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A characterization of $PSL(2, 11)$ and S_5

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§1. Introduction.

The symmetric group S_5 of degree five and the two dimensional projective special linear group $PSL(2, 11)$ over the field of eleven elements are doubly transitive permutation groups of degree five and eleven, respectively, in which the stabilizer of two points is isomorphic to the symmetric group S_3 of degree three.

Let Ω be the set of points $1, 2, \cdots, n$, where n is odd. Let $\mathfrak{G}$ be a doubly transitive permutation group in which the stabilizer $\mathfrak{G}_{1,2}$ of the points 1 and 2 has even order and a Sylow 2-subgroup $\mathfrak{K}$ of $\mathfrak{G}_{1,2}$ is cyclic. In the case $\mathfrak{G}_{1,2}$ is cyclic, Kantor-O'Nan-Seitz and the author proved independently that $\mathfrak{G}$ contains a regular normal subgroup ([**5**] and [8]). In this paper we shall study the case $\mathfrak{G}_{1,2}$ is not cyclic. Let τ be the unique involution in $\mathfrak{K}$. By a theorem of Witt ([**10**, Th. 9.4]) the centralizer $C_{\mathfrak{G}}(\tau)$ of τ in $\mathfrak{G}$ acts doubly transitively on the set $\mathfrak{J}(\tau)$ consisting of points in Ω fixed by τ.

The purpose of this paper is to prove the following theorem.

THEOREM. *Let $\mathfrak{G}$, $\mathfrak{G}_{1,2}$, τ and $\mathfrak{J}(\tau)$ be as above. Assume that all Sylow subgroups of $\mathfrak{G}_{1,2}$ are cyclic, the image of the doubly transitive permutation representation of $C_{\mathfrak{G}}(\tau)$ on $\mathfrak{J}(\tau)$ contains a regular normal subgroup and that $\mathfrak{G}$ does not contain a regular normal subgroup. If $\mathfrak{G}$ has two classes of involutions, then $\mathfrak{G}$ is isomorphic to S_5 and $n=5$. If $\mathfrak{G}$ has one class of involutions and τ is not contained in the center of $\mathfrak{G}_{1,2}$, then $\mathfrak{G}$ is isomorphic to $PSL(2, 11)$ and $n=11$.*

In [**7**] we proved this theorem in the case that the order $\mathfrak{G}_{1,2}$ equals $2p$ for an odd prime number p.

Let $\mathfrak{X}$ be a subset of a permutation group. Let $\mathfrak{J}(\mathfrak{X})$ denote the set of all the fixed points of $\mathfrak{X}$ and let $\alpha(\mathfrak{X})$ be the number of points in $\mathfrak{J}(\mathfrak{X})$. The other notion is standard.

§2. On the degree of $\mathfrak{G}$.

Let $\mathfrak{G}$ be a doubly transitive permutation group on $\Omega=\{1, 2, \cdots, n\}$. Let $\mathfrak{G}_1$ and $\mathfrak{G}_{1,2}$ be the stabilizers of the point 1 and the points 1 and 2.

respectively. In this paper we assume that a Sylow 2-subgroup $\mathfrak{R}$ ($\neq 1$) of $\mathfrak{G}_{1,2}$ is cyclic. Let us denote the unique involution in $\mathfrak{R}$ by τ. By the Burnside argument $\mathfrak{G}_{1,2}$ has a normal 2-complement $\mathfrak{H}$. Let I be an involution with the cycle structure $(1, 2)\cdots$. Then I is contained in $N_{\mathfrak{G}}(\mathfrak{G}_{1,2})$ and we have the following decomposition of $\mathfrak{G}$:

$$\mathfrak{G}=\mathfrak{G}_1+\mathfrak{G}_1 I\mathfrak{G}_1 .$$

By Frattini argument it may be assumed that I normalizes $\mathfrak{R}$. Let d be the number of elements in $\mathfrak{G}_{1,2}$ inverted by I. Let $g(2)$ and $g_1(2)$ denote the numbers of involutions in $\mathfrak{G}$ and $\mathfrak{G}_1$, respectively. Then the following equality is obtained:

$$g(2)=g_1(2)+d(n-1) . \tag{2.1}$$

(See [**4**] or [**6**].)

Let τ fix i ($i\geqq 2$) points of Ω, say $1, 2, \cdots, i$. By a theorem of Witt ([**10**, Th. 9.4]) $C_{\mathfrak{G}}(\tau)$ acts doubly transitively on $\mathfrak{J}(\tau)$. Let $\chi_1(\tau)$ and $\chi(\tau)$ be the kernel of this permutation representation and its image, respectively. In general, let $\mathfrak{X}$ be a subgroup of $\mathfrak{G}_{1,2}$ satisfying the condition of Witt. Then $N_{\mathfrak{G}}(\mathfrak{X})$ acts doubly transitively on $\mathfrak{J}(\mathfrak{X})$. Let $\chi_1(\mathfrak{X})$ and $\chi(\mathfrak{X})$ be the kernel of this permutation representation and its image, respectively. Let us denote $[\mathfrak{G}_{1,2} : C_{\mathfrak{G}_{1,2}}(\tau)]$ by γ.

