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On Eichler's trace formula

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Let $G=GL(2, R)^{+}$ be the subgroup of $GL(2, R)$ consisting of the elements of $GL(2, R)$ such as $\det>0$. Let H be the complex upper half plane. We regard G as a group of transformations on H. Let $Z(G)$ be the center of G. Let Γ be a subgroup of G operating on H discontinuously with a fundamental domain of finite volume. Let ι be the canonical homomorphism of G onto $G/Z(G)$. We fix once for all an element α in G such that $\alpha\Gamma\alpha^{-1}$ is commensurable with Γ and denote by Γ' the subgroup of G generated by Γ and α. Let χ be a linear character of Γ'. We assume that $\chi(\varepsilon)=1$ for $\varepsilon\in Z(\Gamma)=Z(G)\cap\Gamma$ and that the kernel Γ_{χ} of χ in Γ is of finite index in Γ. Let k be an even positive integer. We put

$$j(g, \tau)=(c\tau+d)^{k}(\det g)^{-\frac{k}{2}}$$

where $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. By a cusp form of type (Γ, k, χ), we understand a function $\varphi(\tau)$ on H taking values in the complex number field which satisfies the following conditions;

i) $\varphi(\tau)$ is holomorphic on H.

ii) $\varphi(\gamma\tau)=j(\gamma, \tau)\chi(\gamma)^{-1}\varphi(\tau)$ for any $\gamma\in\Gamma$.

iii) In case $H/\iota(\Gamma)$ is not compact, $\varphi(\tau)$ is regular at every parabolic point P of Γ_{χ} and the constant term in the Fourier expansion of φ at P vanishes.

The set of all such $\varphi(\tau)$ is denoted by $S(\Gamma, k, \chi)$. Let $\Gamma\alpha\Gamma=\bigcup_{\nu=1}^{d}\Gamma\alpha_{\nu}$ be a disjoint sum. For $\varphi\in S(\Gamma, k, \chi)$ we define a linear transformation $T=T(\Gamma\alpha\Gamma)$ in $S(\Gamma, k, \chi)$ by

$$(T(\Gamma\alpha\Gamma)\varphi)(\tau)=\sum_{\nu=1}^{d}j(\alpha_{\nu}, \tau)^{-1}\chi(\alpha_{\nu})\varphi(\alpha_{\nu}\tau)\,.$$

In principle, its trace $\mathrm{tr}\,T$ should be computable following the general method of A. Selberg [3]. It has been actually carried out by Shimizu [5] (in the more general setting than ours), but for some technical difficulties, the case of weight 2 is omitted. In this note, we follow the method of Eichler, and compute $\mathrm{tr}\,T$ including the case of weight 2. Since the case where χ is trivial on Γ is already treated by Eichler [1], we assume χ is non-trivial on Γ.

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Let $S(\Gamma_\chi, k)$ be the space of cusp forms with respect to Γ_χ of weight k, i.e. the forms satisfying the above (i), (ii) and (iii) with $\chi=$ identity in (ii). Let $\{\chi_i \mid 1 \leqq i \leqq r\}$ be the set of distinct characters of Γ such that $\ker \chi_i \supset \Gamma_\chi$. We take χ_1 to be the restriction of χ to Γ. Let S_i $(1 \leqq i \leqq r)$ be the subspaces of $S(\Gamma_\chi, k)$ consisting of the forms satisfying (ii) for $\chi=\chi_i$. Then $S(\Gamma_\chi, k)$ is the direct sum of S_i $(1 \leqq i \leqq r)$. Let $\varphi^i_{i(k)}$ $(1 \leqq i(k) \leqq \dim S_i)$ be a basis of S_i.

