J. Math. Soc. Japan
Vol. 25, No. 2, 1973

Families of holomorphic maps into the projective space omitting some hyperplanes

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(Received Jan. 10, 1972)

§ 1. Introduction.

In [1], as a contribution to the Picard-Borel-Nevanlinna theory of value distributions of holomorphic functions, H. Cartan gave some properties of systems of holomorphic functions which vanish nowhere and whose sum vanish identically. Afterwards, one of his results was improved and applied to the study of algebroid functions by J. Dufresnoy ([2]). Using this, the author showed in [4] that the N-dimensional complex projective space $P_N(\boldsymbol{C})$ omitting $2N+1$ hyperplanes in general position is taut in the sense of H. Wu ([11]) and, as a consequence of it, hyperbolic in the sense of S. Kobayashi ([9]), which gives an affirmative answer to the conjecture in [12], p. 216. The main purpose of this paper is, in this connection, to study families of holomorphic maps into $P_N(\boldsymbol{C})$ omitting h hyperplanes in general position in the case $N+2 \leqq h \leqq 2N$ and to give some function-theoretic properties of such spaces.

Let $\{H_i\,;\, 0 \leqq i \leqq N+t\}$ $(t \geqq 1)$ be $N+t+1$ hyperplanes in general position in $P_N(\boldsymbol{C})$. For the space $X_t := P_N(\boldsymbol{C}) - \bigcup_i H_i$, we shall show that there exists a special analytic set C_t of dimension $\leqq N-t$ in X_t called the critical set (cf., Definition 2.1) with the following properties:

Any sequence $\{f^{(\nu)}\}$ of holomorphic maps of a complex manifold[1] M into X_t has a compactly convergent subsequence if there are some compact sets K in M and L in $X_t - C_t$ such that $f^{(\nu)}(K) \cap L \neq \phi$ $(\nu = 1, 2, \cdots)$ (cf., Theorem 4.2).

In the case $t \geqq N$, it will be proved that $C_t = \phi$, which implies that X_N is taut, namely, the result in the previous paper [4] stated above.

By virtue of the above main result, we can give some properties of families of holomorphic maps into X_t. For any complex manifolds M and N, we denote by $\mathrm{Hol}(M, N)$ the space of all holomorphic maps of M into N with compact-open topology. It will be shown that the set of all maps in

1) In this paper, a complex manifold is always assumed to be connected and σ-compact.

$\mathrm{Hol}(M, X_t)$ which are of rank $\geqq r$ somewhere is a locally compact subset of $\mathrm{Hol}(M, X_t)$ if $r \geqq N-t+1$. Moreover, we shall give generalizations of the classical Schottky's and Landau's theorems to the case of holomorphic maps into X_t.

We shall study also the Kobayashi pseudo-distance d_{X_t} on X_t. We shall prove that $d_{X_t}(p, q) > 0$ for any $p \in X_t - C_t$ and $q \in X_t$ $(p \neq q)$. By the use of this, it will be shown that the holomorphic automorphism group of an arbitrary subdomain of the space X_1, namely, $P_N(\boldsymbol{C})$ omitting $N+2$ hyperplanes in general position is a real Lie group. In this connection, we shall investigate the holomorphic automorphism group of X_1 itself and show that it is isomorphic with the symmetric group S_{N+2} on $N+2$ elements in the last section.

§ 2. Preliminaries.

Let us consider $N+t+1$ hyperplanes $\{H_i : 0 \leqq i \leqq N+t\}$ $(t \geqq 1)$ in general position in $P_N(\boldsymbol{C})$. Choosing homogeneous coordinates $w_0 : w_1 : \cdots : w_N$ suitably, we can write

$$(*) \qquad \begin{aligned} &H_i : w_i = 0 && (0 \leqq i \leqq N) \\ &H_{N+s} : \alpha_s^0 w_0 + \alpha_s^1 w_s + \cdots + \alpha_s^N w_N = 0 && (1 \leqq s \leqq t), \end{aligned}$$

where we may assume $\alpha_1^0 = \alpha_1^1 = \cdots = \alpha_1^N = 1$. Put $X_t := P_N(\boldsymbol{C}) - \bigcup_{i=0}^{N+t} H_i$. In the following sections, we use always these notations unless stated to the contrary.

Let $J = (J_1, J_2, \cdots, J_k)$ be a partition of indices $I := \{0, 1, \cdots, N\}$ into k classes $(2 \leqq k \leqq N+1)$, which means that $I = J_1 \cup J_2 \cup \cdots \cup J_k$, $J_l \neq \phi$ $(1 \leqq l \leqq k)$ and $J_l \cap J_m = \phi$ $(l \neq m)$. Taking a map $\chi : \{1, 2, \cdots, t\} \to \{1, 2, \cdots, k\}$, we define the set

$$C_{J,\chi} := \{w_0 : w_1 : \cdots : w_N \in P_N(\boldsymbol{C}) ; \sum_{i \in J_{\chi(s)}} \alpha_s^i w_i = 0,\ 1 \leqq s \leqq t\} .$$

DEFINITION 2.1. We shall call the union C_t of all sets $C_{J,\chi} \cap X_t$ constructed as above to be the *critical set* for X_t.

LEMMA 2.2. *The critical set C_t is an analytic set of dimension $\leqq N-t$ in X_t. In the particular case $N=t$, it holds that $C_N = \phi$.*

PROOF. Take arbitrary J and χ as above and put $m_l := \sharp\{s : \chi(s) = l\}$ for each l $(1 \leqq l \leqq k)$, where we denote the number of elements in a set A by $\sharp A$. Obviously, $m_1 + m_2 + \cdots + m_k = t$. Since any minor of the matrix (α_s^i) $(0 \leqq i \leqq N,$ $1 \leqq s \leqq t)$ does not vanish by the assumption, the space $\{(w_i)_{i \in J_l} : \sum_{i \in J_{\chi(s)}} \alpha_s^i w_i = 0,$ $\chi(s) = l\}$ in $\boldsymbol{C}^{N_l}$ is of dimension $\max(N_l - m_l, 0)$ for each l $(1 \leqq l \leqq k)$, where $N_l = \sharp J_l$. If $N_l \leqq m_l$ for some l, we have $C_{J,\chi} \cap X_t = \phi$. In the case that

$N_l > m_l$ for any l, the inverse image of $C_{J,\chi}$ by the canonical map of $\boldsymbol{C}^{N+1}-\{0\}$ onto $P_N(\boldsymbol{C})$ is of dimension

$$d:=\sum_{l=1}^{k}(N_l-m_l)=N+1-t .$$

So, we have $\dim C_{J,\chi} \leqq d-1=N-t$. In the case $N=t$, we have $N_l \leqq m_l$ for some l and so $C_{J,\chi} \cap X_t=\phi$. Indeed, if not, $d=\sum_{l=1}^{k}(N_l-m_l)\geqq 2$ because $k\geqq 2$. This completes the proof. q. e. d.

