J. Math. Soc. Japan
Vol. 24, No. 2, 1972

On transcendency of special values of arithmetic automorphic functions

By Yasuo MORITA*

(Received Sept. 28, 1971)

§1. Introduction.

Let Γ be the modular group $SL(2, \boldsymbol{Z})$ and $\tilde{\Gamma}=GL^{+}(2, \boldsymbol{Q})$. Let H be the complex upper half plane $\{z \in \boldsymbol{C};\ \mathrm{Im}\, z>0\}$. We define the action of an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $GL^{+}(2, \boldsymbol{R})$ on H by

$$z \longmapsto \frac{az+b}{cz+d}$$

for $z \in H$. Then Γ and $\tilde{\Gamma}$ operate on H. Let $J(z)$ be the standard modular function of level one. Then the classical theory of complex multiplication shows:

THEOREM C. *If $z \in H$ is fixed by some non-scalar element of $\tilde{\Gamma}$, z is an algebraic number and $J(z)$ generates an abelian extension of $\boldsymbol{Q}(z)$.*

On the other hand, T. Schneider obtained the following theorem:

THEOREM T. *Let $z \in H$ be an algebraic number. Suppose that z is not fixed by any non-scalar element of Γ. Then $J(z)$ is a transcendental number.*

In this paper, we shall give a generalization of Theorem T.

Let B be an indefinite quaternion algebra over the rational number field $\boldsymbol{Q}$, $\mathcal{O}$ a maximal order of B, Γ the group of all the units of $\mathcal{O}$ of reduced norm one, and $\tilde{\Gamma}$ the group of all the invertible elements of B with positive reduced norm. Now we fix an irreducible representation χ of B into $M_2(\boldsymbol{R})$ so that the image $\chi(B)$ is contained in $M_2(\bar{\boldsymbol{Q}})$, where $\bar{\boldsymbol{Q}}$ is the algebraic closure of $\boldsymbol{Q}$ in $\boldsymbol{C}$. Then we may regard Γ and $\tilde{\Gamma}$ as subgroups of $GL^{+}(2, \boldsymbol{R})$ acting on H. As a generalization of the function J, G. Shimura has constructed a holomorphic map φ from H into a projective space $\boldsymbol{P}^{l}$, satisfying the following conditions (cf. Shimura [4], §9): (i) φ induces a biregular isomorphism from $\Gamma \backslash H$ onto an algebraic curve in $\boldsymbol{P}^{l}$; (ii) if z is fixed by some non-scalar element of $\tilde{\Gamma}$, $\varphi(z)$ generates an abelian extension over a certain imaginary quadratic field. We shall call the map φ the *Shimura map*.

Now our main result can be stated as follows:

---

* This paper was partially supported by Sakkokai Foundation.

THEOREM 1. *Let $z \in H$ be an algebraic number. Suppose z is not fixed by any non-scalar element of $\tilde{\Gamma}$. Then $\varphi(z)$ is not algebraic.*

It should be noted that the generalization from Theorem T to our theorem is not trivial. We use the fact that the commutor of $\chi(B)$ in $M_2(C)$ is the set of scalar matrices. Therefore our method cannot be applied to a more general case in which Γ is the Siegel modular group or the unit group of a quaternion algebra over a totally real algebraic number field of degree >1.

§ 2. A reformulation of Lang's result.

In [2], S. Lang considered the transcendency of the moduli of abelian varieties. In this section, we shall prove a theorem about the endomorphisms of abelian varieties, which is, though stronger than the corresponding Theorem 2 of Lang [2], essentially proved in his paper.

Let K be a finite algebraic number field, A an abelian variety defined over K. Moreover suppose every endomorphism of A is defined over K. Let $T_0(A)$ be the tangent space of A at its origin. Let $\{e_1, \cdots, e_n\}$ be a K-base of $T_0(A)$, and identify $T_0(A)$ with $\boldsymbol{C}^n$ by

$$T_0(A) \ni z_1e_1 + \cdots + z_ne_n \longleftrightarrow (z_1, \cdots, z_n) \in \boldsymbol{C}^n .$$

Then $\boldsymbol{C}^n$ can be considered as a covering of A in a natural manner. Let $\iota_0 : \boldsymbol{C}^n \to A$ be the covering map and $D = \iota_0^{-1}(0)$. Then ι_0 induces a biregular isomorphism $\boldsymbol{C}^n/D \simeq A$. Let M be the set of meromorphic functions on $\boldsymbol{C}^n$ which are invariant under the translations of the elements of D and K-rational as functions on A.