Let us recall fundamental definitions and notation of Eichler: a meromorphic automorphic form which vanishes at every parabolic point of Γ and has the principal part of the form $\sum \frac{c_m}{(\tau-\tau_0)^{k+m}}$ at τ_0 is called a cusp form of the second kind. To avoid a confusion (when it seems necessary) we shall refer to the cusp form of usual sense as to be "of the first kind". The integral $\frac{1}{(k-2)!}\int_{\tau_0}^{\tau}(\tau-\sigma)^{k-2}\varphi(\sigma)d\sigma$ is independent of its path, and if we replace τ_0 by τ_0', then the difference $\int_{\tau_0}^{\tau}-\int_{\tau_0'}^{\tau}$ is a polynomial of degree $k-2$ in τ. Hence as in [1, p. 272] we write simply

$$\Phi(\tau)=\frac{1}{(k-2)!}\int_{\tau_0}^{\tau}(\tau-\sigma)^{k-2}\varphi(\sigma)d\sigma \tag{1}$$

and understand that Φ is determined by φ up to addition of a polynomial of degree $k-2$. For any pair of cusp forms φ, ψ of the first or second kind, let

$$I(\varphi, \psi)=\frac{1}{2\pi i}\int_{\partial\mathfrak{F}_\chi}\Phi(\tau)\psi(\tau)d\tau$$

where $\mathfrak{F}_\chi$ is a fundamental domain for Γ_χ and $\partial\mathfrak{F}_\chi$ is its boundary. The value of this integral depends only on φ and ψ. Further we have $I(\varphi, \psi) = -I(\psi, \varphi)$. By [1, p. 282], we can find cusp forms of the second kind $\{\psi^i_{i(k)}\}$ satisfying

$$I(\varphi^i_{i(k)}, \psi^j_{j(k')})=\delta_{ij}\delta_{i(k)j(k')}\,,$$

$$I(\psi^i_{i(k)}, \psi^j_{j(k')})=0\,.$$

LEMMA 1. *Let $f_i(\tau)$ $(i=1, 2)$ be cusp forms of the first or second kind with respect to Γ_χ which satisfy*

$$f_i(\gamma\tau)=j(\gamma, \tau)\chi_i^{-1}(\gamma)f_i(\tau) \qquad \textit{for} \quad \gamma\in\Gamma\,.$$

Then

$$I(f_1, f_2)=0 \quad \textit{if} \quad \chi_1\neq\chi_2^{-1}\,.$$

PROOF. Let $\mathfrak{F}$, $\mathfrak{F}_\chi$ be the fundamental domain of Γ, Γ_χ, respectively. Let $\{\varepsilon_i\}$ be the representatives of Γ/Γ_χ, and hereafter we fix them. Then,

$$I(f_1, f_2)=\frac{1}{2\pi i}\int_{\partial\mathcal{F}_\chi} F_1(\tau)f_2(\tau)d\tau$$
$$=\frac{1}{2\pi i}\sum\int_{\partial(\varepsilon_i\mathcal{F})} F_1(\tau)f_2(\tau)d\tau$$

where $F_1(\tau)$ is defined by (1) from $f_1(\tau)$. By the definition

$$\int_{\partial(\varepsilon_i\mathcal{F})} F_1(\tau)f_2(\tau)d\tau=\chi_1(\varepsilon_i)^{-1}\chi_2(\varepsilon_i)^{-1}\int_{\partial\mathcal{F}} F_1(\tau)f_2(\tau)d\tau\,.$$

So we have

$$I(f_1, f_2)=(\sum\chi_1(\varepsilon_i)^{-1}\chi_2(\varepsilon_i)^{-1})\int_{\partial\mathcal{F}} F_1(\tau)f_2(\tau)d\tau$$
$$=0 \quad\text{if}\quad \chi_1\neq\chi_2^{-1}\,.$$

With this lemma we may assume that

$$\phi^i_{i(k)}(\gamma\tau)=j(\gamma, \tau)\chi_i(\gamma)\phi^i_{i(k)}(\tau) \qquad \text{for} \quad \gamma\in\Gamma\,.$$

