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# On elliptic modular surfaces

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## Introduction

The purpose of this paper is to study a certain class of algebraic elliptic surfaces called elliptic modular surfaces from both analytic and arithmetic point of view. Our results are based on the general theory of elliptic surfaces due to Kodaira [**11**].

Let $B$ denote an (algebraic) elliptic surface having a global section over its base curve $\Delta$. We denote by $J$ and $G$ the functional and homological invariants of $B$ over $\Delta$, and by $\mathcal{F}(J, G)$ the family of (not necessarily algebraic) elliptic surfaces over $\Delta$ with the functional and homological invariants $J$ and $G$. We assume throughout the paper that $J$ is non-constant and that the fibres of $B$ over $\Delta$ contain no exceptional curves of the first kind. The part I is devoted to the generalities on such an elliptic surface $B$. In § 1, we give an explicit description of the structure of the Néron-Severi group of $B$; for the sake of later use, the results are formulated over an arbitrary algebraically closed ground field. In § 2, we compute the cohomology groups $H^i(\Delta, G)$ of the base curve (or Riemann surface) $\Delta$ with coefficients in the sheaf $G$ following Kodaira. This gives an analytic proof of the so-called Ogg-Šafarevič's formula. In § 3, it is shown that, in the family $\mathcal{F}(J, G)$ of analytic elliptic surfaces, algebraic surfaces are "dense" (Theorem 3.2); this answers a question raised by Kodaira. The results in § 1 or 2 must be well-known, but they are included here because we could not find a suitable reference.

In the part II, we develop the analytic theory of elliptic modular surfaces. First in § 4 we define the elliptic modular surface $B_\Gamma$ for each subgroup $\Gamma$ of finite index of $SL(2, \boldsymbol{Z})$ such that $\Gamma \not\ni -1$, and examine its singular fibres and numerical characters. The base $\Delta_\Gamma$ of $B_\Gamma$ is the compact Riemann surface associated with $\Gamma$. In § 5, we show that the group of global sections of an elliptic modular surface $B_\Gamma$ over $\Delta_\Gamma$ is a finite group (Theorem 5.1). In other words, the generic fibre of $B_\Gamma$ is an elliptic curve defined over the field $K_\Gamma$ of modular functions with respect to $\Gamma$, and it has only a finite

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number of $K_{\Gamma}$-rational points. A few examples are given. In § 6, we prove that the space $S_3(\Gamma)$ of $\Gamma$-cusp forms of weight 3 is canonically isomorphic to the space of holomorphic 2-forms on $B_{\Gamma}$ (Theorem 6.1). In § 7, we give a geometric interpretation of Shimura's complex torus attached to $S_3(\Gamma)$; namely it is essentially the parameter space of the family $\mathcal{F}(J, G)$ containing $B_{\Gamma}$ (Theorem 7.3 and Remark 7.5). Moreover, the group of division points of this complex torus has an algebro-geometric (or rather arithmetic) meaning as essentially the group of locally trivial algebraic principal homogeneous spaces for $B_{\Gamma}$ over $\Delta_{\Gamma}$.

In the appendix, we shall consider arithmetic questions concerning elliptic modular surfaces. As is well-known, the fibre systems of (self-products of) elliptic curves or abelian varieties parametrized by a curve $\Gamma\backslash\mathfrak{H}$ ($\Gamma$ being a certain arithmetic subgroup of $SL(2, \boldsymbol{R})$) have been considered by several people—notably by Sato, Kuga, Shimura, Ihara, Morita and Deligne ([**14**], [**8**], [**16**], [**2**]). Their main result was to establish the relation between the local zeta functions of the fibre varieties and the Hecke polynomials, and thereby to reduce the Ramanujan-Petersson conjecture on the eigenvalues of Hecke operators acting on the space $S_w(\Gamma)$ of $\Gamma$-cusp forms to the Weil conjecture on the absolute values of the zeroes and poles of the zeta function of a (non-singular complete) variety defined over a finite field. In these treatments the fibre varieties seem to play a somewhat auxiliary role (except for [**14**]). Now we think it worthwhile to study a suitable compactification of such a fibre variety as an example of a non-singular complete variety. In particular, the elliptic modular surfaces $B_{\Gamma}$ (attached to certain groups $\Gamma$) will provide important examples for the arithmetic theory of non-singular complete surfaces. From such a viewpoint, we recall the algebraic formulation of the elliptic modular surface of level $n$ ($n \geqq 3$) in section A, and the arithmetic theory of surfaces in section B. Then we determine the zeta function of the elliptic modular surface of level $n$ in characteristic $p$, using the results of Deligne [**2**] or Ihara [**9**]. We discuss the validity of various conjectures for such a surface.

The main results of this paper were announced in two short notes [**24**].

We wish to take this opportunity to thank Professor Kodaira for his interest in this work and for showing us his notes on the results of § 2. We also wish to thank Professor Shimura for several valuable remarks.

## Part I. Generalities.

### § 1. Néron-Severi group of an elliptic surface.

We fix an algebraically closed field $k$. Let $\Delta$ denote a non-singular projective curve over $k$ and let $B$ denote a non-singular projective surface having a structure of an elliptic surface over $\Delta$ with the canonical projection $\Phi: B \to \Delta$. We assume that $B$ admits a section $o$ over $\Delta$ and that the fibres of $B$ contain no exceptional curves of the first kind. In the following, we shall describe the structure of the Néron-Severi group $NS(B)$ of the surface $B$, i. e. the group of algebraic equivalence classes of divisors on $B$.