Let us assume that n is odd. Let $g_1^*(2)$ be the number of involutions in $\mathfrak{G}_1$ which fix only the point 1. Then from (2.1) the following equality is obtained:

$$g_1^*(2)n+\gamma n(n-1)/i(i-1)=g_1^*(2)+\gamma(n-1)/(i-1)+d(n-1) . \tag{2.2}$$

It follows from (2.2) that $d>g_1^*(2)$ and $n=i(\beta i-\beta+\gamma)/\gamma$, where $\beta=d-g_1^*(2)$ equals the number of involutions with cycle structures $(1, 2)\cdots$ which are conjugate to τ.

Next let us assume that n is even. Let $g^*(2)$ be the number of involutions in $\mathfrak{G}$ which fix no point of Ω. Then the following equality is obtained:

$$g^*(2)+\gamma n(n-1)/i(i-1)=\gamma(n-1)/(i-1)+d(n-1) . \tag{2.3}$$

Since $\mathfrak{G}$ is doubly transitive on Ω, $g^*(2)$ is a multiple of $n-1$. It follows from (2.3) that $d(n-1)>g^*(2)$ and $n=i(\beta i-\beta+\gamma)/\gamma$, where $\beta=d-g^*(2)/(n-1)$ equals the number of involutions with the cycle structures $(1, 2)\cdots$ which are conjugate to τ (see [**7**]).

Let $\mathfrak{R}_0$ be the set of elements in $\mathfrak{R}$ inverted by I. For an element K of $\mathfrak{R}_0$, let $\mathfrak{D}(IK)$ be the set of elements in $\mathfrak{H}$ inverted by IK and $d(IK)$ be the number of elements in $\mathfrak{D}(IK)$.

LEMMA 1. *$d=\sum_{K\in\mathfrak{R}_0} d(IK)$ and $d(IK)$ is odd.*

PROOF. Let KH be an element of $\mathfrak{G}_{1,2}=\mathfrak{K}\mathfrak{H}$ inverted by I. Then $(KH)^I=H^{-1}K^{-1}=K^{-1}KH^{-1}K^{-1}=K^IH^I$. Therefore $K^I=K^{-1}$ and $H^{IK}=H^{-1}$. This proves the first assertion. For the second, see [2, Lem. 10.4.1].

LEMMA 2. *Every involution in* $I\mathfrak{G}_{1,2}$ *is conjugate to* I *or* $I\tau$ *and* $\mathfrak{G}$ *has one or two classes of involutions.*

PROOF. For an element K of $\mathfrak{K}_0$ every involution in $IK\mathfrak{H}$ is conjugate to IK. Every involution in $I\mathfrak{K}_0$ is conjugate to I or $I\tau$ and every involution in G is conjugate to an involution in $I\mathfrak{G}_{1,2}$ since $\mathfrak{G}$ is doubly transitive. This proves the lemma.

LEMMA 3. *d is even and so is β if $|\mathfrak{K}_0|>2$.*

PROOF. Trivial.

LEMMA 4. *Assume $|\mathfrak{K}|>2$. Then let $\mathfrak{V}$ be a subgroup of $\mathfrak{K}$ of order* 4. *If $\langle\mathfrak{V}, I\rangle$ is dihedral, then $\langle\mathfrak{V}, J\rangle$ dihedral for every involution $J(\neq\tau)$ in $N_{\mathfrak{G}}(\mathfrak{V})$.*

PROOF. Since a Sylow 2-subgroup of $\mathfrak{G}_{1,2}$ is cyclic, $\alpha(\langle I, \mathfrak{V}\rangle)=\alpha(\langle J, \mathfrak{V}\rangle)\leqq 1$. A doubly transitive permutation group $\mathfrak{M}$ of odd degree such that the stabilizer $\mathfrak{M}_{1,2}$ of two points is of odd order has one class of involutions since all involutions are conjugate in $I'\mathfrak{M}_{1,2}$, where I' is an involution of $\mathfrak{M}$ with the cycle structure $(1, 2)\cdots$. From this and Lemma 2 $I\chi_1(\mathfrak{V})$ and $J\chi_1(\mathfrak{V})$ are conjugate under $\chi(\mathfrak{V})$. Thus $I=Y^{-1}JXY$, where X and Y are elements of $N_{\mathfrak{G}_{1,2}}(\mathfrak{V})$ and $N_{\mathfrak{G}}(\mathfrak{V})$, respectively. Since $N_{\mathfrak{G}}(\mathfrak{V})=\langle I, C_{\mathfrak{G}}(\mathfrak{V})\rangle$, X and Y are contained in $C_{\mathfrak{G}}(\mathfrak{V})$. Thus $V^I=V^J$ for every element V of $\mathfrak{V}$. This proves the lemma.