THEOREM 2. *Let L be a $\boldsymbol{C}$-linear endomorphism of $\boldsymbol{C}^n$. Then the following two statements are equivalent.*

(i) *L maps $D \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ into $D \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$, i.e., L is an element of* $\mathrm{End}(A) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$.

(ii) *There are n elements $x_1, \cdots, x_n$ of $D \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ which are linearly independent over $\boldsymbol{C}$ and which are mapped into $D \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ by L. Moreover the matrix representation of L by the $\boldsymbol{C}$-base $\{e_1, \cdots, e_n\}$ of $\boldsymbol{C}^n$ is contained in $M_n(K)$.*

PROOF. First we shall show that (i) implies (ii). Multiplying by some natural number if necessary, we may assume that L is an endomorphism of A. Then, if f belongs to M, $f \circ L$ also belongs to M. For $z \in \boldsymbol{C}^n$, we define its components $z_1, \cdots, z_n$ by $z = \sum_{k=1}^{n} z_k e_k$. Since $\{e_1, \cdots, e_n\}$ gives a K-base of the tangent space of the origin of A, $\left[\frac{\partial}{\partial z_i} f(z_1, \cdots, z_n)\right]_{z=0}$ belongs to K whenever $f(z) = f(z_1, \cdots, z_n)$ belongs to M and $\frac{\partial}{\partial z_i} f(z_1, \cdots, z_n)$ is finite at $z_1 = \cdots = z_n = 0$. Let $f_1, \cdots, f_n$ be n elements of M satisfying $\left(\frac{\partial}{\partial z_i} f_j\right)(0) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Let (α_{ij}) be the matrix representation of

L by the $\boldsymbol{C}$-base $e_1, \cdots, e_n$ of $\boldsymbol{C}^n$. Then

$$\left[\frac{\partial}{\partial z_i} f_j(Lz)\right]_{z=0} = \sum_{k=1}^{n} \alpha_{ki}\left(\frac{\partial}{\partial z_k} f_j\right)(0) = \alpha_{ji}.$$

Therefore α_{ji} belongs to K. Since the first assertion of (ii) is obvious, we see that (i) implies (ii).

For the proof of the fact that (ii) implies (i), we need a few preparatory lemmas. Let $g(z)$ be a meromorphic functions on $\boldsymbol{C}^n$. Then we say that the order of $g(z)$ is not greater than ρ if there exist a constant c and two entire functions $g_i(z)$ $(i=1, 2)$ such that $g(z)=g_1(z)/g_2(z)$, $g_2(z)\neq 0$ and $|g_i(z)| \leqq \exp(c|z|^\rho)$, where $|z|^2 = \sum_{\nu=1}^{n} |z_\nu|^2$ for $z=(z_1, \cdots, z_n)$.

LEMMA 1. *Let*

$$\theta(z) = \sum_{m\in \boldsymbol{Z}^n} \exp 2\pi i\left\{\frac{1}{2}\tau[m+g] + {}^t(m+g)(z+h)\right\}$$

be a θ-function, where g and h are real n-vectors, τ is a complex symmetric matrix with positive imaginary part and $\tau[m+g] = {}^t(m+g)\tau(m+g)$. Then there is a constant c satisfying $|\theta(z)| \leqq \exp(c|z|^2)$.

The proof of this lemma is easy and left to the reader.

COROLLARY. *Let $\boldsymbol{C}^n/D$ be an abelian variety. Let $f(z)$ be a meromorphic function on $\boldsymbol{C}^n$ invariant under the translations by the elements of D. Then the order of $f(z)$ is not greater than 2.*

PROOF. Since $f(z)$ is a meromorphic function on the abelian variety $\boldsymbol{C}^n/D$, it can be written as a rational function of some θ-functions of the above form (cf. ex., [1], § 2). Therefore the order of $f(z)$ is not greater than 2.