For that purpose we first consider the fibre $E=\Phi^{-1}(u)$ of $B$ over the generic point $u$ of $\Delta$. $E$ is an elliptic curve defined over the function field $K=k(u)$ of $\Delta$, given with a $K$-rational point $o=o(u)$. Let $E(K)$ denote the group of $K$-rational points of $E$. Then, by the Mordell-Weil theorem ([**15**] p. 71), $E(K)$ is a finitely generated abelian group provided that the absolute invariant $J$ of $E$ is transcendental over the constant field $k$; we always assume that this condition is satisfied in what follows. Let $r$ be the rank of $E(K)$ and take $r$ generators $s_1, \cdots, s_r$ of $E(K)$ modulo the torsion subgroup $E(K)_{\mathrm{tor}}$. $E(K)_{\mathrm{tor}}$ is generated by at most two elements $t_1$, $t_2$ of order $e_1$, $e_2$ with $1 \leqq e_2$, $e_2 | e_1$; $|E(K)_{\mathrm{tor}}| = e_1 e_2$. Now the group $E(K)$ of $K$-rational points of $E$ is canonically identified with the group of sections of $B$ over $\Delta$. For each $s \in E(K)$, we denote by $(s)$ the image (curve) in $B$ of the section corresponding to $s$. We put

$$(1.1) \qquad D_\alpha = (s_\alpha) - (o), \qquad 1 \leqq \alpha \leqq r,$$

$$D'_\beta = (t_\beta) - (o), \qquad \beta = 1, 2.$$

Next we consider the singular fibres of $B$ over $\Delta$. The classification of singular fibres are given in Kodaira [**11**] (cited as [**K**] in the following) or in Néron [**17**]. We shall follow Kodaira's notation. Let $\Sigma$ denote the finite set of points $v$ of $\Delta$ for which $C_v = \Phi^{-1}(v)$ is a singular fibre. For each $v \in \Sigma$, we denote by $\Theta_{v,i}$ $(0 \leqq i \leqq m_v - 1)$ the irreducible components of the divisor $C_v$, $m_v$ being the number of irreducible components. We take $\Theta_{v,0}$ to be the unique components of $C_v$ containing the identity $o(v)$. Then we have

$$(1.2) \qquad C_v = \Theta_{v,0} + \sum_{i \geqq 1} \mu_{v,i} \Theta_{v,i}, \qquad \mu_{v,i} \geqq 1.$$

Let $A_v$ denote the square matrix of size $(m_v - 1)$ whose $(i, j)$-coefficient is $(\Theta_{v,i} \Theta_{v,j})$ $(i, j \geqq 1)$, where $(DD')$ denotes the intersection number of the divisors $D$ and $D'$ on $B$. Finally we take and fix a non-singular fibre $C_{u_0}$ $(u_0 \notin \Sigma)$. With these notations, we can state

THEOREM 1.1. *The Néron-Severi group $NS(B)$ of the elliptic surface $B$ is generated by the following divisors:*

$$C_{u_0}, \quad \Theta_{v,i}\ (1 \leqq i \leqq m_v - 1,\ v \in \Sigma), \tag{1.3}$$
$$(o), \quad D_\alpha\ (1 \leqq \alpha \leqq r) \text{ and } D_\beta'\ (\beta = 1, 2).$$

*The fundamental relations among them are given by (at most) two relations $(\beta = 1, 2)$:*

$$e_\beta D_\beta' \approx e_\beta(D_\beta'(o)) \cdot C_{u_0} + \sum_v (\Theta_{v,1}, \cdots, \Theta_{v,m_v-1}) e_\beta A_v^{-1} \begin{pmatrix} (D_\beta'\Theta_{v,1}) \\ \vdots \\ (D_\beta'\Theta_{v,m_v-1}) \end{pmatrix}^{*)} . \tag{1.4}$$

LEMMA 1.2. *Any two fibres $C_{u_1}$ and $C_{u_2}$ ($u_1, u_2 \in \Delta$) are algebraically equivalent to each other. In particular, for $v \in \Sigma$,*

$$C_{u_0} \approx \Theta_{v,0} + \sum_{i \geqq 1} \mu_{v,i} \Theta_{v,i} . \tag{1.5}$$

PROOF. This is clear from the definition of algebraic equivalence. (1.5) follows from (1.2).

LEMMA 1.3. *The matrix*

$$A_v = ((\Theta_{v,i}\Theta_{v,j}))_{1 \leqq i,j \leqq m_v - 1}$$

*is negative definite and the absolute value of* $\det(A_v)$ *is equal to the number $m_v^{(1)}$ of simple components of $C_v$. Moreover, the group $A_v^{-1}\boldsymbol{Z}^{m_v-1}/\boldsymbol{Z}^{m_v-1}$ is isomorphic to the finite abelian group $C_v^\#/\Theta_{v,0}^\#$ attached to the singular fibre $C_v$* (see below and [**K**] p. 604).

PROOF. This can be checked case by case for each type of singular fibre ([**K**], § 6). For example, for the singular fibre $C_v$ of type II*, $-A_v$ gives a positive definite even integral quadratic form of discriminant 1 in 8 variables.

LEMMA 1.4. *Suppose that a divisor $D$ on $B$ does not meet the generic fibre $E$. Then the algebraic equivalence class of $D$ is uniquely expressed as a linear combination of $C_{u_0}$ and $\Theta_{v,i}$ ($v \in \Sigma$, $i \geqq 1$):*

$$D \approx (D(o)) C_{u_0} + \sum_v (\Theta_{v,1}, \cdots, \Theta_{v,m_v-1}) A_v^{-1} \begin{pmatrix} (D\Theta_{v,1}) \\ \vdots \\ (D\Theta_{v,m_v-1}) \end{pmatrix} .$$

PROOF. By assumption, each component of $D$ is contained in a fibre. Hence the assertion follows immediately from Lemmas 1.2 and 1.3.