From now on, throughout this paper, we assume n is odd and $\chi(\tau)$ contains a regular normal subgroup. Then i equals a power of a prime number, say p^m.

THEOREM 1. *Let $\mathfrak{G}$ be a doubly transitive permutation group of odd degree n such that a Sylow 2-subgroup $\mathfrak{K}$ of $\mathfrak{G}_{1,2}$ is cyclic. If $|\mathfrak{K}|>2$ and $\langle\mathfrak{K}, I\rangle$ is dihedral or quasi-dihedral, then $\mathfrak{G}$ contains a regular normal subgroup. If $|\mathfrak{K}|=2$ and $\mathfrak{G}$ has one class of involutions, then it contains a regular normal subgroup or it is isomorphic to $PSL(2, 11)$ with $n=11$.*

PROOF. $n-1=(i-1)(\beta i+\gamma)/\gamma$ and γ is odd. By Lemma 3 β is even. Therefore a Sylow 2-subgroup of $C_{\mathfrak{G}}(\tau)$ is that of $\mathfrak{G}$. By Lemma 4 a proof of the theorem is similar to the case that $\mathfrak{G}_{1,2}$ is cyclic ([8]).

§ 3. Proof of Theorem.

Let $\mathfrak{G}$ be as in Theorem. By Theorem 1 we may assume $\mathfrak{K}_0=\langle\tau\rangle$ and $d=d(I)+d(I\tau)$. If all involutions are conjugate, then $\beta=d$ is even and if $\mathfrak{G}$ has two classes of involutions, then let us assume $\alpha(I)=1$ and $\beta=d(I\tau)$. Let $\mathfrak{H}_q$ be a Sylow q-subgroup of $\mathfrak{H}$. Since all Sylow subgroups of $\mathfrak{H}$ are cyclic, we may assume that $N_{\mathfrak{G}}(\mathfrak{H}_q)$ contains $\langle\mathfrak{K}, I\rangle$ and $\mathfrak{H}_r$ for $r<q$.

LEMMA 5. *If $\mathfrak{G}$ has two classes of involutions and if $\alpha(\mathfrak{H}_q)$ is odd for every $\mathfrak{H}_q$ such that $\langle \mathfrak{H}_q, I\rangle$ is dihedral, then $\mathfrak{G}$ contains a regular normal subgroup.*

PROOF. Let a be the unique element in $\mathfrak{J}(I)$. Assume $\langle \mathfrak{H}_q, I\rangle$ is dihedral. Let $\mathfrak{H}'_q$ be a Sylow q-subgroup of $\mathfrak{H}$ normalized by I. Then $\langle \mathfrak{H}'_q, I\rangle$ must be dihedral. Since $\alpha(\mathfrak{H}_q)$ is odd, so is $\alpha(\mathfrak{H}'_q)$. Since $\mathfrak{J}(\mathfrak{H}'_q)^I=\mathfrak{J}(\mathfrak{H}'_q)$, it contains a. Let X be an element of $\mathfrak{D}(I)$. Then X is a product of elements of $\mathfrak{D}(I)$, $X_1, \cdots, X_{r-1}$ and X_r, where $|X_j|$ is a power of a prime number and $(|X_j|, |X_k|)=1$ for $j\neq k$. From the above $\mathfrak{J}(X_j)$ contains a. Thus $\mathfrak{J}(X)$ contains a and so does $\mathfrak{J}(\mathfrak{D}(I))$. Since $g_1^*(2)=d(I)$, the set of involutions fixing only the point a is that of involutions in $\langle \mathfrak{D}(I), I\rangle$. It is trivial that I is a unique involution in $\langle \mathfrak{D}(I), I\rangle$ which is commutative with I. Since $C_{\mathfrak{G}}(I)$ is contained in $\mathfrak{G}_a$, there is no involution $(\neq I)$ in $C_{\mathfrak{G}}(I)$ which is commutative with I and conjugate to I. By [1] $\mathfrak{G}$ contains a solvable normal subgroup. This proves the lemma.

By this lemma we may assume that if $\mathfrak{G}$ has two classes of involutions, then there exists $\mathfrak{H}_q$ $(\neq 1)$ such that $\langle \mathfrak{H}_q, I\rangle$ is dihedral and $\alpha(\mathfrak{H}_q)$ is even.