Now, we give

THEOREM 2.3 (J. Dufresnoy). *Let D be a domain in the complex plane $\boldsymbol{C}$. Any sequence $\{f^{(\nu)}\}$ in* $\mathrm{Hol}(D, X_t)$ *has a subsequence $\{f^{(\nu_\lambda)}\}$ satisfying one of the following conditions:*

(a) $\{f^{(\nu_\lambda)}\}$ *converges in* $\mathrm{Hol}(D, P_N(\boldsymbol{C}))$,

(b) *it can be chosen suitable J and χ as stated in the above such that, for any compact subset K of D and any neighborhood U of $C_{J,\chi}$ in $P_N(\boldsymbol{C})$, there exists some λ_0 with $f^{(\nu_\lambda)}(K)\subset U$ $(\lambda \geqq \lambda_0)$.*

The proof is given by the same argument as in the proof of Critère Fondamental and Théorème VI in [2], pp. 18~21. Since the statements are slightly modified from the original, we describe the proof here. We use the following

THEOREM 2.4 (H. Cartan). *Let $\Phi^{(\nu)}:=(\phi_1^{(\nu)}, \phi_2^{(\nu)}, \cdots, \phi_k^{(\nu)})$ $(\nu=1, 2, \cdots)$ be a sequence of systems of k holomorphic functions on a domain D in $\boldsymbol{C}$ such that $\phi_l^{(\nu)}(z)\neq 0$ $(1\leqq l\leqq k)$ and $\phi_1^{(\nu)}+\cdots+\phi_k^{(\nu)}\neq 0$ everywhere on D. Then we can find a subsequence $\{\Phi^{(\nu_\mu)}\}$ of $\{\Phi^{(\nu)}\}$ such that for suitable indices l and m $(1\leqq l< m\leqq k)$ $\{\phi_l^{(\nu_\mu)}\phi_m^{(\nu_\mu)-1}\}$ converges compactly on D.*

This is an immediate consequence of Théorème VII in [1], p. 312, because for each function $\phi_{k+1}^{(\nu)}:=-(\phi_1^{(\nu)}+\cdots+\phi_k^{(\nu)})$ the system $(\phi_1^{(\nu)}, \cdots, \phi_{k+1}^{(\nu)})$ satisfies the conditions $\phi_l^{(\nu)}(z)\neq 0$ $(1\leqq l\leqq k+1)$ and $\phi_1^{(\nu)}(z)+\cdots+\phi_{k+1}^{(\nu)}(z)\equiv 0$ on D.

PROOF OF THEOREM 2.3. Using the homogeneous coordinates with the property (*), we may write

$$f^{(\nu)}=f_0^{(\nu)}:f_1^{(\nu)}:\cdots:f_N^{(\nu)} \qquad (\nu=1, 2, \cdots),$$

where each $f_i^{(\nu)}$ is a nowhere zero holomorphic function on D and $\sum_{i=0}^{N}\alpha_s^i f_i\neq 0$ everywhere for any s $(1\leqq s\leqq t)$. Let us consider a partition $J=(J_1, \cdots, J_k)$ of $\{0, 1, \cdots, N\}$ such that, for a subsequence $\{f^{(\nu_\mu)}\}$ of $\{f^{(\nu)}\}$ and a suitable fixed $p(l)\in J_l$, each $\{f_i^{(\nu_\mu)}/f_{p(l)}^{(\nu_\mu)}\}$ converges compactly to a holomorphic function g_{il} on D $(i\in J_l,\ 1\leqq l\leqq k)$. For example, if we put $J_l:=\{l\}$ $(1\leqq l\leqq N)$, $J=(J_1, J_2, \cdots, J_N)$ is such a partition. Among partitions with the above properties, we choose here $J=(J_1, J_2, \cdots, J_k)$ so that the number k of classes of J is as small as possible. Then, for a suitable $\{\nu_\mu\}$ and $p(l)$ as above, each

$F_{s,l}^{(\nu_\mu)}:=\sum_{i\in J_l}\alpha_s^i(f_i^{(\nu_\mu)}/f_{p(l)}^{(\nu_\mu)})$ converges on D to a holomorphic function $F_{s,l}:=\sum_{i\in J_l}\alpha_s^i g_{il}$.

We discuss first the case that, for each s $(1\leqq s\leqq t)$, there is an index $\chi(s)$ such that $F_{s,\chi(s)}\equiv 0$. Consider the above partition $J=(J_1, J_2, \cdots, J_k)$ and the map $\chi: s\mapsto\chi(s)$. Then, we claim that $\{f^{(\nu_\mu)}\}$ satisfies the condition (b) of Theorem 2.3 for the set $C_{J,\chi}$. Assume the contrary. We can take a sequence $\{z_\lambda\}$ in D such that $\lim_{\lambda\to\infty} z_\lambda=z_0\in D$ and $\lim_{\lambda\to\infty} f^{(\nu_\lambda)}(z_\lambda)=w\in P_N(\boldsymbol{C})-C_{J,\chi}$ for a suitable sequence $\{f^{(\nu_\lambda)}\}$ of $\{f^{(\nu_\mu)}\}$. Let $f^{(\nu_\lambda)}(z_\lambda)=w_0^{(\lambda)}:w_1^{(\lambda)}:\cdots:w_N^{(\lambda)}$ and $w=w_0:w_1:\cdots:w_N$. By the definition of $C_{J,\chi}$, $\sum_{i\in J_{\chi(s)}}\alpha_s^i w_i\neq 0$ for some s and so $w_{i_0}\neq 0$ for some $i_0\in J_{\chi(s)}$. Then, we have

$$\begin{aligned}\sum_{i\in J_{\chi(s)}}\alpha_s^i\frac{w_i}{w_{i_0}}&=\sum_{i\in J_{\chi(s)}}\alpha_s^i\cdot\left(\lim_{\lambda\to\infty}\frac{w_i^{(\lambda)}}{w_{i_0}^{(\lambda)}}\right)\\&=\lim_{\lambda\to\infty}\sum_{i\in J_{\chi(s)}}\alpha_s^i\frac{f_i^{(\nu_\lambda)}(z_\lambda)}{f_{i_0}^{(\nu_\lambda)}(z_\lambda)}\\&=\lim_{\lambda\to\infty}\frac{f_{p(\chi(s))}^{(\nu_\lambda)}(z_\lambda)}{f_{i_0}^{(\nu_\lambda)}(z_\lambda)}F_{s,\chi(s)}^{(\nu_\lambda)}(z_\lambda)\\&=\frac{w_{p(\chi(s))}}{w_{i_0}}F_{s,\chi(s)}(z_0)=0\end{aligned}$$

and so $\sum_{i\in J_{\chi(s)}}\alpha_s^i w_i=0$. This is a contradiction.