PROOF OF THEOREM 1.1. First we show that an arbitrary divisor $D$ on $B$ is algebraically equivalent to a linear combination of divisors in (1.3). By Lemma 1.2 we may assume that no component of $D$ is contained in a fibre. If we put $d = (DC_u)$, the divisor $D - d(o)$ cuts out on the generic fibre $E$ a divisor $\mathfrak{d}$ of degree zero. The sum $S(\mathfrak{d})$ of points in $\mathfrak{d}$ gives a $K$-rational

*) The symbol $\approx$ indicates algebraic equivalence.

point of $E$, say $s$. Since $E(K)$ is generated by $s_1, \cdots, s_r$ and $t_1$, $t_2$, we can write

$$s=\sum_{\alpha=1}^{r} a_\alpha s_\alpha + \sum_{\beta=1}^{2} b_\beta t_\beta ,$$

where $a_\alpha$, $b_\beta$ are integers. Putting

$$D'=\sum_{\alpha} a_\alpha D_\alpha + \sum_{\beta} b_\beta D'_\beta ,$$

we see that $S(\mathfrak{d})=S(D'\cdot E)$. By Abel's theorem on an elliptic curve, the divisor $\mathfrak{d}$ is linearly equivalent to $D'\cdot E$ on $E$. Therefore the divisor $D-d(o)-D'$ does not meet the generic fibre. Applying Lemma 1.4, we conclude that $D$ is algebraically equivalent to a linear combination of divisors in (1.3).

To prove the second part of the theorem, suppose that there is a relation:

$$\sum_{\alpha} a_\alpha D_\alpha + \sum_{\beta} b_\beta D'_\beta + cC_{u_0} + \sum_{v}\sum_{i\geqq 1} d_{v,i}\Theta_{v,i} + e(o) \approx 0 , \tag{1.6}$$

with integers $a_\alpha, b_\beta, \cdots, e$. By considering the intersection number with $C_{u_0}$, we get $e=0$. Since the Picard variety of $B$ is canonically isomorphic to the Jacobian variety of $\Delta$, the left side of (1.6) is linearly equivalent to a divisor of the form $\Phi^{-1}(\mathfrak{c})$, where $\mathfrak{c}$ is a divisor of degree zero on $\Delta$. Restricting it to the generic fibre $E$, we get

$$\sum_{\alpha} a_\alpha D_\alpha\cdot E + \sum_{\beta} b_\beta D'_\beta\cdot E \sim 0 .^{**)}$$

Again by Abel's theorem, this is equivalent to

$$\sum_{\alpha} a_\alpha s_\alpha + \sum_{\beta} b_\beta t_\beta = 0 .$$

Hence we get $a_\alpha=0$ $(1\leqq\alpha\leqq r)$ and $b_\beta\equiv 0 \mod e_\beta$ $(\beta=1, 2)$. On the other hand Lemma 1.4 implies the relations (1.4) for $\beta=1, 2$ and also that $C_{u_0}$ and $\Theta_{v,i}$ $(v\in\Sigma,\ i\geqq 1)$ are independent modulo algebraic equivalence. Therefore the relation (1.6) is a consequence of the relations (1.4) for $\beta=1$ and 2. This completes the proof of Theorem 1.1.

The rank of Néron-Severi group $NS(B)$ is called the Picard number of $B$. Obviously we get (cf. [27] p. 15)

COROLLARY 1.5. *The Picard number $\rho$ of the surface $B$ is given by*

$$\rho=r+2+\sum_{v\in\Sigma}(m_v-1) .$$

Let $C_v^\#$ denote the set of points of multiplicity one on the divisor $C_v$. As is shown in [**K**] § 9 or [17] Ch. III, $C_v^\#$ is a commutative algebraic group with identity $o(v)$, in which $\Theta_{v,0}^\#=\Theta_{v,0}\cap C_v^\#$ is the connected component of the identity. If $C_v$ is a singular fibre of type $I_b$ $(b\geqq 1)$, then $\Theta_{v,0}^\#$ is a multi-

**) The symbol $\sim$ indicates linear equivalence.

plicative group and the quotient group $C_v^{\#}/\Theta_{v,0}^{\#}$ is a cyclic group of order $b$. If $C_v$ is a singular fibre of other type, then $\Theta_{v,0}^{\#}$ is an additive group and $C_v^{\#}/\Theta_{v,0}^{\#}$ is a group of order at most 4.

PROPOSITION 1.6. *Let $E(K)_0$ be the subgroup of $E(K)$ consisting of $s$ such that $s(v)\in\Theta_{v,0}^{\#}$ for all $v\in\Sigma$. Then $E(K)_0$ is a torsion-free subgroup of finite index in $E(K)$.*

PROOF. Suppose that $s$ is an element of $E(K)_0$ of finite order $n>1$. Applying Lemma 1.4 to the divisor $D=n[(s)-(o)]$, we get

$$n[(s)-(o)]\approx n([(s)-(o)](o))C_{u_0},$$

since $D$ does not meet $\Theta_{v,i}$ for $i\geqq 1$. By taking the intersection number of both side with the divisor $(s)$, we have

$$((s)(s))+((o)(o))=2((s)(o))\geqq 0.$$

This contradicts to the fact that $((s)(s))=((o)(o))=-(p_a+1)<0$ (cf. [**K**] p. 15). Hence $E(K)_0$ is torsion-free. It is clear that $E(K)_0$ is a subgroup of finite index in $E(K)$.

COROLLARY 1.7. *Let $\Gamma_1,\cdots,\Gamma_\rho$ be a basis of $NS(B)$ modulo torsion. If $E(K)=E(K)_0\oplus E(K)_{\text{tor}}$, then*

$$\frac{|\det((\Gamma_i\Gamma_j))|}{|NS(B)_{\text{tor}}|^2}=\frac{|\det((D_\alpha D_{\alpha'}))|\cdot\prod m_v^{(1)}}{|E(K)_{\text{tor}}|^2}$$

*where $D_\alpha$ is defined in* (1.1) *and $m_v^{(1)}$ is the number of simple components of $C_v$ $(v\in\Sigma)$.*

PROOF. This is an immediate consequence of Theorem 1.1 and of the following elementary fact.