LEMMA 2. *Let K be a finite algebraic number field. Let $g_1, \cdots, g_M$ be meromorphic functions on $\boldsymbol{C}^n$ whose orders are not greater than a certain real number ρ. Suppose that the partial derivation $\frac{\partial}{\partial z_i}$ maps the ring $K[g_1, \cdots, g_M]$ into itself for every i. Moreover suppose that there are n $\boldsymbol{C}$-linearly independent elements $x_1, x_2, \cdots, x_n$ of $\boldsymbol{C}^n$ such that $g_i(z)$ $(i=1, 2, \cdots, M)$ belongs to K for any $z \in \boldsymbol{Z}x_1+\boldsymbol{Z}x_2+\cdots+\boldsymbol{Z}x_n$. Then the transcendental degree of $K(g_1, \cdots, g_M)$ over K is not greater than n.*

PROOF. This lemma is a special case of Lang [2], p. 181, Theorem 1.

LEMMA 3. *Let $\boldsymbol{C}^n/D$ be a complex torus. Let $g(z)$ and $f_1(z), f_2(z), \cdots, f_m(z)$ be meromorphic functions on $\boldsymbol{C}^n$. Suppose $f_1(z), \cdots, f_m(z)$ are invariant under the translations by the elements of D, and*

$$g(z)^m + f_1(z)g(z)^{m-1} + \cdots + f_m(z) = 0.$$

Then there is a natural number d such that $g(dz)$ is invariant under the translations by the elements of D.

PROOF. Let V be a proper analytic subset of $\boldsymbol{C}^n$ such that $g(z), f_1(z), \cdots, f_m(z)$ are holomorphic on $\boldsymbol{C}^n - V$. By the assumption, $f_j(z+l\omega) = f_j(z)$ for any $\omega \in D$, $l \in \boldsymbol{Z}$ and $j = 1, \cdots, m$. Therefore

$$g(z+l\omega)^m + f_1(z)g(z+l\omega)^{m-1} + \cdots + f_m(z) = 0 .$$

Now we fix ω and put

$$S_{l_1,l_2} = \{z \in \boldsymbol{C}^n - V \mid g(z+l_1\omega) = g(z+l_2\omega)\} .$$

Then, since the number of the distinct roots of

$$X^m + f_1(z)X^{m-1} + \cdots + f_m(z) = 0$$

is at most m, we see that the sets S_{l_1,l_2} $(0 \leqq l_1 < l_2 \leqq m)$ cover $\boldsymbol{C}^n - V$. Since these S_{l_1,l_2} are analytic subsets of $\boldsymbol{C}^n - V$, there are some $l_1, l_2 \in \boldsymbol{Z}$ satisfying $S_{l_1,l_2} = \boldsymbol{C}^n - V$. Therefore there are $l_1 = l_1(\omega)$, $l_2 = l_2(\omega) \in \boldsymbol{Z}$ such that $g(z+l_1\omega) = g(z+l_2\omega)$ for all $z \in \boldsymbol{C}^n - V$, hence $g(z+l_1\omega) = g(z+l_2\omega)$ for all $z \in \boldsymbol{C}^n$. Putting $k(\omega) = l_2(\omega) - l_1(\omega)$, we have $g(z+k(\omega)\omega) = g(z)$ for all $z \in \boldsymbol{C}^n$. Let $\{\omega_1, \cdots, \omega_{2n}\}$ be a $\boldsymbol{Z}$-base of D, and d be the least common multiple of $k(\omega_1), \cdots, k(\omega_{2n})$. Then d has the required property of our lemma. Q. E. D.

Now we shall start the proof of the fact that (ii) implies (i).

Let M be as before. If $f \in M$, we can write

$$(*) \qquad f(z+w) = \sum_i a_i(z)b_i(w) / \sum_j c_j(z)d_j(w)$$

with $a_i, b_i, c_j, d_j \in M$, since M is the function field of the abelian variety $\boldsymbol{C}^n/D$. Let w_0 be a point on $\boldsymbol{C}^n$ which gives a generic point of $\boldsymbol{C}^n/D$ over K. We see that the left hand side of (*) is defined at $z=0$, $w=w_0$. Therefore the right hand side of (*) is also defined at $z=0$, $w=w_0$. Hence it belongs to the local ring at $z=0$, $w=w_0$. Therefore we may assume that $a_i(0)$, $c_j(0) \in K$ and $\sum_j c_j(0)d_j(w_0) \neq 0$. Then, since $\{z_k\}$ corresponds to a K-base of $T_0(A)$, $\dfrac{\partial a_i}{\partial z_k}(0)$ and $\dfrac{\partial c_j}{\partial z_k}(0) \in K$. From (*), we have