Let us further recall the results of Eichler with some comments. Set

$$K_1(\tau, \sigma)=\sum\varphi^i_{i(k)}(\tau)\Psi^i_{i(k)}(\sigma)$$

where $\Psi^i_{i(k)}(\sigma)$ is defined by (1) from $\phi^i_{i(k)}(\sigma)$. For $K_1(\tau, \sigma)$, there exists a function $K_0(\tau, \sigma)$ which satisfies the following conditions i)~v) [**1**, p. 290], [**2**, p. 222]. Let $K(\tau, \sigma)=-K_1(\tau, \sigma)+K_0(\tau, \sigma)$, then

i) $K_0(\tau, \sigma)$ is a modular form of weight $-(k-2)$ with respect to Γ_χ in σ.

ii) For every cusp form $\varphi(\tau)$ of the first kind

$$\frac{1}{2\pi i}\int_{\partial\mathcal{F}_\chi} K(\tau, \sigma)\varphi(\sigma)d\sigma=\varphi(\tau)\,.$$

iii) $\dfrac{\partial^{k-1}}{\partial\sigma^{k-1}}K(\tau, \sigma)=\dfrac{\partial^{k-1}}{\partial\tau^{k-1}}K(\sigma, \tau)$.

(This formula is not proved in [1], but can be obtained easily by considering the integral $\frac{1}{2\pi i}\int_{\partial\mathcal{F}_\chi} K(\tau_1, \sigma)\frac{\partial^{k-1}}{\partial\sigma^{k-1}}K(\tau_2, \sigma)d\sigma.$)

iv) $K(\tau, \sigma)$ is a holomorphic function in τ and σ except at such points as $\sigma=\gamma\tau$, $\gamma\in\Gamma_\chi$ and at the point $\sigma=\tau$, $K(\tau, \sigma)$ is of the form

$$K(\tau, \sigma)=\frac{1}{\sigma-\tau}+\cdots\cdots\cdots\cdots$$

where······denotes a holomorphic function in τ and σ.

v) For $k>2$, $K_0(\tau, \sigma)$ is a cusp form in τ of weight k with respect to Γ_χ; in case $k=2$, $K(\tau, \sigma)-K(\tau, \sigma_0)$ is a cusp form in τ of weight 2 for any σ_0. (It is not stated explicitly in [1] that $K(\tau, \sigma)$ is a meromorphic function and a cusp form in τ. But this fact can be proved easily by noting iii) and considering the integral $\int_{\sigma_0}^{\sigma}(\sigma-\xi)^{k-2}\frac{\partial^{k-1}}{\partial\tau^{k-1}}K(\xi, \tau)d\xi.$)

We put

$$K'(\tau, \sigma)=\frac{1}{[\Gamma:\Gamma_\chi]}\sum\chi(\varepsilon_i)K(\tau, \sigma)[\varepsilon_i]_\tau^{-k}$$

where

$$\varphi(\tau)[\rho]^{-l}=\varphi(\rho\tau)(c\tau+d)^{-l}(\det\rho)^{\frac{l}{2}} \qquad \text{for} \quad \rho\in G\,,$$

and for an integer l. For simplicity let us write $\varphi_i(\tau)$, $\Psi_i(\sigma)$ instead of $\varphi_i^1(\tau)$, $\Psi_i^1(\sigma)$, then

$$K'(\tau, \sigma)=-\sum\varphi_i(\tau)\Psi_i(\sigma)+K_0'(\tau, \sigma)$$

where

$$K_0'(\tau, \sigma)=\frac{1}{[\Gamma:\Gamma_\chi]}\sum\chi(\varepsilon_i)K_0(\tau, \sigma)[\varepsilon_i]_\tau^{-k}\,.$$

LEMMA 2. *For $k>2$, $K_0'(\tau, \sigma)$ satisfies*

$$K_0'(\tau, \sigma)[\gamma]_\tau^{-k}=\chi(\gamma)^{-1}K_0'(\tau, \sigma)\,,$$

$$K_0'(\tau, \sigma)[\gamma]_\sigma^{k-2}=\chi(\gamma)K_0'(\tau, \sigma) \qquad \textit{for} \quad \gamma\in\Gamma\,.$$