LEMMA 1.8. *Let $N$ denote a free submodule of finite index in $NS(B)$ and let $\Gamma_1',\cdots,\Gamma_\rho'$ be a basis of $N$. Then the quantity $|\det((\Gamma_i'\Gamma_j'))|/[NS(B):N]^2$ is independent of the choice of the submodule $N$.*

REMARK 1.9. Actually it can be verified that the Néron-Severi group $NS(B)$ of $B$ is torsion-free, by studying the fundamental relations (1.4) more closely.

REMARK 1.10. As to the torsion subgroup $E(K)_{\text{tor}}$ of $E(K)$, we add the following remark. It is known (cf. [**18**], [**20**]) that the canonical homomorphism of $E(K)$ to $C_v^{\#}$ defined by $s\mapsto s(v)$ induces an injection:

$$E(K)_{\text{tor}}\hookrightarrow(C_v^{\#})_{\text{tor}}.$$

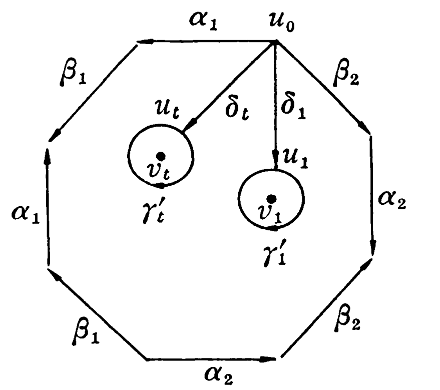
Hence, by the structure of $C_v^{\#}$ recalled before Proposition 1.6, the order of $E(K)_{\text{tor}}$ is at most 4 unless all singular fibers $C_v$ are of multiplicative type (i.e. of type $I_b$ $(b\geqq 1)$).

## § 2. The cohomology groups $H^i(\Delta, G)$.

From now on (until the end of § 7) we take $k=\boldsymbol{C}$, the field of complex numbers. Let $B$ denote an elliptic surface over $\Delta$ with a section $o$; $\Delta$ is a non-singular projective curve with the function field $K=\boldsymbol{C}(\Delta)$. Since the generic fibre $E$ of $B$ over $\Delta$ is an elliptic curve defined over $K$, its absolute invariant $J$ is contained in $K$; $J$, viewed as a meromorphic function on the Riemann surface $\Delta$, is the functional invariant of $B$ over $\Delta$ ([**K**] § 7). As before, $J$ is assumed to be non-constant. Let $G$ denote the homological invariant of $B$ over $\Delta$ ([**K**] § 7); $G$ is a sheaf over $\Delta$ and its restriction to $\Delta'=\Delta-\Sigma$ is locally constant, $\Sigma$ being defined as in § 1. The stalk $G_{u_0}$ of $G$ over a point $u_0 \in \Delta'$ is the first homology group $H_1(C_{u_0}, \boldsymbol{Z})$ of the fibre $C_{u_0}$. The stalk $G_v$ over $v \in \Sigma$ is the group of "invariant cycles" around $v$ and it is isomorphic to $\boldsymbol{Z}$ or (0) according to whether the singular fibre $C_v$ is of type $I_b$ $(b \geqq 1)$ or not ([**K**] § 11). The sheaf $G$ determines (and is determined by) a representation $\varphi$ of the fundamental group $\pi_1(\Delta')$ of $\Delta'$ in $SL(2, \boldsymbol{Z})$. Now we shall compute the cohomology groups $H^i(\Delta, G)$ of $\Delta$ with coefficients in $G$ following Kodaira. Note that the cohomology group $H^i(\Delta, G)$ is isomorphic to the homology group $H_{2-i}(\Delta, G)$ by duality (cf. § 7).

Let $g$ be the genus of $\Delta$ and let $t$ be the total number of singular fibres; put $\Sigma=\{v_1, v_2, \cdots, v_t\}$. Choosing a base point $u_0 \in \Delta'$, we represent each element of the fundamental group $\pi_1(\Delta')$ of $\Delta'$ by a closed path starting from $u_0$. As is well-known, there are standard generators $\alpha_i$, $\beta_i$ $(1 \leqq i \leqq g)$ and $\gamma_j$ $(1 \leqq j \leqq t)$ of $\pi_1(\Delta')$ with a single relation:

$$\gamma_t \cdots \gamma_1 \beta_g^{-1} \alpha_g^{-1} \beta_g \alpha_g \cdots \beta_1^{-1} \alpha_1^{-1} \beta_1 \alpha_1 = 1. \tag{2.1}$$




We take a small (oriented) disk $E_j$ around each $v_j$ and put $\gamma_j' = -\partial E_j$. Choose a point $u_j$ on $\gamma_j'$ and a path $\delta_j$ connecting $u_0$ and $u_j$ such that $\delta_j^{-1}\gamma_j'\delta_j$ is homotopic to $\gamma_j$. Thus we obtain a cell decomposition of $\Delta$ in which the Riemann surface $\Delta$ is decomposed into 0-cells $u_j$ $(0 \leqq j \leqq t)$, 1-cells $\alpha_i$, $\beta_i$ $(1 \leqq i \leqq g)$, $\delta_j$, $\gamma_j'$ $(1 \leqq j \leqq t)$ and 2-cells $E_j$ $(1 \leqq j \leqq t)$ and $\Delta_0 = \Delta - \cup_j E_j$. On the other hand, we shall identify the stalk $G_{u_0}$ of $G$ over the base point $u_0$ with $G_0 = \boldsymbol{Z} \oplus \boldsymbol{Z}$ so that the action of an element $\gamma \in \pi_1(\Delta')$ on $G_{u_0}$ corresponds to the right multiplication of ${}^t\varphi(\gamma)$ on $\boldsymbol{Z} \oplus \boldsymbol{Z}$, $\varphi$ being the representation of $\pi_1(\Delta')$ in $SL(2, \boldsymbol{Z})$ associated with $G$. We put

$$A_i = {}^t\varphi(\alpha_i)\,, \qquad B_i = {}^t\varphi(\beta_i)\,, \qquad C_j = {}^t\varphi(\gamma_j)\,. \tag{2.2}$$

The stalk $G_{v_j}$ $(1 \leqq j \leqq t)$ is then identified with the subgroup $G_j$ of $\boldsymbol{Z} \oplus \boldsymbol{Z}$ consisting of elements invariant under $C_j$.