It remains to discuss the case that $F_{s_0,l}\not\equiv 0$ $(1\leqq l\leqq k)$ for a suitable s_0. In this case, we shall prove $k=1$, which means that $\{f^{(\nu_\mu)}\}$ satisfies the condition (a) of Theorem 2.3. Assume that $k\geqq 2$. We take an arbitrary domain $\tilde{D}$ with $\tilde{D}\Subset D':=D-\bigcup_{l=1}^{k}\{z\in D; F_{s_0,l}(z)=0\}$. Then, we may assume that, for the functions $\phi_l^{(\nu_\mu)}:=f_{p(l)}^{(\nu_\mu)}F_{s_0,l}^{(\nu_\mu)}$, $\phi_l^{(\nu_\mu)}\neq 0$ and $\sum_{l=1}^{k}\phi_l^{(\nu_\mu)}\neq 0$ on $\tilde{D}$. By Theorem 2.4 $\{\phi_l^{(\nu_\lambda)}/\phi_m^{(\nu_\lambda)}\}$ converges on $\tilde{D}$ for a subsequence $\{\nu_\lambda\}$ of $\{\nu_\mu\}$ and suitable l, m. Then, $\{f_{p(l)}^{(\nu_\lambda)}/f_{p(m)}^{(\nu_\lambda)}\}$ is also convergent on $\tilde{D}$ because $f_{p(l)}^{(\nu_\lambda)}/f_{p(m)}^{(\nu_\lambda)}=(\phi_l^{(\nu_\lambda)}/\phi_m^{(\nu_\lambda)})(F_{s_0,m}^{(\nu_\lambda)}/F_{s_0,l}^{(\nu_\lambda)})$. Using the diagonal argument and changing indices if necessary, we may assume that $\{f_{p(l)}^{(\nu_\lambda)}/f_{p(m)}^{(\nu_\lambda)}\}$ converges on D'. Moreover, it is easily shown by the maximum principle for holomorphic functions that $\{f_{p(l)}^{(\nu_\lambda)}/f_{p(m)}^{(\nu_\lambda)}\}$ converges on D. This contradicts the property of the number k. Thus we conclude $k=1$. q. e. d.

REMARK 2.5. As is easily seen, in the case (a) of Theorem 2.3, the limit of $\{f^{(\nu_\mu)}\}$ has the image included either in X_t or in some H_i. In case of $t=N$, Theorem 2.3 means that X_N is taut in the sense of H. Wu because of Lemma 2.2 (cf., [**4**], Theorem 5.1).

§ 3. The Kobayashi pseudo-distance on X_t.

In [9], S. Kobayashi defined an intrinsic pseudo-distance d_M on M for every complex manifold M, which is uniquely determined by the conditions that (a) we have

$$\tanh \frac{1}{2} d_{D_1}(z, z') = \frac{|z-z'|}{|1-z\bar{z}'|}$$

for the unit disc $D_1 := \{|z|<1\}$ in $\boldsymbol{C}$, (b) $d_N(f(p), f(q)) \leqq d_M(p, q)$ $(p, q \in M)$ for any holomorphic map $f: M \to N$ and (c) $d'(p, q) \leqq d_M(p, q)$ for any $p, q \in M$ if d' is a pseudo-distance on M with the property $d'(f(z), f(z')) \leqq d_{D_1}(z, z')$ $(z, z' \in D_1)$ for any holomorphic map $f: D_1 \to M$.

By the condition (b) we see easily

(3.1). (i) *For any submanifold M in N, we have $d_M(p, q) \geqq d_N(p, q)$ $(p, q \in M)$.*

(ii) *If f is a holomorphic automorphism of M, then $d_M(f(p), f(q)) = d_M(p, q)$ for any $p, q \in M$.*

The Kobayashi pseudo-distance d_{X_t} on the space X_t stated in the previous section has the following property.

THEOREM 3.2. *It holds that $d_{X_t}(p, q) > 0$ if $p \in X_t - C_t$ and $q \in X_t$ $(p \neq q)$, where C_t is the critical set for X_t.*

PROOF. On X_t, we can consider a system of global coordinates $z_1, \cdots, z_N$. It may be assumed that, for the unit ball $B := \{(z_1, \cdots, z_N); \sum_{i=1}^{N} |z_i - a_i|^2 < 1\}$ with center at $p = (a_1, \cdots, a_N)$, we have $B \cap C_t = \phi$ and $q \notin B$. In view of Lemma in [8], p. 50, it suffices to show the existence of a pair (r, δ) with $0 < r, \delta < 1$ such that any holomorphic map $f: D_1 := \{|z|<1\} \to X_t$ with $f(0) \in B_r := \{(z_1, \cdots, z_N); \sum_{i=1}^{N} |z_i - a_i|^2 < r^2\}$ satisfies the condition $f(\{|z|<\delta\}) \subset B$. Assume the contrary. As in the proof of Proposition 2 in [8], p. 51, we can take a holomorphic map f_ν of D_1 into X_t with $f_\nu(0) \in B_{1/2}$ and $f_\nu(\{|z| < 1/\nu\}) \not\subset B$ for any $\nu = 1, 2, \cdots$. Obviously, $\{f_\nu\}$ has no subsequence which satisfies the condition either (a) or (b) of Theorem 2.3. This is a contradiction. Thus we have Theorem 3.2. q. e. d.

By definition, a hyperbolic manifold is a complex manifold on which the Kobayashi pseudo-distance is a true distance. As an immediate consequence, we see

COROLLARY 3.3. *The space $X_t - C_t$ is hyperbolic.*

REMARK 3.4. For any $k \geqq N-t+1$, the space X_t is of type MH_k in the sense of A. Eisenman [3], p. 54. Indeed, for any k-dimensional real analytic submanifold N of X_t and non-empty subdomain N' of N, $N' - N' \cap C_t$ is also of dimension k because of Lemma 2.2. By Theorem 3.2, the k-dimensional Hausdorff measure $d_{X_t}^k$ with respect to d_{X_t} is a Borel measure on N (cf.,

[3], Propositions 1.24 and 1.28 etc.).

Now, we apply Theorem 3.2 to the study of holomorphic automorphism groups. Let D be an arbitrary subdomain of the space X_1, namely, $P_N(\boldsymbol{C})$ omitting $N+2$ hyperplanes in general position. By $\mathrm{Aut}(D)$, we denote the space of all holomorphic automorphisms of D with compact-open topology. Let us consider the domain $D' := \bigcup_{f \in \mathrm{Aut}(D)} f(D-C_t)$. Obviously, $D-C_t \subset D' \subset D$.