In case $k=2$, we consider

$$\frac{1}{[\Gamma:\Gamma_\chi]}\sum\chi(\varepsilon_i)^{-1}(K_0(\tau, \sigma)-K_0(\tau, \sigma_0))[\varepsilon_i]_\sigma^0$$
$$=\frac{1}{[\Gamma:\Gamma_\chi]}\sum\chi(\varepsilon_i)^{-1}K_0(\tau, \sigma)[\varepsilon_i]_\sigma^0$$

and

$$\frac{1}{[\Gamma:\Gamma_\chi]}\sum\chi(\varepsilon_i)^{-1}(K(\tau, \sigma)-K_0(\tau, \sigma_0))[\varepsilon_i]_\sigma^0$$
$$=\frac{1}{[\Gamma:\Gamma_\chi]}\sum\chi(\varepsilon_i)^{-1}K(\tau, \sigma)[\varepsilon_i]_\sigma^0$$

and denote anew the left hand sides of the above equations by $K_0'(\tau, \sigma)$ and $K'(\tau, \sigma)$, respectively. Then we have

$$K'(\tau, \sigma)=-\sum\varphi_i(\tau)\Psi_i(\sigma)+K_0'(\tau, \sigma)$$

and $K_0'(\tau, \sigma)$ satisfies the first two equations with $k=2$.

PROOF. The first formula follows from the definition. To prove the second we consider

$$\chi(\varepsilon_j)K'(\tau, \sigma)[\varepsilon_j^{-1}]_\sigma^{k-2}-K'(\tau, \sigma)\,.$$

Then by the same argument as in [1, p. 290], we can prove

$$\chi(\varepsilon_j)K_0'(\tau, \sigma)[\varepsilon_j^{-1}]^{k-2}-K_0'(\tau, \sigma)$$

is a holomorphic modular form of weight $-(k-2)$ in σ with respect to Γ_χ. So we obtained the second formula. As for the case $k=2$, the statement is obvious.

Now we calculate the trace of $T=T(\Gamma\alpha\Gamma)$. By the definition

$$\begin{aligned}\operatorname{tr} T&=\sum I(\varphi_i T,\ \phi_i)\\&=\frac{-1}{2\pi i}\int_{\partial\mathfrak{F}_\chi}\Big(\sum\varphi_i(\tau)T\Psi_i(\tau)\Big)d\tau\,.\end{aligned}$$

By the definition

$$-\sum\varphi_i(\tau)T\Psi_i(\tau)=[K'(\tau,\ \sigma)T_\tau-K'_0(\tau,\ \sigma)T_\tau]_{\tau=\sigma}\,.$$

So we have

$$\operatorname{tr} T=\frac{1}{2\pi i}\int_{\partial\mathfrak{F}_\chi}[K'(\tau,\ \sigma)T_\tau-K'_0(\tau,\ \sigma)T_\tau]_{\tau=\sigma}d\tau\,.$$

By lemma 2, and by the same calculation as in lemma 1

$$\operatorname{tr} T=\frac{[\Gamma:\Gamma_\chi]}{2\pi i}\int_{\partial\mathfrak{F}}[K'(\tau,\ \sigma)T_\tau-K'_0(\tau,\ \sigma)T_\tau]_{\tau=\sigma}d\tau\,.$$

If $\Gamma\alpha\Gamma\cap Z(G)=\emptyset$, then $[K'_0(\tau,\ \sigma)T_\tau]_{\tau=\sigma}d\tau$ is a differential form on the Riemann surface $H/\iota(\Gamma)$. So we have

$$\operatorname{tr} T=\frac{[\Gamma:\Gamma_\chi]}{2\pi i}\int_{\partial\mathfrak{F}}[K'(\tau,\ \sigma)T_\tau]_{\tau=\sigma}d\tau\,.$$