Now the $i$-chains $\sigma_i$ with coefficients in the sheaf $G$ are given as follows:

$$\begin{aligned} \sigma_0 &= \sum_{j=0}^{t} l_j u_j\,, \\ \sigma_1 &= \sum_{i=1}^{g} (a_i\alpha_i + b_i\beta_i) + \sum_{j=1}^{t} (c_j\gamma'_j + d_j\delta_j)\,, \\ \sigma_2 &= e\Delta_0 + \sum_{j=1}^{t} e_j E_j\,, \end{aligned} \tag{2.3}$$

where the coefficients $l_j, a_i, \cdots, e$ belong to $G_0$ and $e_j \in G_j$, i.e. $e_j C_j = e_j$. If we denote by $\partial$ the boundary operator, we have

$$\begin{aligned} &\partial(a_i\alpha_i) = a_i(A_i - 1)u_0\,, \qquad \partial(b_i\beta_i) = b_i(B_i - 1)u_0\,, \\ &\partial(c_j\gamma'_j) = c_j(C_j - 1)u_j\,, \qquad \partial(d_j\delta_j) = d_j u_j - d_j u_0\,, \\ &\partial(e\Delta_0) = \sum_{i=1}^{g} eK^{(i-1)}[(1 - A_iB_iA_i^{-1})\alpha_i + (A_i - K_i)\beta_i] \\ &\qquad\qquad + \sum_{j=1}^{t} eK[(C^{(j-1)} - C^{(j)})\delta_j + C^{(j-1)}\gamma'_j] \\ &\partial(e_jE_j) = -e_j\gamma'_j\,, \end{aligned} \tag{2.4}$$

where we put $K_i = A_iB_iA_i^{-1}B_i^{-1}$, $K^{(i)} = K_1K_2 \cdots K_i$ and $C^{(j)} = C_1 \cdots C_j$, $(K^{(0)} = C^{(0)} = 1$, $K^{(g)} = K)$. Therefore we can define a complex of modules

$$M_0 \xrightarrow{\partial_1} M_1 \xrightarrow{\partial_2} M_2 \tag{2.5}$$

where

$$M_0 = G_0 \oplus \sum_{j=1}^{t} G_j$$

$$M_1 = G_0^{2g+t}\,, \qquad M_2 = G_0$$

and

$$\partial_1(e, e_1, \cdots, e_t) = (a_i, b_i, c_j) \qquad (1 \leqq i \leqq g,\ 1 \leqq j \leqq t)$$

with

$$\begin{aligned} a_i &= eK^{(i-1)}(1 - A_iB_iA_i^{-1})\,, \\ b_i &= eK^{(i-1)}(A_i - K_i)\,, \\ c_j &= eKC^{(j-1)} - e_j\,; \end{aligned}$$

$\partial_2$ is defined by

$$\partial_2(a_i, b_i, c_j) = \sum_{i=1}^{g} [a_i(A_i - 1) + b_i(B_i - 1)] + \sum_{j=1}^{t} c_j(C_j - 1)\,.$$

Obviously the homology group $H_{2-i}(\Delta, G)$ is isomorphic to the cohomology group $H^i(M)$ of the complex (2.5). Hence $H^i(\Delta, G) \cong H^i(M)$.

PROPOSITION 2.1. *$H^0(\Delta, G) = (0)$ if $t \geqq 1$.*

PROOF. It is immediate that $H^0(M) = \mathrm{Ker}(\partial_1)$ is isomorphic to the subgroup $H$ of $G_0$ consisting of elements invariant under $\varphi(\pi_1(\Delta'))$. Assume $H \neq (0)$. Since $H$ is contained in $G_j$ $(1 \leqq j \leqq t)$, $H \neq (0)$ will imply that all singular fibres are of type $I_b$ $(b \geqq 1)$ and that $H \cong \boldsymbol{Z}$. If we take a suitable basis of $G_0$, all $\varphi(\gamma)$ $(\gamma \in \pi_1(\Delta'))$ are of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. (Namely, any generator of $H$ is of the form $(m, n)$ with relatively prime integers $m$, $n$. Therefore there is a basis of $G_0$ containing a generator of $H$.) Hence $\varphi(\pi_1(\Delta'))$ is an abelian group and we have

$$ {}^tC_j = \varphi(\gamma_j) = \begin{pmatrix} 1 & b_j \\ 0 & 1 \end{pmatrix}, \qquad b_j > 0 . $$

This contradicts the relation (cf. (2.1))

$$ A_1 B_1 A_1^{-1} B_1^{-1} \cdots C_1 \cdots C_t = 1 . $$

PROPOSITION 2.2. *$H^2(\Delta, G)$ is a finite group.*

For the proof, see [**K**] Theorem 11.7. (Note that the sheaf $G$ is non-trivial since the functional invariant $J$ is non-constant.) We remark that $H^2(\Delta, G)$ is isomorphic to $H^2(M) = G_0/\mathrm{Im}(\partial_2)$ where $\mathrm{Im}(\partial_2)$ is the subgroup of $G_0 = \boldsymbol{Z} \oplus \boldsymbol{Z}$ generated by