LEMMA 6. *If $\alpha(\mathfrak{H}_q)$ is even, then $\langle \mathfrak{H}_q, \tau\rangle$ is dihedral.*

PROOF. Assume $\langle \mathfrak{H}_q, \tau\rangle$ is abelian. If $\mathfrak{J}(\mathfrak{H}_q)$ contains $\mathfrak{J}(\tau)$, then $\alpha(\mathfrak{H}_q)$ is odd since $\alpha(\tau)$ is odd. Therefore $\mathfrak{H}_q$ is not contained in $\chi_1(\tau)$. Since $\chi(\tau)$ contains a regular normal subgroup, so does $\chi(H_q\chi_1(\tau))$ and its degree $\alpha(\langle \mathfrak{H}_q, \tau\rangle)$ is a power of p. Since the stabilizer in $\chi(\mathfrak{H}_q)$ of any two points of $\mathfrak{J}(\mathfrak{H}_q)$ is of even order, $\alpha(\mathfrak{H}_q)=i'(\beta'(i'-1)+\gamma')/\gamma'$, where $i'=\alpha(\langle \mathfrak{H}_q, \tau\rangle)$, γ' is odd and β' is some integer. Therefore $\alpha(\mathfrak{H}_q)$ is odd, which is a contradiction.

LEMMA 7. *If $\alpha(I)=1$, $\alpha(\mathfrak{H}_q)$ is even and if $\langle \mathfrak{H}_q, I\rangle$ is dihedral, then $q=p=|\mathfrak{H}_q|$.*

PROOF. By Lemma 6 $\langle \mathfrak{H}_q, \tau\rangle$ is dihedral. Therefore $\langle \mathfrak{H}_q, I\tau\rangle$ is abelian. If $\alpha(\langle \mathfrak{H}_q, I\tau\rangle)\geqq 2$, then $\langle \mathfrak{H}_q, I\tau\rangle$ must be conjugate to a subgroup of $\langle \mathfrak{H}_q, \mathfrak{K}\rangle$ and it is dihedral. Therefore $\alpha(\langle \mathfrak{H}_q, I\tau\rangle)\leqq 1$. Assume $\alpha(\langle \mathfrak{H}_q, I\tau\rangle)=1$. Since $\alpha(I)=1$ and $\alpha(\mathfrak{H}_q)$ is even, $\mathfrak{J}(I)$ is not contained in $\mathfrak{J}(\mathfrak{H}_q)$. Let a be an element of $\mathfrak{J}(\langle \mathfrak{H}_q, I\tau\rangle)$. Then $a^I\neq a$ and $a^I=a^\tau$ is an element of $\mathfrak{J}(\mathfrak{H}_q)$. Therefore $(a^I)^{I\tau}=a^\tau=a^I$ and it is an element of $\mathfrak{J}(\langle \mathfrak{H}_q, I\tau\rangle)$, which is a contradiction. Thus $\alpha(\langle \mathfrak{H}_q, I\tau\rangle)=0$. Since $C_{\mathfrak{G}}(I\tau)$ is conjugate to $C_{\mathfrak{G}}(\tau)$, $q=p$. Since $|C_{\mathfrak{G}_{1,2}}(\tau)|$ is not divisible by p, a Sylow p-subgroup of $C_{\mathfrak{G}}(I\tau)$ is elementary abelian. Thus $|\mathfrak{H}_p|=p$.

LEMMA 8. *If $\mathfrak{G}$ has one class of involutions and $\langle \mathfrak{H}_q, \tau\rangle$ is dihedral, or if $\alpha(I)=1$ and $\langle \mathfrak{H}_q, \tau\rangle$ and $\langle \mathfrak{H}_q, I\rangle$ are dihedral, then $q=p=|\mathfrak{H}_q|$ and $\alpha(\mathfrak{H}_p)$ is even.*

PROOF. Assume by way of contradiction that $q\neq p$. Let $\mathfrak{H}'_q$ be a subgroup of $\mathfrak{H}_q$ of order q. If all involutions are conjugate, we may assume that

$\langle \mathfrak{H}'_q, I\tau \rangle$ is abelian. Since $\langle \mathfrak{H}'_q, \tau \rangle$ is dihedral and $q \neq p$, $\alpha(\langle \mathfrak{H}'_q, I\tau \rangle) = 1$. Thus q is a factor of $i-1$. Since τ normalizes $\langle \mathfrak{H}'_q, I\tau \rangle$, $\mathfrak{Z}(\langle \mathfrak{H}'_q, I\tau \rangle)$ is contained in $\mathfrak{Z}(\tau)$. Therefore $\alpha(\langle \mathfrak{H}'_q, I\tau, \tau \rangle) = 1$ and $\langle \mathfrak{H}'_q, \tau \rangle \chi_1(I\tau)$ is a complement of a Frobenius subgroup of $\chi(I\tau)$. By a property of Frobenius groups $\langle \mathfrak{H}'_q, \tau \rangle \chi_1(I\tau)$ must be cyclic. Since it is isomorphic to $\langle \mathfrak{H}'_q, \tau \rangle$, $\langle \mathfrak{H}'_q, \tau \rangle$ must be cyclic, which is a contradiction. Thus $q = p$. In the same way as in the proof of Lemma 7, $\mathfrak{H}_p = \mathfrak{H}'_p$. Since $\alpha(\langle \mathfrak{H}_p, I\tau \rangle) = 0$ and $I(\mathfrak{H}_p)^{I\tau} = I(\mathfrak{H}_p)$, $\alpha(\mathfrak{H}_p)$ is even.