In the other case we take some $g_0\in\Gamma\alpha\Gamma\cap Z(G)$ and fix it, then

$$\begin{aligned}\operatorname{tr} T=\frac{[\Gamma:\Gamma_\chi]}{2\pi i}\Big(&\int_{\partial\mathfrak{F}}\Big[K'(\tau,\ \sigma)T_\tau-\frac{1}{[\Gamma:\Gamma_\chi]}\cdot\frac{\chi(g_0)}{\sigma-\tau}\Big]_{\tau=\sigma}d\tau\\&-\int_{\partial\mathfrak{F}}\Big[K'_0(\tau,\ \sigma)T_\tau-\frac{1}{[\Gamma:\Gamma_\chi]}\cdot\frac{\chi(g_0)}{\sigma-\tau}\Big]_{\tau=\sigma}d\tau\Big)\,.\end{aligned}$$

To compute the second term of the above equation, we divide $\partial\mathfrak{F}$ into disjoint several pieces, each of which being paired to another by some $\varepsilon_\mu\in\Gamma$, $\partial\mathfrak{F}=\cup(A_\mu\cup\varepsilon_\mu A_\mu)$. Then

$$\begin{aligned}&[\Gamma:\Gamma_\chi]\int_{\partial\mathfrak{F}}\Big[K'_0(\tau,\ \sigma)T_\tau-\frac{1}{[\Gamma:\Gamma_\chi]}\cdot\frac{\chi(g_0)}{\sigma-\tau}\Big]_{\tau=\sigma}d\tau\\&=\sum_\mu\Big(\int_{A_\mu}+\int_{\varepsilon_\mu A_\mu}\Big)\\&=-\frac{k-1}{2}\sum\chi(g_0)\int_{A_\mu}d\log\frac{d\varepsilon_\mu(\tau)}{d\tau}\\&=-\frac{k-1}{2}i\,v(\mathfrak{F})\chi(g_0)\end{aligned}$$

where $v(\mathfrak{F})$ is the measure of $\mathfrak{F}$ with respect to the invariant measure $\frac{dx\,dy}{y^2}$, $\tau=x+iy$. So we have

$$\operatorname{tr} T=\frac{k-1}{4\pi}v(\mathcal{F})\chi(g_0)+\frac{[\Gamma:\Gamma_\chi]}{2\pi i}\int_{\partial\mathcal{F}}\Big[K'(\tau,\sigma)T_\tau-\frac{1}{[\Gamma:\Gamma_\chi]}\cdot\frac{\chi(g_0)}{\sigma-\tau}\Big]_{\tau=\sigma}d\tau\,.$$

Let $K_T(\tau)=[\Gamma:\Gamma_\chi][K'(\tau,\sigma)T_\tau]_{\tau=\sigma}$ (resp. $[\Gamma:\Gamma_\chi][K'(\tau,\sigma)T_\tau-[\Gamma:\Gamma_\chi]^{-1}\times\chi(g_0)(\sigma-\tau)^{-1}]_{\tau=\sigma}$), if $\Gamma\alpha\Gamma$ has no element in $Z(G)$ (resp. $\Gamma\alpha\Gamma$ has an element in $z(G)$). Then $K_T(\tau)d\tau$ has poles of the first order at the fixed points of elements of $\Gamma\alpha\Gamma$ and nowhere else. We put

(2) $$S_0=\begin{cases}\dfrac{k-1}{4\pi}v(\mathcal{F})\chi(g_0) & \text{if}\quad \Gamma\alpha\Gamma\cap Z(G)\neq\emptyset\\ 0 & \text{otherwise}.\end{cases}$$