LEMMA 3.5. *The domain D' is hyperbolic and hence* $\mathrm{Aut}(D')$ *is a real Lie group.*

PROOF. Take two distinct points p and q in D'. We may describe $p=f_0(p_0)$ with a suitable $p_0 \in D-C_t$ and $f_0 \in \mathrm{Aut}(D)$. By (3.1), we have

$$d_{D'}(p, q) \geqq d_D(f_0(p_0), f_0(f_0^{-1}(q))) = d_D(p_0, f_0^{-1}(q))$$
$$\geqq d_{X_1}(p_0, f_0^{-1}(q)).$$

On the other hand, since $p_0 \neq f_0^{-1}(q)$ and $p_0 \in D-C_t$, we see $d_{X_1}(p_0, f_0^{-1}(q)) > 0$ by Theorem 3.2. So, D' is hyperbolic. The last assertion is a direct result of S. Kobayashi [9], Theorem 6.2. q. e. d.

THEOREM 3.6. *For the above domains D and D',* $\mathrm{Aut}(D)$ *is topologically isomorphic with a closed subgroup of* $\mathrm{Aut}(D')$ *by the canonical restriction map* $\rho: f \in \mathrm{Aut}(D) \mapsto \rho(f) := f|D' \in \mathrm{Aut}(D')$.

PROOF. According to Lemma in [7], since D is certainly K-complete and $D-D'$ is a thin analytic subset of D, $\rho: \mathrm{Aut}(D) \mapsto \mathrm{Aut}(D')$ is a homeomorphism onto a subgroup of $\mathrm{Aut}(D')$ and, moreover, the image of ρ is closed because $\mathrm{Aut}(D)$ is complete (cf., the proof of Theorem 3 in [7]). q. e. d.

COROLLARY 3.7. *For any domain D in X_1,* $\mathrm{Aut}(D)$ *is a real Lie group.*

REMARK 3.8. The space $X_0 := P_N(\boldsymbol{C}) - \bigcup_{i=0}^{N} H_i$ defined by $N+1$ hyperplanes $\{H_i\}$ in general position is biholomorphically isomorphic with $(\boldsymbol{C}-\{0\})^N$. As is easily seen, $\mathrm{Aut}(X_0)$ is not a real Lie group if $N \geqq 2$.

§ 4. A generalization of J. Dufresnoy's theorem.

LEMMA 4.1. *Consider polydiscs $D := \{|z_i| < r_i, 1 \leqq i \leqq n\}$ and $G := \{|z_i| < r_i', 1 \leqq i \leqq n\}$ in $\boldsymbol{C}^n$, where $0 < r_i' \leqq r_i$. If a sequence $\{f^{(\nu)}\}$ in* $\mathrm{Hol}(D, X_t)$ *converges compactly on G to a map $g \in$* $\mathrm{Hol}(G, X_t)$ *with $g(G) \not\subset C_t$, then it converges in* $\mathrm{Hol}(D, X_t)$.

For the proof, we need the following T. Nishino's result [10], Lemma I.

THEOREM. *Let $\{f_\nu(z)\}$ be a sequence of holomorphic functions on $D := \{|z_i| < r_i, 1 \leqq i \leqq n\}$ which converges compactly on $\{|z_1| < r_1'\} \cap D$ $(0 < r_1' < r_1)$ and, moreover, on $\{|z_1| < r_1\}$ as functions of z_1 for any fixed $z' = (z_2, \cdots, z_n)$ $(|z_i| < r_i, 2 \leqq i \leqq n)$. Then, it converges compactly on D.*

PROOF OF LEMMA 4.1. It suffices to prove Lemma 4.1 in the case that $r'_1 < r_1$, $r'_2 = r_2, \cdots, r'_n = r_n$. Indeed, if it is shown, $\{f^{(\nu)}\}$ converges compactly on $\{|z_1| < r_1, |z_i| < r'_i, 2 \leqq i \leqq n\}$ and then by the same argument we see that it converges compactly on $\{|z_1| < r_1, |z_2| < r_2, |z_i| < r'_i, 3 \leqq i \leqq n\}$, hence, on $\{|z_i| < r_i, |z_j| < r'_j, 1 \leqq i \leqq 3, 4 \leqq j \leqq n\}$ and so on. Let $f^{(\nu)} = f_0^{(\nu)} : f_1^{(\nu)} : \cdots : f_N^{(\nu)}$ and $g = g_0 : g_1 : \cdots : g_N$ with holomorphic functions $f_i^{(\nu)}$ and g_i, where we may choose $f_0^{(\nu)} \equiv 1$ and $g_0 \equiv 1$. For any partition $J = (J_1, \cdots, J_k)$ of $I := \{0, 1, \cdots, N\}$, we can take an index $s(J)$ with the property that

$$g_{s(J),l}(z) := \sum_{i \in J_l} \alpha^i_{s(J)} g_i(z) \not\equiv 0$$

for any l $(1 \leqq l \leqq k)$. Because, if not, there is a map $\chi: \{1, 2, \cdots, t\} \to \{1, 2, \cdots, k\}$ with $g_{s,\chi(s)}(z) \equiv 0$ for any s, i. e., $g(G) \subset C_{J,\chi}$. We denote anew all functions $g_{s(J),1}, g_{s(J),2}, \cdots, g_{s(J),k}$ constructed as above for each partition $J = (J_1, \cdots, J_k)$ of I by $G_1, G_2, \cdots, G_{\iota_0}$. Let $D' := \{z' = (z_2, \cdots, z_n)\,;\ |z_i| < r_i, 2 \leqq i \leqq n\}$. Consider a thin analytic set

$$V := \bigcup_{\iota=1}^{\iota_0} \{z' \in D'\,;\ G_\iota(z_1, z') \equiv 0 \text{ as a function of } z_1\}$$

in D'.