COROLLARY 9. *If every involution is conjugate to* τ, *then* $d = (1+p)d'$, *where* $d' = [C_{\mathfrak{H}}(\tau) : C_{\mathfrak{H}}(\langle \tau, I \rangle)]$ *and* $\gamma = p$.

PROOF. Since τ is not contained in the center of $\mathfrak{G}_{1,2}$, there exists a Sylow q-subgroup $\mathfrak{H}_q$ such that $\langle \mathfrak{H}_q, \tau \rangle$ is dihedral. By Lemma 8 if $\langle \mathfrak{H}_q, \tau \rangle$ is dihedral, then $q = p = |\mathfrak{H}_q|$. Thus $\gamma = [\mathfrak{H} : C_{\mathfrak{H}}(\tau)] = p$. Since $\langle \mathfrak{H}_p, \tau \rangle$ is dihedral, we may assume that $\langle \mathfrak{H}_p, I \rangle$ is dihedral. Let $\mathfrak{H}'_r$ be a Sylow r-subgroup of $C_{\mathfrak{H}}(I)$. Then $\langle \mathfrak{H}_r, I \rangle$ is abelian since $\mathfrak{H}'_r$ and $\mathfrak{H}_r$ are conjugate by an element of $C_{\mathfrak{H}}(I)$, and $\langle \mathfrak{H}_r, \tau \rangle$ is also abelian. Thus $[\mathfrak{H} : C_{\mathfrak{H}}(I)] = p[C_{\mathfrak{H}}(\tau) : C_{\mathfrak{H}}(\langle I, \tau \rangle)] = pd'$. Similarly $[\mathfrak{H} : C_{\mathfrak{H}}(I\tau)] = d'$.

COROLLARY 10. *If* $\alpha(I) = 1$, *then* γ *is a factor of* $P\beta$.

LEMMA 11. *If* $\langle \mathfrak{H}_q, I \rangle$ *or* $\langle \mathfrak{H}_q, \tau \rangle$ *is dihedral for* $q \neq p$, *then* $\langle \mathfrak{H}_q, \mathfrak{H}_p \rangle$ *is abelian.*

PROOF. Assume $\langle \mathfrak{H}_q, I \rangle$ is dihedral. If $q < p$, then $\langle I, \mathfrak{H}_q \rangle$ is contained in $N_{\mathfrak{G}}(\mathfrak{H}_p)$. Since $N_{\mathfrak{G}}(\mathfrak{H}_p)/C_{\mathfrak{G}}(\mathfrak{H}_p)$ is cyclic, $\langle \mathfrak{H}_q, \mathfrak{H}_p \rangle$ is abelian. If $q > p$, then $\langle I, \mathfrak{H}_p \rangle$ is contained in $N_{\mathfrak{G}}(\mathfrak{H}_q)$. Thus $\langle \mathfrak{H}_q, \mathfrak{H}_p \rangle$ is abelian.

LEMMA 12. $i = p = 3$ *and* $\alpha(\mathfrak{H}_p) = 2$.

PROOF. If $\alpha(I) = 1$, then $\alpha(I\tau) = i$ and $\langle \mathfrak{H}_p, I \rangle$ is dihedral by Lemma 7. If $\alpha(I) = \alpha(I\tau) = i$, then we may assume that $\langle \mathfrak{H}_p, I \rangle$ is dihedral. Since $C_{\mathfrak{G}}(I\tau)$ is conjugate to $C_{\mathfrak{G}}(\tau)$ and it contains $\mathfrak{H}_p$, $C_{\mathfrak{G}}(\tau)$ contains a subgroup of order p which is conjugate to $\mathfrak{H}_p$. Let $\mathfrak{N}$ be a normal subgroup of $C_{\mathfrak{G}}(\tau)$ containing $\chi_1(\tau)$ such that $\mathfrak{N}/\chi_1(\tau)$ is a regular normal subgroup of $\chi(\tau)$ of order i. Since Sylow 2-subgroup of $\mathfrak{N}$ is cyclic, $\mathfrak{N}$ has a normal 2-complement, which is normalized by I. Let $\mathfrak{P}$ be a Sylow p-subgroup of $C_{\mathfrak{G}}(\tau)$ which is invariant by I and let $\mathfrak{P}'$ be a subgroup of $\mathfrak{P}$ of order p which is conjugate to $\mathfrak{H}_p$. Then $\mathfrak{P}\chi_1(\tau)$ is a regular normal subgroup of $\chi(\tau)$.

(1) $\mathfrak{P}$ is normal in $\mathfrak{P}\chi_1(\tau)$.