Here g_0 is an arbitrary element in $\Gamma\alpha\Gamma\cap Z(G)$. Let S_1 (resp. S_2) denote the sum of residue at elliptic points (resp. at cusps), then

$$\operatorname{tr} T=S_0+S_1+S_2\,.$$

To compute S_1, S_2 explicitly let C_1 (resp. C_2, C_3) denote the complete system of inequivalent elliptic element (resp. hyperbolic, parabolic elements which fix a cusp of Γ) in $\Gamma\alpha\Gamma$ by the equivalence relation defined by $g\sim g'\Leftrightarrow g'=\varepsilon\gamma g\gamma^{-1}$, $\gamma\in\Gamma$, $\varepsilon\in\Gamma\cap Z(G)$. Let $\Gamma(g)=\{\gamma\in\Gamma\mid g=\varepsilon\gamma g\gamma^{-1}$ for some $\varepsilon\in\Gamma\cap Z(G)\}$. Then by the same way as in [**1**, p. 292],

(3) $$S_1=\sum_{g\in c_1}\chi(g)\frac{1}{[\Gamma(g):Z(\Gamma)]}\cdot\frac{\rho(g)^{*k-1}}{\rho(g)-\rho(g)^*}(\det g)^{1-\frac{k}{2}}$$

where $\rho(g)$, $\rho(g)^*$ are the characteristic roots of g and they are determined in the following way; if τ_0 is the fixed point of g in H, then

$$\rho(g)=\frac{a+d+\sqrt{(a+d)^2-4(ad-bc)}}{2}=c\tau_0+d\,,$$

$$\rho(g)^*=\frac{ad-bc}{\rho(g)}\qquad\text{for}\quad g=\begin{pmatrix}a&b\\c&d\end{pmatrix}.$$

For a cusp P, we denote by $\Gamma(P)$ (resp. $\Gamma_\chi(P)$) the union of parabolic elements in Γ (resp. Γ_χ) which fix P and $Z(\Gamma)$. To compute S_2 we transform P to ∞ by an element in G so that the generator of $\iota(\Gamma(P))$ is $\begin{pmatrix}1&1\\0&1\end{pmatrix}$. Let q be the index of $\Gamma_\chi(P)$ in $\Gamma(P)$, and $g_j=\begin{pmatrix}1&c(g_j)\\0&1\end{pmatrix}$ be the representatives of $\Gamma(P)/\Gamma_\chi(P)$. Then the principal part of $[\Gamma:\Gamma_\chi]K'(\tau,\sigma)$ at P is of the form

$$\frac{2\pi i}{q}\sum_{g_j}\frac{\chi(g_j)e^{2\pi i\frac{\tau+c(g_j)}{q}}}{e^{2\pi i\frac{\sigma}{q}}-e^{2\pi i\frac{\tau+c(g_j)}{q}}}\,.$$

An element which fix P in $\Gamma\alpha\Gamma$ is of the form $\begin{pmatrix}a&b\\0&d\end{pmatrix}$ and we may assume that a and d are positive integers. If $a\neq d$, by the same way as in [**1**, p. 293], the residue at P corresponding to the equivalence class in C_2 which contains $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is 0 or $-\chi(g)\dfrac{a^{k-1}}{d-a}(\det g)^{1-\frac{k}{2}}$ according to $a>d$, or $a<d$. We note here that the other fixed point is also a cusp of Γ [5, p. 9], [4, p. 60] and the residue of $K_T(\tau)d\tau$ at this cusp corresponding to the class of $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is 0, or $-\chi(g)\dfrac{d^{k-1}}{a-d}(\det g)^{1-\frac{k}{2}}$ according to $a<d$, $a>d$ respectively. We put

$$S_2' = -\sum_{g\in C_2}\chi(g)\frac{\mathrm{Min.}(|a|^{k-1}, |d|^{k-1})}{|d-a|}(\det g)^{1-\frac{k}{2}} \tag{4}$$

where a, d are the characteristic roots of g. If $a=d$, we note that

$$\frac{2\pi i}{q}\left[\frac{e^{2\pi i\frac{\tau}{q}}}{e^{2\pi i\frac{\sigma}{q}}-e^{2\pi i\frac{\tau}{q}}}-\frac{q}{2\pi i}\frac{1}{\sigma-\tau}\right]_{\tau=\sigma}=-\frac{1}{2q}.$$