Now, we take an arbitrary point z' in $D' - V$. Apply Theorem 2.3 to the sequence of maps

$$h^{(\nu)}(z_1) := f_0^{(\nu)}(z_1, z') : \cdots : f_N^{(\nu)}(z_1, z')$$

of $\{|z_1| < r_1\}$ into X_t. If $|z_1| < r'_1$, then $\lim_{\nu\to\infty} h^{(\nu)}(z_1) = g(z_1, z')$ by the assumption. And, since $z' \in D' - V$, it holds $G_\iota(z_1, z') \not\equiv 0$ $(1 \leqq \iota \leqq \iota_0)$, which means $g(\{(z_1, z') : |z_1| < r'_1\}) \not\subset C_t$. So, $\{h^{(\nu)}\}$ has no subsequence satisfying the condition (b) of Theorem 2.3. In view of the condition (a) of Theorem 2.3, it has a subsequence converging to some g in $\mathrm{Hol}(\{|z_1| < r_1\}, P_N(\boldsymbol{C}))$. Moreover, by the same reason, any subsequence of $\{h^{(\nu)}\}$ has a convergent subsequence, whose limit g^* is necessarily equal to g by the fact that $g^*(z_1) = g(z_1, z') = g(z_1)$ on $\{|z_1| < r'_1\}$ and by the theorem of identity. Then, as is easily seen, $\{h^{(\nu)}\}$ itself converges to g in $\mathrm{Hol}(\{|z_1| < r_1\}, P_N(\boldsymbol{C}))$. In conclusion, each $\{f_i^{(\nu)}(z_1, z')\}$ $(1 \leqq i \leqq N)$ converges compactly in $\{|z_1| < r_1\}$ as a sequence of holomorphic functions of z_1 for any arbitrarily fixed z' in $D' - V$. Then, by the above Theorem, it converges compactly on $\{|z_1| < r_1\} \times (D' - V)$ as functions of n complex variables. Moreover, we can easily conclude that $\{f^{(\nu)}\}$ converges in $\mathrm{Hol}(D, P_N(\boldsymbol{C}))$ by the maximum principle. On the other hand, we know that the limit of any convergent subsequence of nowhere zero holomorphic functions vanishes identically or vanishes nowhere. This concludes that the image of the limit of $\{f^{(\nu)}\}$ is included in X_t. Thus we have Lemma 4.1.

q. e. d.

THEOREM 4.2. *Let M be a complex manifold and $\{f^{(\nu)}\}$ be a sequence in* $\mathrm{Hol}(M, X_t)$ *such that there are some compact sets K in M and L in $X_t - C_t$ with $f^{(\nu)}(K) \cap L \neq \phi$, where C_t is the critical set for X_t. Then $\{f^{(\nu)}\}$ has a subsequence which converges in* $\mathrm{Hol}(M, X_t)$.

PROOF. By the assumption, we can find a sequence $\{p_\mu\}$ in M and a subsequence $\{f^{(\nu_\mu)}\}$ of $\{f^{(\nu)}\}$ such that $\lim_{\mu\to\infty} p_\mu = p_0 \in M$ and $\lim_{\mu\to\infty} f^{(\nu_\mu)}(p_\mu) = q_0 \in X_t - C_t$ exist. Take the unit ball B with center at p_0 with respect to a system of local coordinates in a neighborhood of p_0 and an open neighborhood U of q_0 with $U \Subset X_t - C_t$. Let $\delta := \min_{q \in \partial U} d_{X_t}(q, q_0)$, $W := \{p \in B\,;\, d_B(p, p_0) < \delta/2\}$ and V be the connected component of the set $\{q \in X_t\,;\, d_{X_t}(q, q_0) < \delta\}$ with $q_0 \in V$, where ∂U denotes the boundary of U. We have then $p_\mu \in W$, $f^{(\nu_\mu)}(p_\mu) \in V$ and $d_{X_t}(f^{(\nu_\mu)}(p_\mu), q_0) < \delta/2$ for sufficiently large μ. So, for any $p \in W$,

$$\begin{aligned} d_{X_t}(f^{(\nu_\mu)}(p), q_0) &\leqq d_{X_t}(f^{(\nu_\mu)}(p), f^{(\nu_\mu)}(p_\mu)) + d_{X_t}(f^{(\nu_\mu)}(p_\mu), q_0) \\ &\leqq d_B(p, p_\mu) + d_{X_t}(f^{(\nu_\mu)}(p_\mu), q_0) \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta\,. \end{aligned}$$

This shows that $f^{(\nu_\mu)}(W) \subset \{q \in X_t\,;\, d_{X_t}(q, q_0) < \delta\}$. Since $f^{(\nu_\mu)}(W) \cap V \neq \phi$ and $f^{(\nu_\mu)}(W)$ is connected, we obtain $f^{(\nu_\mu)}(W) \subset V$. On the other hand, we may consider V as a bounded domain in $\boldsymbol{C}^N = P_N(\boldsymbol{C}) - H_0$. We can choose a subsequence $\{f^{(\nu_\lambda)}\}$ of $\{f^{(\nu_\mu)}\}$ converging compactly on W. Obviously, $g := \lim_{\lambda\to\infty} (f^{(\nu_\lambda)} | W)$ satisfies the conditions $g(W) \subset X_t$ and $g(W) \not\subset C_t$. Now, let us consider the set M' of all points p in M such that $\{f^{(\nu_\lambda)}\}$ converges on some neighborhood W_p of p to a map g_p in $\mathrm{Hol}(W_p, X_t)$ with $g_p(W_p) \subset X_t$ and $g_p(W_p) \not\subset C_t$. As was shown in the above, M' is not empty. Moreover, Lemma 4.1 implies that M' is open and closed in M. It follows that $M' = M$ because M is connected, whence we have Theorem 4.2 by the usual diagonal argument. q. e. d.

§ 5. Families of holomorphic maps into X_t.

There are some applications of Theorem 4.2 to the study of families of holomorphic maps into X_t.

THEOREM 5.1. *Let M be a complex manifold and $r \geqq N - t + 1$. Then,*

$$\mathrm{Hol}^r(M, X_t) := \{f \in \mathrm{Hol}(M, X_t)\,;\ f \text{ is of rank } \geqq r \text{ somewhere}\}$$

is a locally compact subset of $\mathrm{Hol}(M, X_t)$, *where the rank of f means the rank of the Jacobian matrix of f.*

PROOF. Take an arbitrary $f_0 \in \mathrm{Hol}^r(M, X_t)$. By Lemma 2.2, there is a point p in M such that $f_0(p) \notin C_t$ and f_0 is of rank $\geqq r$ at p. We make use of the Jacobian matrix of f at p with respect to arbitrarily fixed local coordinates in a neighborhood of p and fixed global coordinates on X_t. For an open neighborhood V of $f_0(p)$ with $V \Subset X_t - C_t$ and a real number $\delta > 0$, $\mathfrak{U}_\delta$ be the set of all maps $f \in \mathrm{Hol}^r(M, X_t)$ satisfying the conditions that $f(p) \in V$ and the Jacobian matrix of f at p has a minor of order r with the absolute value $> \delta$. Clearly, $\mathfrak{U}_\delta$ is an open subset of $\mathrm{Hol}^r(M, X_t)$. By the assumption, at least one minor of order r of the Jacobian matrix of f_0 at p has the absolute value $> \delta_0$ (> 0). Then, $\mathfrak{U}_{\delta_0}$ is a neighborhood of f_0. Moreover, any sequence $\{f^{(\nu)}\}$ in $\mathfrak{U}_{\delta_0}$ satisfies the assumption of Theorem 4.2 and hence has a subsequence $\{f^{(\nu_\mu)}\}$ converging in $\mathrm{Hol}(M, X_t)$. The limit of $\{f^{(\nu_\mu)}\}$ is obviously contained in $\mathrm{Hol}^r(M, X_t)$. This shows that $\mathfrak{U}_{\delta_0}$ is relatively compact in $\mathrm{Hol}(M, X_t)$. q. e. d.

