & a^1_{q+1} \cdots\cdots a^1_n \\ \vdots \qquad\quad \vdots & \vdots \qquad\qquad \vdots \\ 0 \cdots\cdots 0 & a^n_{q+1} \cdots\cdots a^n_n \end{bmatrix}$$

of rank $n-q$. Then we have

$$\left[\frac{\partial^2 \bar{\varphi}(z+\Gamma)}{\partial z_i \partial \bar{z}_j}\right] = \left[\frac{1}{4} \frac{\partial^2 \varphi(z)}{\partial y_i \partial y_j}\right] = \frac{1}{2} BB^t.$$

The matrix $BB^t$ is positive semi-definite and of rank $n-q$. This proves the theorem.

## § 3. Application.

A. Morimoto [5] has constructed a complex Lie group, on which every holomorphic function is a constant and which contains no complex torus of positive dimension. Such a group is called an (H, C)-group. Since all (H, C)-groups are abelian, every (H, C)-group is always pseudoconvex. It is known that there exist some examples of non-compact pseudoconvex manifolds without non-constant holomorphic functions (cf. H. Grauert [3]). By the theorem and Morimoto's result [5], it is shown that there exist a number of such manifolds even in the case of group manifolds.

A complex manifold $X$ of dimension $n$ is called a *strongly $q$-pseudoconvex* manifold if there exist a real-valued $C^{\infty}$ function $\varphi$ on $X$ and a compact subset $K$ of $X$ such that the Levi form $L(\varphi, x)$ of $\varphi$ has at least $n-q$ positive eigenvalues at every point $x$ of $X-K$ and $X_c=\{x\in X: \varphi(x)<c\}$ is a relatively compact subset of $X$ for any $c\in \boldsymbol{R}$. Moreover if we can take the empty set as $K$, $X$ is called a *$q$-complete* manifold.

We recall the well-known result of A. Andreotti and H. Grauert [**1**].

Let $\mathfrak{F}$ be a coherent analytic sheaf on a complex manifold $X$. If $X$ is strongly $q$-pseudoconvex, then we have $\dim H^i(X, \mathfrak{F})<+\infty$ for $i\geqq q+1$. In particular if $X$ is $q$-complete, then we have $H^i(X, \mathfrak{F})=0$ for $i\geqq q+1$.

Since the complex abelian Lie group $G$ in the theorem is $q$-complete, *we have*

$$H^i(G, \mathfrak{F})=0\,, \qquad i\geqq q+1$$

*for every coherent analytic sheaf $\mathfrak{F}$ on $G$.*

H. Grauert gave the following conjecture at page 347 of [**2**]: Let $X$ be a complex space with countable topology. If $\dim_C X\leqq n$, then $X$ is strongly $(n-1)$-pseudoconvex.

In case that $X$ is a complex Lie group the above conjecture is valid.

COROLLARY. *Let $G$ be a connected complex Lie group of dimension $n$. If $G$ is non-compact, then $G$ is $(n-1)$-complete and we have*

$$H^n(G, \mathfrak{F})=0$$

*for every coherent analytic sheaf $\mathfrak{F}$ on $G$.*

PROOF. We put $G^0=\{g\in G: f(g)=f(e) \text{ for all } f\in H^0(G, \mathfrak{O})\}$. Morimoto [**5**] proved that $G^0$ is a complex abelian Lie subgroup of $G$. If $G=G^0$, then $G$ is $(n-1)$-complete by the theorem. When $G\neq G^0$, we put $q=\dim_C G^0<n$. Then by the consequence of a previous paper [**4**], $G$ is $q$-complete. In any way we have the above corollary.

REMARK. Take $(z_1, z_2, \cdots, z_n)\in \boldsymbol{C}^n-\boldsymbol{R}^n$ such that $1, z_1, z_2, \cdots, z_n$ are linearly independent over the ring of all rational numbers. And we put $v=(z_1, z_2, \cdots, z_n)$. Then Morimoto [**6**] proved that $G=\boldsymbol{C}^n/\Gamma(e^1, \cdots, e^n, v)$ is an (H, C)-group. Therefore every holomorphic function on $G$ is a constant. But the theorem asserts that $G$ is pseudoconvex and 1-complete.

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## References

[1] A. Andreotti and H. Grauert, Théorèmes de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. France, **90** (1962), 193–259.

[2] H. Grauert, Une notion de dimension cohomologique dans la théorie des espaces complexes, Bull. Soc. Math. France, **87** (1959), 341–350.

[3] H. Grauert, Bemerkenswerte pseudokonvexe Mannigfaltigkeiten, Math. Zeit., **81** (1963), 377–391.

[4] H. Kazama, $q$-Complete complex Lie groups and holomorphic fibre bundles over a Stein manifold, Mem. Fac. Sci. Kyushu Univ., **26** (1971), 102–118.

[5] A. Morimoto, Non-compact complex Lie groups without non-constant holomorphic functions, Proc. Conf. on Complex Analysis at Univ. of Minneapolis, 1965, 256-272.

[6] A. Morimoto, On the classification of non-compact complex abelian Lie groups, Trans. Amer. Math. Soc., **123** (1966), 200–228.