$$ G_0({}^t\varphi(\gamma) - 1), \qquad \gamma \in \pi_1(\Delta') . $$

PROPOSITION 2.3. *$H^1(\Delta, G)$ is a finitely generated group of rank*

$$ 4g - 4 + 2t - t_1 , \tag{2.6} $$

*where $t_1$ is the number of singular fibres of types $I_b$ $(b \geqq 1)$.*

PROOF. We consider $H^1(M) = \mathrm{Ker}(\partial_2)/\mathrm{Im}(\partial_1)$. By Proposition 2.1 the map $\partial_1$ is injective. The rank of $M_0 \cong \mathrm{Im}(\partial_1)$ is $2 + t_1$, since $G_j$ $(1 \leqq j \leqq t)$ is of rank 1 or 0 according to whether the singular fibre $C_{v_j}$ is of types $I_b$ $(b \geqq 1)$ or not. Obviously $M_1$ is of rank $2(2g+t)$, while the rank of $\mathrm{Im}(\partial_1)$ is 2 by the remark after Proposition 2.2. Hence the rank of $H^1(M) \cong H^1(\Delta, G)$ is equal to

$$ 2(2g+t) - 2 - (2 + t_1) = 4g - 4 + 2t - t_1 . $$

REMARK 2.4. Actually $H^1(\Delta, G)$ is a free module. This follows from the exact sequence below (2.8) and Proposition 1.6.

Let $B^\#$ denote the group scheme over $\Delta$ associated with the elliptic surface $B$ and let $B_0^\#$ be the connected component of the identity section $o$ in $B^\#$; we have

$$B^{\#}=\bigcup_{v\in\Delta} C_v^{\#}, \qquad B_0^{\#}=\bigcup_{u\in\Delta'} C_u\cup(\bigcup_{v\in\Sigma}\Theta_{v,0}^{\#})$$

in the notation of § 1 (cf. [**K**] § 9). Let $\mathfrak{f}$ be the line bundle over $\Delta$ defined in [**K**] § 11; $\mathfrak{f}$ is the normal bundle of the image (curve) $o(\Delta)$ of the section $o$ in $B$. We denote by $\mathcal{O}(\mathfrak{f})$, $\Omega(B^{\#})$ or $\Omega(B_0^{\#})$ the sheaves of germs of holomorphic sections of $\mathfrak{f}$, $B^{\#}$ or $B_0^{\#}$ over $\Delta$. By Theorem 11.2 of [**K**], we have then the exact sequence:

$$0\longrightarrow G\xrightarrow{i}\mathcal{O}(\mathfrak{f})\xrightarrow{h}\Omega(B_0^{\#})\longrightarrow 0. \tag{2.7}$$

It induces the exact sequence of cohomology groups:

$$\begin{aligned}0\longrightarrow H^0(\Delta,\Omega(B_0^{\#}))\longrightarrow H^1(\Delta,G)&\xrightarrow{i^*}H^1(\Delta,\mathcal{O}(\mathfrak{f}))\\ &\xrightarrow{h^*}H^1(\Delta,\Omega(B_0^{\#}))\longrightarrow H^2(\Delta,G)\longrightarrow 0.\end{aligned}\tag{2.8}$$

Note that the group of sections $H^0(\Delta,\Omega(B^{\#}))$ (respectively $H^0(\Delta,\Omega(B_0^{\#}))$) may be identified with the group $E(K)$ of $K$-rational points of the generic fibre $E$ (respectively, with the subgroup $E(K)_0$ (cf. § 1)). (Incidentally (2.7) gives another proof of Proposition 2.1 since the degree of $\mathfrak{f}$ is negative and hence $H^0(\Delta,\mathcal{O}(\mathfrak{f}))=0$.) Let $r$ be the rank of $E(K)$ and let $r'$ be the rank of the image group $i^*H^1(\Delta,G)$; equivalently $r'$ can be defined as the maximum number $\nu$ such that the group $H^1(\Delta,\Omega(B_0^{\#}))$ contains a subgroup isomorphic to $(\boldsymbol{Q}/\boldsymbol{Z})^\nu$, product of $\nu$ copies of the group of rational numbers modulo integers. As an immediate consequence of Proposition 2.3 and (2.8), we get

THEOREM 2.5.

$$r+r'=4g-4+2t-t_1.$$

This is a special case of Ogg-Šafarevič's formula, which has been proved for abelian varieties over function fields of arbitrary characteristic (cf. [**18**], [**19**], [**20**]).

Let $b_i$ denote the $i$-th Betti number of $B$. Then we have

$$b_1=2g,\qquad b_2=c_2+2b_1-2, \tag{2.9}$$

where $c_2$ is the Euler number of $B$. By [**K**] Theorem 12.2,

$$\begin{aligned}c_2=12(p_a+1)&=\sum_{v\in\Sigma}\varepsilon_v\\ &=\mu+6\sum_{b\geqq 0}\nu(I_b^*)+2\nu(II)+10\nu(II^*)\\ &\quad+3\nu(III)+9\nu(III^*)+4\nu(IV)+8\nu(IV^*),\end{aligned}\tag{2.10}$$

where $p_a$ is the arithmetic genus of $B$; $\varepsilon_v$ is the Euler number of singular fibre $C_v$; $\nu(T)$ is the number of singular fibres of type $T$: and $\mu$ is the total multiplicity of $J$, i.e. the degree of the map $J: \Delta \to \boldsymbol{P}^1$. On the other hand, the Picard number $\rho$ of $B$ is given by Corollary 1.5. By a direct computation, one can check that $\varepsilon_v - m_v = 0$ or 1 according to whether $C_v$ is of type $I_b$ ($b \geqq 1$) or not. Hence one gets (cf. [19] VI)

COROLLARY 2.6.

$$b_2 - \rho = r'.$$

Further we know from the Lefschetz-Hodge theory that

$$b_2 = 2p_g + h^{1,1} \qquad (h^{p,q} = \dim H^q(B, \Omega^p)), \tag{2.11}$$

$$\rho \leqq h^{1,1},$$

where $p_g = h^{0,2} = h^{2,0}$ is the geometric genus of $B$. Hence we get

COROLLARY 2.7.

$$r' \geqq 2p_g; \quad \textit{or equivalently,}$$

$$r \leqq 4g - 4 + 2t - t_1 - 2p_g.$$

The following result is due to Kodaira.