$$\left(\frac{\partial}{\partial z_k} f\right)(w_0) = \left[\frac{\partial}{\partial z_k} f(z+w_0)\right]_{z=0}$$

$$= \frac{\left(\sum_i \frac{\partial a_i}{\partial z_k}(0)b_i(w_0)\right)\left(\sum_j c_j(0)d_j(w_0)\right) - \left(\sum_i a_i(0)b_i(w_0)\right)\left(\sum_j \frac{\partial c_j}{\partial z_k}(0)d_j(w_0)\right)}{\left\{\sum_j c_j(0)d_j(w_0)\right\}^2} .$$

Therefore $\dfrac{\partial f}{\partial z_k} \in M$ whenever $f \in M$.

Now let $f_1(z), \cdots, f_n(z)$ be n elements of M which are algebraically independent over K and defined at 0. Since M is finite algebraic over $K(f_1, \cdots, f_n)$, the integral closure of $K[f_1, \cdots, f_n]$ in M is a finite $K[f_1, \cdots, f_n]$-module. Let $\{h_1, \cdots, h_m\}$ be a finite set of elements of M such that $K[h_1, \cdots, h_m]$

is the integral closure of $K[f_1, \cdots, f_n]$ in M. Then $h_j(0)$ $(j=1, \cdots, m)$ is also defined. Since $\frac{\partial}{\partial z_i} h_j(z)$ $(z \in \boldsymbol{C}^n)$ is defined whenever $h_j(z)$ is defined, $\frac{\partial}{\partial z_i} h_j$ belongs to the integral closure of $\boldsymbol{C}[h_1, \cdots, h_n]$ in $M \otimes_K \boldsymbol{C}$. But we have seen above that $\frac{\partial}{\partial z_i} h_j$ belongs to M. Therefore it belongs to $K[h_1, \cdots, h_m]$. Therefore $\frac{\partial}{\partial z_i}$ maps $K[h_1, \cdots, h_m]$ into itself.

Now let $g_1(z)=h_1(z), \cdots, g_m(z)=h_m(z)$, $g_{m+1}(z)=h_1(Lz), \cdots, g_{2m}(z)=h_m(Lz)$. Since L can be represented by the base $z_1, \cdots, z_n$ as (α_{ij}) with $\alpha_{ij} \in K$, $\frac{\partial}{\partial z_i}$ $(i=1, \cdots, n)$ maps $K[g_1, \cdots, g_{2m}]$ into itself. Let $x_1, \cdots, x_n$ be n elements of $D \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ with the property of (ii) of Theorem 2. Here we may assume $x_1, \cdots, x_n, Lx_1, \cdots, Lx_n \in D$, since we can multiply $x_1, \cdots, x_n$, $Lx_1, \cdots, Lx_n$ by any large natural number. By the Corollary of Lemma 1 we can apply Lemma 2. Then we see that the transcendental degree of $K(g_1, \cdots, g_{2m})$ is not greater than n. Since the transcendental degree of $K(h_1, \cdots, h_m)$ is n, we see that $h_1(Lz), \cdots, h_m(Lz)$ are algebraically dependent on $K(h_1, \cdots, h_m)$. Therefore, by Lemma 3, there is a natural number d such that $h_1(dLz), \cdots, h_m(dLz)$ are meromorphic functions on $\boldsymbol{C}^n/D$. Since $M \otimes_K \boldsymbol{C} = \boldsymbol{C}(h_1, \cdots, h_m)$, we see that the map $f(z) \mapsto f(dLz)$ induces a rational map of $M \otimes_K \boldsymbol{C}$ into itself. Therefore dL induces an endomorphism of $\boldsymbol{C}^n/D$, so that L belongs to $\mathrm{End}(A) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$. Q. E. D.

§ 3. The proof of Theorem 1.