PROOF. Let $\mathfrak{H}'_q$ be a Sylow q-subgroup of $\chi_1(\tau)$ contained in $\mathfrak{H}_q$. We may assume that by the Frattini argument $\mathfrak{P}$ is contained in $N_{\mathfrak{G}}(\mathfrak{H}'_q)$. We shall prove that $\mathfrak{P}$ is contained in $C_{\mathfrak{G}}(\mathfrak{H}'_q)$. Since $C_{\mathfrak{G}}(\tau) = \chi_1(\tau)(N_{\mathfrak{G}}(\mathfrak{H}'_q) \cap C_{\mathfrak{G}}(\tau))$, $N_{\mathfrak{G}}(\mathfrak{H}'_q) \cap C_{\mathfrak{G}}(\tau)$ acts doubly transitively on $\mathfrak{Z}(\tau)$ and hence so does $N_{\mathfrak{G}}(\mathfrak{H}'_q) \cap C_{\mathfrak{G}}(\tau) \cap N_{\mathfrak{G}}(\mathfrak{P})$. Thus $N_{\mathfrak{G}}(\mathfrak{H}'_q) \cap C_{\mathfrak{G}_1}(\tau) \cap N_{\mathfrak{G}}(\mathfrak{P})$ acts transitively on $\mathfrak{P} - \{1\}$. Assume that $\mathfrak{P}$ is not contained in $C_{\mathfrak{G}}(\mathfrak{H}'_q)$. Since $\mathrm{Aut}(\mathfrak{H}'_q)$ is cyclic, $i = p$ and it is a factor of $q-1$. If $\langle \mathfrak{H}'_q, I\rangle$ is dihedral, then $\langle \mathfrak{H}'_q, \mathfrak{P}\rangle$ must be abelian since $\langle \mathfrak{P}, \mathfrak{J}\rangle$ is dihedral and $\mathrm{Aut}(\mathfrak{H}'_q)$ is cyclic. Thus $\langle \mathfrak{H}'_q, I\rangle$ is abelian and so is $\langle \mathfrak{H}'_q, I\tau\rangle$. If $\mathfrak{J}(\mathfrak{H}'_q)=\mathfrak{J}(\tau)$, then q is a factor of $i-1$ since $I\tau$ is conjugate to τ. This is a contradiction and $\mathfrak{J}(\tau)$ is a proper subset of $\mathfrak{J}(\mathfrak{H}'_q)$. Since $p<q$, $\mathfrak{H}_p$ is contained in $N_{\mathfrak{G}}(\mathfrak{H}'_q)$. If $X_1(\mathfrak{H}'_q)$ contains $\mathfrak{H}_p$, then $\mathfrak{J}(\langle \mathfrak{H}_p, \tau\rangle)=\mathfrak{J}(\tau)$. Since $\mathfrak{J}(\mathfrak{H}_p)^\tau=\mathfrak{J}(\mathfrak{H}_p)$, $\alpha(\mathfrak{H}_p)$ is odd. On the other hand, $\alpha(\mathfrak{H}_p)$ is even by Lemma 8 since $\langle H_p, \tau\rangle$ and $\langle H_p, I\rangle$ are dihedral. Thus $\mathfrak{H}_p$ is not contained in $X_1(\mathfrak{H}'_q)$. Thus $X(\mathfrak{H}'_q)_{1,2}$ contains a dihedral subgroup $\langle \tau, \mathfrak{H}_p\rangle X_1(\mathfrak{H}'_q)$. Thus $X(\mathfrak{H}'_q)$ is a doubly transitive permutation group on $\mathfrak{J}(\mathfrak{H}'_q)$ in which the stabilizer of two points contains at least two involutions. Since $N_{\mathfrak{G}}(\mathfrak{H}'_q)\cap C_{\mathfrak{G}}(\tau)$ acts doubly transitively on $\mathfrak{J}(\tau)$ and $(N_{\mathfrak{G}}(\mathfrak{H}'_q)\cap C_{\mathfrak{G}}(\tau))X_1(\tau)/X_1(\tau)=X(\tau)$, $X(\mathfrak{H}'_q)$ satisfies the conditions in Theorem. By the inductive hypothesis $X(\mathfrak{H}'_q)$ is isomorphic to one of S_5 and $PSL(2, 11)$ or contains a regular normal subgroup. Since $\langle I, \mathfrak{H}'_q\rangle$ is abelian and $|X_1(\mathfrak{H}'_q)|$ is not divisible by p, $C_{\mathfrak{G}}(\mathfrak{H}'_q)X_1(\mathfrak{H}'_q)$ is a proper subgroup of $X(\mathfrak{H}'_q)$. Thus $X(\mathfrak{H}'_q)$ contains a regular normal subgroup, which is contained in $C_{\mathfrak{G}}(\mathfrak{H}'_q)X_1(\mathfrak{H}'_q)$. Let $\tilde{\mathfrak{P}}$ be a Sylow p-subgroup of $C_{\mathfrak{G}}(\mathfrak{H}'_q)$. Since $|\mathfrak{H}_p|=p$ and $\langle \mathfrak{P}, \mathfrak{H}'_q\rangle$ is non abelian, $\tilde{\mathfrak{P}}$ is isomorphic to a regular normal subgroup of $X(\mathfrak{H}'_q)$. By the Frattini argument τ normalizes $\tilde{\mathfrak{P}}$. Since $\alpha(\langle \tau, \mathfrak{H}'_q\rangle)=i$, we may assume that $\tilde{\mathfrak{P}}$ contains $\mathfrak{P}$, which is a contradiction. Therefore $|C_{\mathfrak{G}}(\mathfrak{H}'_q)\cap \mathfrak{P}X_1(\tau)|$ is divisible by i. By the Burnside's splitting theorem $\mathfrak{P}$ is normal in $\mathfrak{P}X_1(\tau)$.