PROPOSITION 2.8. *Let $h^1 =$ the rank of $H^1(\Delta, G)$. Then*

$$h^1 - 2p_g \geqq \nu(I_0^*) + \nu(II) + \nu(III) + \nu(IV).$$

PROOF. We consider the meromorphic differential $\omega = dJ/J$ on $\Delta$. If we denote respectively by $\nu^0$, $\nu^1$ and $\nu^\infty$ the number of zeroes of $J$, of zeroes of $J-1$, and the number of poles of $J$, then the divisor of poles of $\omega$ has degree $\nu^0 + \nu^\infty$, while the divisor of zeroes of $\omega$ has degree $\geqq \mu - \nu^1$. Hence we have

$$2g - 2 \geqq \mu - \nu^1 - (\nu^0 + \nu^\infty). \tag{2.12}$$

For $i = 1, 2, 3$, let $\nu^0(i)$ denote the number of zeroes of $J$ whose order is congruent to $i$ mod 3. Similarly $\nu^1(i)$ ($i = 1, 2$) denotes the number of zeroes of $J-1$ whose order is congruent to $i$ mod 2. Obviously we have

$$\mu \geqq \nu^0(1) + 2\nu^0(2) + 3\nu^0(3), \qquad \mu \geqq \nu^1(1) + 2\nu^1(2). \tag{2.13}$$

From (2.12) and (2.13), we get

$$\frac{1}{6}\mu \leqq 2g - 2 + \frac{1}{2}\nu^1(1) + \frac{2}{3}\nu^0(1) + \frac{1}{3}\nu^0(2) + \nu^\infty. \tag{2.14}$$

On the other hand, we know from [K] § 8 that

$$\nu^\infty = \sum_{b \geqq 1} (\nu(I_b) + \nu(I_b^*)),$$

$$\nu^0(1) = \nu(II) + \nu(IV^*), \qquad \nu^0(2) = \nu(II^*) + \nu(IV), \tag{2.15}$$

$$\nu^1(1) = \nu(III) + \nu(III^*).$$

Then, computing $h^1 - 2p_g$ by (2.6), (2.10) and comparing it with (2.14), (2.15), we get the assertion.

## § 3. A density theorem.

Let $\mathscr{F}(J, G)$ denote the family of all (not necessarily algebraic) elliptic surfaces over $\Delta$ having the same functional and homological invariants $J$, $G$ as the elliptic surface $B$ considered in § 2. We refer to [**K**] § 9, 10, 11 for what follows. Let $A$ be one of the sheaves of groups $\Omega(B^{\#})$ or $\Omega(B_0^{\#})$ over $\Delta$. For each cohomology class $\eta \in H^1(\Delta, A)$, let $B^{\eta}$ denote the elliptic surface in $\mathscr{F}(J, G)$ obtained by twisting $B$ with $\eta$. The family $\mathscr{F}(J, G)$ modulo $A$-equivalence is parametrized by the cohomology group $H^1(\Delta, A)$. Moreover $B^{\eta}$ is an algebraic surface if and only if $\eta$ is an element of finite order of $H^1(\Delta, A)$ ([**K**] Theorem 11.5). Now from the exact sequence (2.8) we see that

$$H^1(\Delta, \Omega(B_0^{\#})) \cong h^*H^1(\Delta, \mathcal{O}(\mathfrak{f})) \times H^2(\Delta, G), \tag{3.1}$$

since $h^*H^1(\Delta, \mathcal{O}(\mathfrak{f})) = H^1(\Delta, \mathcal{O}(\mathfrak{f}))/i^*H^1(\Delta, G)$ is a divisible group. For a fixed $c \in H^2(\Delta, G)$, we denote by $\eta(t)$ ($t \in H^1(\Delta, \mathcal{O}(\mathfrak{f}))$) the element of $H^1(\Delta, \Omega(B_0^{\#}))$ corresponding to $(h^*(t), c)$ under the isomorphism (3.1). Then the collection $\{B^{\eta(t)} \mid t \in H^1(\Delta, \mathcal{O}(\mathfrak{f}))\}$ forms a complex analytic family $\mathcal{V}^{(c)}$ ([**K**] Theorem 11.3).

We shall be concerned with the following question: "Are algebraic surfaces dense in the family $\mathcal{V}^{(c)}$ or in the family $\mathscr{F}(J, G)$?" Since $H^2(\Delta, G)$ is a finite group by Proposition 2.2, $B^{\eta(t)}$ is an algebraic surface if and only if $t$ is a rational linear combination of elements of $i^*H^1(\Delta, G)$. Thus the above question is equivalent to whether or not the vector space $H^1(\Delta, \mathcal{O}(\mathfrak{f}))$ is spanned over $\boldsymbol{R}$ (the real numbers) by $i^*H^1(\Delta, G)$. Note that the rank of $i^*H^1(\Delta, G)$ is $r' \geqq 2p_g$ by Corollary 2.7, while $H^1(\Delta, \mathcal{O}(\mathfrak{f}))$ is a complex vector space of dimension $p_g$ ([**K**] p. 15). Let us consider the exponential exact sequence of sheaves on $B$

$$0 \longrightarrow \boldsymbol{Z} \xrightarrow{j} \mathcal{O} \longrightarrow \mathcal{O}^{\times} \longrightarrow 0.$$

In the corresponding exact sequence of cohomology groups, the image of $H^1(B, \mathcal{O}^{\times})$ in $H^2(B, \boldsymbol{Z})$ can be identified with the Néron-Severi group of $B$. Thus we have the exact sequence:

$$0 \longrightarrow NS(B) \longrightarrow H^2(B, \boldsymbol{Z}) \xrightarrow{j^*} H^2(B, \mathcal{O}) \longrightarrow \cdots. \tag{3.2}$$