The following theorem is in a sense considered as a generalization of the classical Schottky's theorem (cf., [2], p. 23).

THEOREM 5.2. *Let M be a complex manifold, K an arbitrarily given compact set, p_0 a point in K and q_0 a point in $X_t - C_t$. Then, there is a compact set L in $X_t - C_t$ such that any holomorphic map f of M into X_t with $f(p_0) = q_0$ satisfies the condition $f(K) \subset L$.*

PROOF. If the conclusion is not valid, we can find compact subsets L_ν $(\nu = 1, 2, \cdots)$ of $X_t - C_t$ with $L_\nu \subset \mathring{L}_{\nu+1}$ and $X_t = \bigcup_\nu L_\nu$ such that there is a holomorphic map $f^{(\nu)}$ of M into X_t with $f^{(\nu)}(p_0) = q_0$ and $f^{(\nu)}(K) \not\subset L_\nu$, where $\mathring{L}_\nu$ denotes the interior of L_ν. The sequence $\{f^{(\nu)}\}$ has no convergent subsequence though it satisfies the assumption of Theorem 4.2, which is absurd. We have therefore Theorem 5.2. q. e. d.

Before we state another application of Theorem 4.2, we give some comments to the results in the previous paper [5].

As in § 2, we consider a partition $J = (J_1, \cdots, J_k)$ $(k \geqq 2)$ of $I := \{0, 1, \cdots, N\}$ and a map $\chi : \{1, 2, \cdots, t\} \to \{1, 2, \cdots, k\}$. Using homogeneous coordinates $w_0 : \cdots : w_N$ on $P_N(\boldsymbol{C})$ with the property $(*)$, we define this time the set

$$E_{J,\chi} := \{w_0 : \cdots : w_N \in P_N(\boldsymbol{C}) ; \sum_{i \in J_l} \alpha_s^i w_i = 0,\ 1 \leqq l \leqq k,\ l \neq \chi(s),\ 1 \leqq s \leqq t\}$$

in $P_N(\boldsymbol{C})$. Obviously, $\bigcup_{J,\chi} E_{J,\chi} \subset \bigcup_{J,\chi} C_{J,\chi}$.

We can prove the following improvement of Theorem B in [5].

THEOREM 5.3. *Every holomorphic map f of $\boldsymbol{C}^n$ into X_t satisfies one of the following conditions* (a) *and* (b):

(a) *f is of constant,*

(b) *$f(\boldsymbol{C}^n) \subset E_{J,\chi}$ for some suitable J and χ.*

Moreover, in the case (b), *$f(\boldsymbol{C}^n)$ is included in a linear subvariety of dimension $k-1$ if J is a partition of I into k subclasses.*

This is easily proved by Lemma 5 in [5] and the same argument as in the proof of Theorem B in it.

REMARK 5.4. If we have the case (b) in Theorem 5.3, the number k is always $\leqq [N/(t+1)]$ (cf., [5], § 7).

We have also the similar improvement of Theorem A in [5].

THEOREM 5.5. *Every holomorphic map of a complex manifold M excluding a regular thin analytic subset S into X_t satisfies one of the conditions:*

(a) *f can be extended to a holomorphic map of M into $P_N(\boldsymbol{C})$.*

(b) *$f(M-S)\subset E_{J,\chi}$ for some suitable J and χ.*

REMARK 5.6. If we have the case (b) in Theorem 5.5, as was shown in the proof of Theorem A in [5], $f(M-S)$ is included in a linear subvariety of dimension $N-(k-1)t\leqq N-t$ because $k\geqq 2$. So, f is of rank $\leqq N-t$ everywhere, which shows Theorem A in [5].

Now, we give a generalization of the classical Landau's theorem.

THEOREM 5.7. *For any given $\delta>0$ and point q in X_t-C_t, there exists a real number R $(0<R<+\infty)$ depending only on δ and q with the following property:*

If $\rho>R$, then there is no holomorphic map $f(z):=1:f_1(z):\cdots:f_N(z)$ of a disc $\{|z|<\rho\}$ in $\boldsymbol{C}$ into X_t such that $f(0)=q$ and

$$|f_1'(0)|^2+\cdots+|f_N'(0)|^2\geqq\delta^2 .$$

PROOF. Assume the contrary. We can find a sequence $\{R_\nu\}$ of real numbers with $R_1<R_2<\cdots$ and $\lim_{\nu\to\infty} R_\nu=\infty$ such that there is a holomorphic map $f^{(\nu)}$ of $D_\nu:=\{|z|<R_\nu\}$ into X_t with $f^{(\nu)}(0)=q$ and

$$|f_1^{(\nu)\prime}(0)|^2+\cdots+|f_N^{(\nu)\prime}(0)|^2\geqq\delta^2$$

for each ν. As is easily seen by Theorem 4.2 and the diagonal argument, we can choose a subsequence $\{f^{(\nu_\mu)}\}$ of $\{f^{(\nu)}\}$ with the property that for any μ the sequence

$$f^{(\nu_\mu)},\ f^{(\nu_{\mu+1})},\ f^{(\nu_{\mu+2})},\ \cdots$$

converges in $\mathrm{Hol}(D_\mu, X_t)$. Then, $\{f^{(\nu_\mu)}\}$ may be considered to converge to a holomorphic map $f:\boldsymbol{C}\to X_t$. Obviously, $f(0)=q\notin E_{J,\chi}$ for any J and χ and f is not of constant. This contradicts Theorem 5.3. We have therefore Theorem 5.7. q. e. d.

§ 6. Holomorphic automorphisms of X_1.

The purpose of this section is to study holomorphic automorphisms of the space X_1, namely, $P_N(C)$ minus $N+2$ hyperplanes in general position. We shall prove

THEOREM 6.1. *Every holomorphic automorphism of X_1 is given by a linear automorphism of $P_N(\boldsymbol{C})$. Moreover,* $\mathrm{Aut}(X_1)$ *is isomorphic with the symmetric group S_{N+2} on $N+2$ elements.*

For the proof, we need the following lemma on polynomials of N variables $z_1, \cdots, z_N$.