From (1) $\mathfrak{P}$ is normal in $C_{\mathfrak{G}}(\tau)$. Since $C_{\mathfrak{G}_1}(\tau)$ acts transitively on $\mathfrak{P}-\{1\}$, $[C_{\mathfrak{G}_1}(\tau): N_{\mathfrak{G}}(\mathfrak{P}')\cap C_{\mathfrak{G}_1}(\tau)]=(i-1)/(p-1)$. And $[N_{\mathfrak{G}}(\mathfrak{P}')\cap C_{\mathfrak{G}_1}(\tau): C_{\mathfrak{G}}(\mathfrak{P}')\cap C_{\mathfrak{G}_1}(\tau)]=p-1$. Next we shall study $|C_{\mathfrak{G}}(\mathfrak{P}')|$.

(2) Let $\mathfrak{S}$ be a Sylow 2-subgroup of $C_{\mathfrak{G}}(\mathfrak{P}')$ containing τ. Then $\mathfrak{S}$ is conjugate to $\mathfrak{R}$ and $[C_{\mathfrak{G}}(\mathfrak{P}'): C_{\mathfrak{G}}(\mathfrak{P}')\cap C_{\mathfrak{G}}(\tau)]$ is odd.

PROOF. Since n is odd, $\mathfrak{J}(\mathfrak{S})$ is non empty. Since $\mathfrak{J}(\mathfrak{S})$ is contained in $\mathfrak{J}(\tau)$ and $\alpha(\langle \tau, \mathfrak{P}'\rangle)=0$, $\alpha(\mathfrak{S})\geqq p$. Thus $\mathfrak{S}$ is conjugate to $\mathfrak{R}$.

(3) Let $\mathfrak{Q}'$ be a Sylow q-subgroup of $C_{\mathfrak{G}}(\mathfrak{P}')$. If $\alpha(\mathfrak{Q}')\geqq 1$, then $\alpha(\mathfrak{Q}')\geqq 2$.

PROOF. From (2) $C_{\mathfrak{G}}(\mathfrak{P}')$ has a normal 2-complement. Therefore it may be assumed that τ normalizes $\mathfrak{Q}'$. If $\mathfrak{J}(\mathfrak{Q}')\cap\mathfrak{J}(\tau)$ is non empty, then $\alpha(\langle \mathfrak{Q}', \tau\rangle)\geqq p$. If $\mathfrak{J}(\mathfrak{Q}')\cap\mathfrak{J}(\tau)$ is empty, then $\alpha(\mathfrak{Q}')\geqq 2$ since $\mathfrak{J}(\mathfrak{Q}')^\tau=\mathfrak{J}(\mathfrak{Q}')$.

(4) $\alpha(\mathfrak{P}')$ is divisible by $p-1$.

PROOF. Let q be a prime factor of $p-1$. By Corollary 9 and 10, $i-1$ is a factor of $n-1$ and so is $p-1$. Let $\mathfrak{Q}$ be a Sylow q-subgroup of $N_{\mathfrak{G}}(\mathfrak{P}')$ containing a Sylow q-subgroup $\mathfrak{Q}'$ of $C_{\mathfrak{G}}(\mathfrak{P}')$. Then $\alpha(\mathfrak{Q})=1$ and $\alpha(\mathfrak{Q}')\geqq 2$ from (3). If $|\mathfrak{G}_{1,2}|$ is not divisible by q, then $\mathfrak{Q}'=1$ and it may be assumed that $\mathfrak{Q}$ is contained in $C_{\mathfrak{G}_1}(\tau)\cap N_{\mathfrak{G}}(\mathfrak{P}')$. Thus every element $(\neq 1)$ of $\mathfrak{Q}$ fixes only the point 1 and hence $|\mathfrak{Q}|$ is a factor of $\alpha(\mathfrak{P}')$. Next assume $\mathfrak{H}_q\neq 1$. If $\langle \mathfrak{H}_q, \tau\rangle$ is abelian, then it may be assumed that $\mathfrak{H}_q$ is contained in $C_{\mathfrak{G}}(\mathfrak{P}')$ since $X(\tau)$ contains a regular normal subgroup and $\mathfrak{P}$ is normal in $C_{\mathfrak{G}}(\tau)$ by