Now we shall prove Theorem 1, using Theorem 2 of § 2. First we shall construct a family of abelian varieties following the methods of G. Shimura. For any $z \in H$, put $D_z = \chi(\mathcal{O})\binom{z}{1} \subset \boldsymbol{C}^2$. Then D_z is a lattice in $\boldsymbol{C}^2$. Let ρ be an element of $\mathcal{O}$ such that ρ^2 is a negative rational integer. Put $E_z\left(\chi(\alpha)\binom{z}{1}, \chi(\beta)\binom{z}{1}\right) = \mathrm{tr}_{B/\boldsymbol{Q}}(\rho\alpha\beta')$, where $\alpha, \beta \in B$, $'$ and $\mathrm{tr}_{B/\boldsymbol{Q}}$ denote the canonical involution and the reduced trace respectively. Then E_z determines a Riemann form on the complex torus $\boldsymbol{C}^2/D_z$. Let ι^* be a projective embedding $\boldsymbol{C}^2/D_z \to A_z \subset \boldsymbol{P}^m$ induced by E_z. Let C_z be the polarization of A_z which is induced by E_z. Since $\mathcal{O}$ is a ring, $\chi(\gamma)$ $(\gamma \in \mathcal{O})$ maps $D_z = \chi(\mathcal{O})\binom{z}{1}$ into itself. Let $\theta_z(\gamma)$ denote the element of $\mathrm{End}(A_z)$ defined by $\chi(\gamma)$. Then θ_z gives a ring isomorphism from $\mathcal{O}$ into $\mathrm{End}(A_z)$. Let P_z be the isomorphism class of the triple (A_z, C_z, θ_z). Then the Shimura map φ has the following property: $\boldsymbol{Q}(\varphi(z))$ is the field of moduli of P_z. (cf. Shimura [4], § 9, [5], § 5 and [7], § 6, Theorem 6.7.) Therefore, if $\varphi(z)$ is algebraic, P_z can be defined over a finite algebraic number field, i. e., A_z, a fixed polar divisor of C_z and all elements of $\theta_z(\mathcal{O})$ can be defined over a common finite algebraic number field (Shimura [**6**], p. 127, Proposition 1.5). Moreover, if z is not fixed by any non-scalar element of $\tilde{\Gamma}$, $\mathrm{End}(A_z)$ coincides with $\theta_z(\mathcal{O})$ (Shimura [**6**], p. 135, Proposition 4.2).

Now assume $z \in H$ and $\varphi(z)$ are both algebraic and that z is not fixed by any non-scalar element of $\tilde{\Gamma}$. Then there is a finite algebraic number field K satisfying the following conditions, (i) $\theta_z(B) = \mathrm{End}(A_z) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$, (ii) $\chi(B) \subseteq M_2(K)$, (iii) A_z, every element of $\mathrm{End}(A_z)$ and z are all rational over K. We shall prove Theorem 1 by showing that these (i), (ii), (iii) lead to a contradiction.

Let φ be the canonical homomorphism from $\boldsymbol{C}^2$ to $\boldsymbol{C}^2/D_z$. Let $\iota^* : \boldsymbol{C}^2/D_z \to A_z$ be as before and ι be the composite map $\iota^* \circ \varphi : \boldsymbol{C}^2 \to A_z$. Now fix a K-base of the tangent space of the origin of A_z and make a holomorphic map ι_0 from $\boldsymbol{C}^2$ to A_z as in § 2. Let β be the linear transformation of $\boldsymbol{C}^2$ satisfying $\iota = \iota_0 \circ \beta$. Then

$$(**) \qquad \chi_0(a) = \beta \chi(a) \beta^{-1} \qquad (a \in \mathcal{O})$$

is the analytic representation of the endomorphism $\theta_z(a)$ with respect to this analytic coordinate system ι_0.