LEMMA 6.2. *If $k \leqq N+1$ and*

$$(1) \qquad \sum_{i=1}^{k} a_i z_1^{l_{i1}} \cdots z_N^{l_{iN}} (1+z_1+\cdots+z_N)^{l_{iN+1}} = 0\,,$$

then we have

$$(2) \qquad \sum_{i=1}^{k} a_i z_1^{l_{i1}} \cdots a_N^{l_{iN}} z_{N+1}^{l_{iN+1}} = 0$$

for another new variable z_{N+1}.

PROOF. The proof is given by double induction on N and k. If $k=1$, Lemma 6.2 is evident for any N. In the case of $k=2$ and $N=1$, from the identity

$$a_1 z_1^{l_{11}} (1+z_1)^{l_{12}} + a_2 z_1^{l_{21}} (1+z_1)^{l_{22}} = 0\,,$$

we conclude easily $a_1 = -a_2$, $l_{11} = l_{21}$ and $l_{12} = l_{22}$. So, Lemma 6.2 is valid in the case of $N=1$. For our purpose, it suffices to prove Lemma 6.2 under the assumption that it is true for polynomials of $\leqq N-1$ variables and polynomials of N variables with $\leqq k-1$ terms, where $2 \leqq N$ and $2 \leqq k \leqq N+1$. Here, it may be assumed that $l_{iN+1}=0$ for some i and $l_{jN+1}>0$ for some j. For, if not, consider the polynomial whose terms are constructed by the division of a common factor in (1). Changing indices, we may assume that $l_{iN+1}=0$ $(1 \leqq i \leqq r)$ and $l_{iN+1}>0$ $(r+1 \leqq i \leqq k)$. In (1), putting $z_N = -(1+z_1+\cdots+z_{N-1})$, we obtain

$$\sum_{i=1}^{r} a_i z_1^{l_{i1}} \cdots z_{N-1}^{l_{iN-1}} (-(1+z_1+\cdots+z_{N-1}))^{l_{iN}} = 0\,.$$

It follows from the induction assumption that

$$\sum_{i=1}^{r} a_i z_1^{l_{i1}} \cdots z_{N-1}^{l_{iN-1}} z_N^{l_{iN}} = 0\,.$$

Then, by (1), we have also

$$\sum_{i=r+1}^{k} a_i z_1^{l_{i1}} \cdots z_N^{l_{iN}} (1+z_1+\cdots+z_N)^{l_{iN+1}} = 0\,.$$

Since $k-r<k$, we get by the induction hypothesis on k

$$\sum_{i=r+1}^{k} a_i z_1^{l_{i1}} \cdots z_N^{l_{iN}} z_{N+1}^{l_{iN+1}} = 0$$

and hence the identity (2). q. e. d.

PROOF OF THEOREM 6.1. According to Theorem 6 in [5], every holomorphic automorphism f of X_1 can be extended to a bimeromorphic map of $P_N(\boldsymbol{C})$ onto itself. As is well-known, every meromorphic function on $P_N(\boldsymbol{C})$ is rational. Therefore, using the inhomogeneous coordinates $z_1=w_1/w_0, \cdots, z_N=w_N/w_0$ and $u_1=v_1/v_0, \cdots, u_N=v_N/v_0$ for homogeneous coordinates $w_0:w_1:\cdots:w_N$ and $v_0:v_1:\cdots:v_N$ on $P_N(\boldsymbol{C})$ with the property $(*)$, we can write $f=(f_1, \cdots, f_N)$ with rational functions $u_i=f_i(z)=P_i(z)/Q_i(z)$ $(1\leqq i\leqq N)$, where $P_i(z)$ and $Q_i(z)$ are mutually prime polynomials in $z_1, \cdots, z_N$. Then, any prime factor of each P_i or Q_i is necessarily a constant multiple of the polynomial $z_1, z_2, \cdots, z_N$ or $1+z_1+\cdots+z_N$. Indeed, if not, we can find easily a point $a=(a_1, \cdots, a_N)$ in X_1 with $f(a)\notin X_1$. Accordingly, we can write

$$u_i=f_i(z)=a_i z_1^{l_{i1}}\cdots z_N^{l_{iN}}(1+z_1+\cdots+z_N)^{l_{iN+1}} \quad (1\leqq i\leqq N), \tag{3}$$

where l_{ij} are not necessarily non-negative integers. Moreover, since a rational function $1+f_1(z)+\cdots+f_N(z)$ has non-zero finite values everywhere on X_1, we can write also

$$1+u_1+\cdots+u_N=a_{N+1}z_1^{l_{N+11}}\cdots z_N^{l_{N+1N}}(1+z_1+\cdots+z_N)^{l_{N+1N+1}}. \tag{4}$$

The same argument can be applicable to the inverse map $g:=f^{-1}$ of f. It can be written $g=(g_1, \cdots, g_N)$ with the rational functions

$$z_j=g_j(u)=b_j u_1^{m_{j1}}\cdots u_N^{m_{jN}}(1+u_1+\cdots+u_N)^{m_{jN+1}} \quad (1\leqq j\leqq N), \tag{5}$$

and moreover we have

$$1+z_1+\cdots+z_N=b_{N+1}u_1^{m_{N+11}}\cdots u_N^{m_{N+1N}}(1+u_1+\cdots+u_N)^{m_{N+1N+1}}. \tag{6}$$

Substituting (3) and (4) into (5) and (6), we get

$$\begin{aligned} &z_j=c_j z_1^{n_{j1}}\cdots z_N^{n_{jN}}(1+z_1+\cdots+z_N)^{n_{jN+1}} \quad (1\leqq j\leqq N)\\ &1+z_1+\cdots+z_N=c_{N+1}z_1^{n_{N+11}}\cdots z_N^{n_{N+1N}}(1+z_1+\cdots+z_N)^{n_{N+1N+1}}, \end{aligned} \tag{7}$$

where $c_j=b_j a_1^{m_{j1}}\cdots a_{N+1}^{m_{jN+1}}$ $(1\leqq j\leqq N+1)$ and $n_{ji}=\sum_{k=1}^{N+1}m_{jk}l_{ki}$ $(1\leqq i, j\leqq N+1)$, i.e., the matrix (n_{ji}) is the product of the matrices (m_{jk}) and (l_{kj}). In (7), by transposing factors with negative powers into the left side of the equations and by observing the coefficients of each term in their expansions, we see $c_j=1$ and $n_{ji}=\delta_{ji}$, where $\delta_{ji}=1$ if $i=j$ and $=0$ if $i\neq j$. In particular, $a_i\neq 0$, $b_j\neq 0$ and $\det(l_{ij})=\det(m_{ij})=\pm 1$. On the other hand, putting $l'_{0j}=-\min\{0, l_{1j}, l_{2j}, \cdots, l_{N+1j}\}$, $l'_{ij}:=l_{ij}+l'_{0j}$ $(\geqq 0)$, $a'_0=1$, $a'_i=a_i$ $(1\leqq i\leqq N)$ and $a'_{N+1}=-a_{N+1}$, we get by (3) and (4)

$$\sum_{i=0}^{N+1}a'_i z_1^{l'_{i1}}\cdots z_N^{l'_{iN}}(1+z_1+\cdots+z_N)^{l'_{iN+1}}=0, \tag{8}$$

where for each j $(1\leqq j\leqq N+1)$ there is at least one index i with $l'_{ij}=0$.