If we denote by $\Gamma\alpha\Gamma(P)$ the union of the parabolic elements which fix P and $\Gamma\alpha\Gamma\cap Z(G)$ and if we write $g=\begin{pmatrix} d & c(g) \\ 0 & d \end{pmatrix}$ for $g\in\Gamma\alpha\Gamma(P)$, then the residue corresponding to parabolic elements is

$$\begin{aligned} R_P &= \frac{1}{q}\Big(\sum_{\substack{g\in(\Gamma\alpha\Gamma(P)/\Gamma_\chi(P))-\\(\Gamma\alpha\Gamma(P)\cap Z(G))/\Gamma_\chi(P)}}\chi(g)\frac{e^{2\pi i\frac{c(g)}{qd}}}{1-e^{2\pi i\frac{c(g)}{qd}}}-\frac{1}{2}\chi(g_0)\Big) \\ &= -\frac{1}{2q}\sum_{g\in\Gamma\alpha\Gamma(P)/\Gamma_\chi(P)}\chi(g)+\frac{1}{2qi}\sum_{\substack{g\in(\Gamma\alpha\Gamma(P)/\Gamma_\chi(P))-\\(\Gamma\alpha\Gamma(P)\cap Z(G))\Gamma_\chi(P)}}\chi(g)\cot\frac{c(g)}{qd}\pi. \end{aligned}$$

We put

$$S_2'' = \sum_P R_P. \tag{5}$$

Then we have

THEOREM.

$$\mathrm{tr}\, T = S_0+S_1+S_2'+S_2''.$$

S_0, S_1, S_2' and S_2'' are given by (2), (3), (4) *and* (5) *respectively.*

REMARK. Suppose there exists an element $\gamma\in GL(2, R)$ satisfying the following conditions

$$\gamma\Gamma\gamma^{-1}=\Gamma, \qquad \gamma\Gamma\alpha\Gamma\gamma^{-1}=\Gamma\alpha\Gamma,$$
$$\chi(\gamma g\gamma^{-1})=\chi(g) \qquad \text{for } g\in\Gamma' \text{ and } \det\gamma<0.$$

i) We can choose C_1 such that $\gamma C_1\gamma^{-1}=C_1$. Then since $\rho(g)=\rho^*(\gamma g\gamma^{-1})$, the above S is equal to the following

$$S_1^* = -\frac{1}{2}\sum_{g\in C_1}\chi(g)\frac{1}{[\Gamma(g):Z(\Gamma)]}\cdot\frac{\rho(g)^{k-1}-\rho(g)^{*k-1}}{\rho(g)-\rho(g)^*}(\det g)^{1-\frac{k}{2}}.$$

ii) We can choose the representatives of cups $\{P\}$ of Γ such that $\gamma\{P\}=\{P\}$. Then

$$R_P+R_{\gamma P}=-\frac{1}{2q}\Big(\sum_{g\in\Gamma\alpha\Gamma(P)/\Gamma_\chi(P)}\chi(g)+\sum_{g\in\Gamma\alpha\Gamma(\gamma P)/\Gamma_\chi(\gamma P)}\chi(g)\Big),$$

hence the above is equal to the following

$$S_2^*=-\frac{1}{2}\sum_P\frac{1}{[\Gamma(P):\Gamma_\chi(P)]}\sum_{g\in\Gamma\alpha\Gamma(P)/\Gamma_\chi(P)}\chi(g).$$

Finally this S can easily be seen to be equal to the corresponding term

$$\lim_{s\to 0}-\frac{s}{4}\sum_{g\in C_3}\chi(g)\Big(\frac{d(g)}{m(g)}\Big)^{1+s}$$

in Shimizu [5, p. 13].

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