Now $\chi(a)$ belongs to $M_2(K)$ by the assumption $\chi(\mathcal{O}) \subseteq M_2(K)$. Moreover, since $\theta_z(a)$ is an element of $\mathrm{End}(A_z)$, $\chi_0(a)$ also belongs to $M_2(K)$ by (ii) of Theorem 2. Since $B \otimes_{\boldsymbol{Q}} K \cong M_2(K)$, we have $M_2(K) = \beta M_2(K) \beta^{-1}$ from (**). Therefore $a \mapsto \beta a \beta^{-1}$ $(a \in \mathcal{O})$ induces an automorphism of $M_2(K)$. Therefore we can write $\beta = \nu\alpha$ with $\nu \in \boldsymbol{C}^\times$ and $\alpha \in GL(2, K)$, and we have

$$\chi_0(a) = \alpha \chi(a) \alpha^{-1},$$

$$D = \left\{ \nu\alpha\chi(\gamma)\begin{pmatrix} z \\ 1 \end{pmatrix} \,\middle|\, \gamma \in \mathcal{O} \right\}.$$

Let $\{\gamma_1 = 1, \gamma_2, \gamma_3, \gamma_4\}$ be a $\boldsymbol{Z}$-base of $\mathcal{O}$. Put

$$x_i = \nu\alpha\chi(\gamma_i)\begin{pmatrix} z \\ 1 \end{pmatrix} \in D \qquad (i = 1, 2, 3, 4).$$

Here we may assume that x_1 and x_2 are linearly independent over $\boldsymbol{C}$. Moreover we may assume

$$\gamma_3\gamma_2 \neq \gamma_4,$$

since we may replace γ_4 by $-\gamma_4$.

Now we note one fact. Let $\gamma, \delta \in B$. Then $\chi_0(\gamma)x_1 = \chi_0(\delta)x_1$ if and only if $\chi(\gamma)\begin{pmatrix} z \\ 1 \end{pmatrix} = \chi(\delta)\begin{pmatrix} z \\ 1 \end{pmatrix}$, hence if and only if $\chi(\delta^{-1}\gamma)\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix}$. But, since $\chi(\delta^{-1}\gamma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\boldsymbol{R})$ and $z \notin \boldsymbol{R}$, the last condition implies $a = 1$, $b = 0$,

$c=0$, $d=1$, hence $\delta^{-1}\gamma=1$. Therefore $\chi_0(\gamma)x_1=\chi_0(\delta)x_1$ if and only if $\gamma=\delta$.

Let L be the $\boldsymbol{C}$-linear endomorphism of $\boldsymbol{C}^2$ which maps x_1 and x_2 onto x_3 and x_4 respectively. Then L satisfies the statement (ii) of Theorem 2. (Observe that the vectors $\nu^{-1}x_i$ have components in K, since $z\in K$.) Therefore, by Theorem 2, L is an element of $\mathrm{End}(A_z)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$. Therefore, by the assumption $\mathrm{End}(A_z)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}=\theta_z(B)$, we may write $L=\chi_0(\gamma)$ with some $\gamma\in B$. Then, by the definition of L, we have

$$\chi_0(\gamma)x_1=L(x_1)=x_3=\chi_0(\gamma_3)x_1$$

and

$$\chi_0(\gamma\gamma_2)x_1=\chi_0(\gamma)x_2=Lx_2=x_4=\chi_0(\gamma_4)x_1\,.$$

Therefore we have

$$\gamma=\gamma_3$$

and

$$\gamma\gamma_2=\gamma_4\,,$$

hence $\gamma_3\gamma_2=\gamma_4$. This contradicts our assumption $\gamma_3\gamma_2\neq\gamma_4$. Q. E. D.

University of Tokyo

References

[1] W. L. Baily, On the theory of θ-functions, the moduli of abelian varieties, and the moduli of curves, Ann. of Math., **75** (1962), 342–381.

[2] S. Lang, Algebraic values of meromorphic functions, Topology, **3** (1965), 183–191.

[3] T. Schneider, Einführung in die transcendenten Zahlen, Springer, Berlin, 1957.

[4] G. Shimura, Construction of class fields and zeta functions of algebraic curves, Ann. of Math., **85** (1967), 58–159.

[5] G. Shimura, On the theory of automorphic functions, Ann. of Math., **70** (1959), 101–144.

[6] G. Shimura, On the field of definition for a field of automorphic functions: II, Ann. of Math., **81** (1965), 124–165.

[7] G. Shimura, Moduli and fibre systems of abelian varieties, Ann. of Math., **83** (1966), 294–338.

[8] G. Shimura, Algebraic number fields and symplectic discontinuous groups, Ann. of Math., **86** (1967), 503–592.