Let $\{0, 1, \cdots, N+1\}=I_1\cup\cdots\cup I_{\kappa_0}$ ($I_\kappa\neq\phi$, $I_\kappa\cap I_{\kappa'}=\phi$ if $\kappa\neq\kappa'$) be a partition with the property that for any κ

$$\sum_{i\in I_\kappa} a'_i z_1^{l'_{i1}}\cdots z_n^{l'_{iN}}(1+z_1+\cdots+z_N)^{l'_{iN+1}}=0,$$

where we choose it so that the number κ_0 of classes is as large as possible. Clearly, $\#I_\kappa\geqq 2$ for any κ $(1\leqq\kappa\leqq\kappa_0)$. After a suitable change of indices, it may be assumed that $I_1=\{0, 1, \cdots, k\}$ $(k\geqq 1)$. Assume that $k\leqq N$. By Lemma 6.2,

$$\sum_{i=0}^{k} a'_i z_1^{l'_{i1}}\cdots z_N^{l'_{iN}}(1+z_1+\cdots+z_N)^{l'_{iN+1}}=0$$

implies

$$\sum_{i=0}^{k} a'_i z_1^{l'_{i1}}\cdots z_N^{l'_{iN}} z_{N+1}^{l'_{iN+1}}=0.$$

In this situation, we have $l'_{0j}=l'_{1j}=\cdots=l'_{kj}$ $(1\leqq j\leqq N+1)$. Indeed, e.g., if $l'_{01}=l'_{11}=\cdots=l'_{k'1}$ and $l'_{i1}\neq l'_{01}$ for any i with $k'+1\leqq i\leqq k$ $(k'\leqq k-1)$, we see easily

$$\sum_{i=0}^{k'} a'_i z_1^{l'_{i1}}\cdots z_{N+1}^{l'_{iN+1}}=0 \quad\text{and}\quad \sum_{i=k'+1}^{k} a'_i z_1^{l'_{i1}}\cdots z_{N+1}^{l'_{iN+1}}=0,$$

which contradicts the property of κ_0. Now, it holds that

$$(11)\qquad \begin{vmatrix} 1 & l'_{01}\cdots & l'_{0N+1} \\ 1 & l'_{11}\cdots & l'_{iN+1} \\ & \cdots & \\ 1 & l'_{N+11}\cdots & l'_{N+1N+1} \end{vmatrix} = \begin{vmatrix} 1 & l'_{01}\cdots & l'_{0N+1} \\ 0 & l_{11}\cdots & l_{1N+1} \\ & \cdots & \\ 0 & l_{N+11}\cdots & l_{N+1N+1} \end{vmatrix} = \det(l_{ij})=\pm 1.$$

On the other hand, this is equal to zero because at least two rows are equal to each other. This is a contradiction. We conclude $k=N+1$ or $\kappa_0=1$.

In (8), after a suitable change of indices, let $l'_{iN+1}=0$ $(0\leqq i\leqq r)$ and $l'_{iN+1}\neq 0$ $(r+1\leqq i\leqq N+1)$. If $r\leqq N-1$, substituting $z_N=-(1+z_1+\cdots+z_{N-1})$ in (8), we obtain

$$\sum_{i=0}^{r} a'_i z_1^{l'_{i1}}\cdots z_{N-1}^{l'_{iN-1}}(-(1+z_1+\cdots+z_{N-1}))^{l'_{iN}}=0$$

and hence by Lemma 6.2 the identity

$$\sum_{i=0}^{r} a'_i z_1^{l'_{i1}}\cdots z_N^{l'_{iN}}=0,$$

which contradicts the above shown fact $\kappa_0=1$. Therefore, there is at most one index i with $l'_{iN+1}>0$. By a suitable non-singular linear transformation of variables, it can be similarly proved that for any j $(1\leqq j\leqq N+1)$ there is at most one index i with $l'_{ij}>0$. Moreover, in this situation we have at most one index j with $l'_{ij}>0$ for any i. Indeed, if not, two distinct rows of the matrix (11) are equal to $(1, 0, \cdots, 0)$ and then $\det(l_{ij})=0$. In conclusion, we have a subset $\{i_1, i_2, \cdots, i_{N+1}\}$ $(i_l\neq i_m$ if $l\neq m)$ of $\{0, 1, \cdots, N+1\}$ such that any l'_{ij} is equal to zero except $l'_{i_1 1}, l'_{i_2 2}, \cdots, l'_{i_{N+1}N+1}$. Then $\det(l_{ij})=l'_{i_1 1}l'_{i_2 2}\cdots l'_{i_{N+1}N+1}=\pm 1$. Since $l'_{ij}\geqq 0$, we get $l'_{i_1 1}=l'_{i_2 2}=\cdots=l'_{i_{N+1}N+1}=1$.

We put $l'_{i0}=1$ if $l'_{i1}=l'_{i2}=\cdots=l'_{iN+1}=0$ and $l'_{i0}=0$ if $l'_{ij}=1$ for some j. Then, rewriting the representation (4) of f by homogeneous coordinates $w_0:w_1:\cdots:w_N$ and $v_0:v_1:\cdots:v_N$, we have

$$v_i=a'_i w_0^{l'_{i0}} w_1^{l'_{i1}}\cdots w_N^{l'_{iN}}(w_0+w_1+\cdots+w_N)^{l'_{iN+1}} \quad (0\leqq i\leqq N) \tag{11}$$

and the identity

$$v_0+v_1+\cdots+v_N=a''_{N+1}w_0^{l'_{N+10}}w_1^{l'_{N+11}}\cdots w_N^{l'_{N+1N}}(w_0+\cdots+w_N)^{l'_{N+1N+1}}, \tag{12}$$

where a'_i $(0\leqq i\leqq N)$ and a''_{N+1} are uniquely determined up to a non-zero constant common factor by the identity (12) and (l'_{ij}) $(0\leqq i, j\leqq N+1)$ is a non-singular matrix whose components are either 1 or 0. The formulas (11) and (12) show that any given f in $\mathrm{Aut}(X_1)$ is a linear automorphism of $P_N(\boldsymbol{C})$ which induces a permutation among the spaces $H_0, H_1, \cdots, H_{N+1}$. This completes the proof of Theorem 6.1. q. e. d.

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