(1). If $\langle \mathfrak{H}_q, \tau \rangle$ is dihedral, then $\langle \mathfrak{H}_q, \mathfrak{H}_p \rangle$ is abelian by Lemma 11. Since $\mathfrak{P}'$ is conjugate to $\mathfrak{H}_p$, $\mathfrak{Q}'$ is conjugate to $\mathfrak{H}_q$. Since $N_{\mathfrak{G}}(\mathfrak{P}')/C_{\mathfrak{G}}(\mathfrak{P}')$ is cyclic, by the Frattini argument τ is contained in $N_{\mathfrak{G}}(\mathfrak{Q})$. If q is a factor of $\alpha(\mathfrak{P}')-1$, then $\alpha(\langle \mathfrak{Q}, \mathfrak{P}' \rangle) \geqq 1$. Since $\alpha(\langle \tau, \mathfrak{P}' \rangle)=0$, $\alpha(\langle \mathfrak{Q}\,; \mathfrak{P}' \rangle) \geqq 2$ and $|\mathfrak{Q}|$ must be a factor of $|\mathfrak{H}_q|$, which is a contradiction and hence q is a factor of $\alpha(\mathfrak{P}')$. Thus $[\mathfrak{Q} : \mathfrak{Q}']$ is a factor of $\alpha(\mathfrak{P}')$. This proves (4).

(5) $i=p=3$ and $\alpha(\mathfrak{H}_p)=2$.

PROOF. $\chi(\mathfrak{P}')$ is a doubly transitive group of degree $\alpha(\mathfrak{P}')$. Let r be a prime factor of $\alpha(\mathfrak{P}')-1$. Let $\mathfrak{R}$ be a Sylow r-subgroup of $N_{\mathfrak{G}}(\mathfrak{P}')$. Then $\alpha(\mathfrak{R}) \geqq 1$. From (4) $\mathfrak{R}$ is contained in $C_{\mathfrak{G}}(\mathfrak{P}')$. From (3) $\alpha(\mathfrak{R}) \geqq 2$. Thus $\alpha(\mathfrak{P}')-1 =1$ and $\alpha(\mathfrak{P}')=2$. From (4) $p-1$ is a factor of $\alpha(\mathfrak{P}')=2$. Thus $p=3$. Since $\alpha(\mathfrak{P}')=2$, $\mathfrak{P}$ is a subgroup of $\chi_1(\mathfrak{P}')$. Thus $\mathfrak{P}=\mathfrak{P}'$ and $i=3$.

COROLLARY 13. $\mathfrak{K}=\langle \tau \rangle$.

PROOF. Since $\mathfrak{K}$ is contained in $N_{\mathfrak{G}}(\mathfrak{H}_p)$ and $\langle \mathfrak{H}_p, \tau \rangle$ is dihedral, $\mathfrak{K}=\langle \tau \rangle$.

LEMMA 14. *If $\mathfrak{G}$ has one class of involutions, then $\mathfrak{G}$ is isomorphic to PSL (2, 11) with $n=11$.*

PROOF. The lemma follows from Theorem 1 and Corollary 13.

LEMMA 15. *If $\mathfrak{G}$ has two classes of involutions, then $\mathfrak{G}$ is isomorphic to S_5 with $n=5$.*

PROOF. Assume that $\langle \mathfrak{H}_q, \tau \rangle$ is abelian and $\langle \mathfrak{H}_q, I \rangle$ is dihedral. If $\mathfrak{J}(\tau)$ does not contain $\mathfrak{J}(\mathfrak{H}_q)$, then there exist points a and b in $\mathfrak{J}(\mathfrak{H}_q)$ such that $a^{\tau}=b$. Let η be an involution of $N_{\mathfrak{G}}(\mathfrak{H}_q) \cap \mathfrak{G}_{a,b}$ which is commutative with τ. Then $\alpha(\tau\eta)=1$ and $\langle \tau\eta, \mathfrak{H}_q \rangle$ is abelian, which is a contradiction. Therefore $\alpha(\mathfrak{H}_q)=3$. Since $\mathfrak{P}$ is normal in $C_{\mathfrak{G}}(\tau)$, $\mathfrak{J}(\mathfrak{P})^{\mathfrak{H}_q}=\mathfrak{J}(\mathfrak{P})$. Since $\alpha(\mathfrak{P})=2$ and $\alpha(\langle \mathfrak{H}_q, \mathfrak{P} \rangle)=0$, $\mathfrak{H}_q=1$. Thus $\gamma=p\beta$ and $n=i\{\beta(i-1)+\gamma\}/\gamma=5$. This proves the lemma.

This completes a proof of Theorem.

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