We shall compare this with the exact sequence

$$0 \longrightarrow H^0(\Delta, \Omega(B_0^{\#})) \longrightarrow H^1(\Delta, G) \xrightarrow{i^*} H^1(\Delta, \mathcal{O}(\mathfrak{f})). \tag{2.8}$$

THEOREM 3.1. *There exists an isomorphism $\phi$ of $H^1(\Delta, \mathcal{O}(\mathfrak{f}))$ onto $H^2(B, \mathcal{O})$ such that $\phi(i^*H^1(\Delta, G))$ is commensurable with $j^*H^2(B, \boldsymbol{Z})$ in $H^2(B, \mathcal{O})$.*

PROOF. Let $\Phi$ denote the canonical projection of $B$ onto $\Delta$. We consider the Leray spectral sequence (cf. [2] p. 30-31):

$$E_2^{p,q}=H^p(\Delta, R^q\Phi_*(\mathcal{O}))\Rightarrow H^{p+q}(B, \mathcal{O}).$$

For a positive integer $m>1$, we denote by $\varphi_m$ the rational map of $B$ to itself over $\Delta$, which induces multiplication by $m$ on the generic fibre $E$. It can be checked that, for suitably chosen $m$, $\varphi_m$ is a holomorphic map of $B$ onto $B$. (For instance, we can take $m$ such that $m\equiv 1 \bmod m_0$, where $m_0$ is the least common multiple of 12 and $b_i'$s ($1\leqq i\leqq t_1$), supposing that $B$ has singular fibres of types $I_b$ for $b=b_1, \cdots, b_{t_1}$.) The map $\varphi_m$ induces an endomorphism $\varphi_m^*$ of the spectral sequence, and $\varphi_m^*$ acts on $E_r^{p,q}$ ($r\geqq 2$) as multiplication by $m^q$ since it acts on $R^q\Phi_*(\mathcal{O})$ as such. The map $d_r: E_r^{p,q}\to E_r^{p+r,q-r+1}$ ($r\geqq 2$) commutes with $\varphi_m^*$ and this implies that $d_r=0$ ($r\geqq 2$) since $E_r^{p,q}$ are modules over a field of characteristic zero. Therefore $E_2^{p,q}=H^p(\Delta, R^q\Phi_*(\mathcal{O}))$ is isomorphic to the submodule of $H^{p+q}(B, \mathcal{O})$ on which $\varphi_m^*$ acts as multiplication by $m^q$. In particular, we get an isomorphism

$$\psi: H^1(\Delta, \mathcal{O}(\mathfrak{f}))\xrightarrow{\sim} H^2(B, \mathcal{O}), \tag{3.3}$$

using the fact $R^1\Phi_*(\mathcal{O})\cong\mathcal{O}(\mathfrak{f})$ and $\dim H^1(\Delta, \mathcal{O}(\mathfrak{f}))=p_g=\dim H^2(B, \mathcal{O})$. Applying the above argument to the constant sheaf $\boldsymbol{Q}$ on $B$ instead of the sheaf $\mathcal{O}$ and noting that $R^1\Phi_*(\boldsymbol{Q})\cong G\otimes\boldsymbol{Q}$, we get a homomorphism

$$\psi': H^1(\Delta, G\otimes\boldsymbol{Q})\longrightarrow H^2(B, \boldsymbol{Q}). \tag{3.4}$$

It follows from (3.3) and (3.4) that $\psi$ maps the subgroup $i^*H^1(\Delta, G)\otimes\boldsymbol{Q}$ of $H^1(\Delta, \mathcal{O}(\mathfrak{f}))$ into the subgroup $j^*H^2(B, \boldsymbol{Z})\otimes\boldsymbol{Q}$ of $H^2(B, \mathcal{O})$. Since the rank $r'$ of $i^*H^1(\Delta, G)$ is equal to the rank $b_2-\rho$ of $j^*H^2(B, \boldsymbol{Z})$ by Corollary 2.6, we conclude that $\psi(i^*H^1(\Delta, G))$ is commensurable with $j^*H^2(B, \boldsymbol{Z})$, which completes the proof.

THEOREM 3.2. *Assume that the functional invariant $J$ is non-constant on $\Delta$. Then algebraic surfaces are dense in the family $\mathcal{F}(J, G)$ of elliptic surfaces over $\Delta$ with the functional and homological invariants $J$ and $G$.*

PROOF. By Theorem 3.1 and the argument preceding it, it is sufficient to show that the vector space $H^2(B, \mathcal{O})$ is spanned over $\boldsymbol{R}$ by the image $j^*H^2(B, \boldsymbol{Z})$ of $H^2(B, \boldsymbol{Z})$. This follows from the following general fact.

LEMMA 3.3. *Let $X$ denote a compact Kähler manifold and let $j: \boldsymbol{Z}\to\mathcal{O}$ be the natural injection of sheaves on $X$. Then, for each $n\geqq 1$, the vector space $H^n(X, \mathcal{O})$ is spanned over $\boldsymbol{R}$ by the image $j^*H^n(X, \boldsymbol{Z})$.*

PROOF. It is enough to show that the map

$$j_1^*: H^n(X, \boldsymbol{R})\longrightarrow H^n(X, \mathcal{O})$$

induced by the canonical map $j_1: \boldsymbol{R}\to\mathcal{O}$ is surjective. $j_1^*$ factors as

$$H^n(X, \boldsymbol{R}) \longrightarrow H^n(X, \boldsymbol{C}) \xrightarrow{j_2^*} H^n(X, \mathcal{O}),$$

where $j_2$ is the canonical map $\boldsymbol{C} \to \mathcal{O}$. The Hodge decomposition $H^n(X, \boldsymbol{C}) = \bigoplus_{p+q=n} H^{p,q}$ and the Dolbeault isomorphism $H^n(X, \mathcal{O}) \cong H^{0,n}$ are related so that $j_2^* : H^n(X, \boldsymbol{C}) \to H^n(X, \mathcal{O})$ corresponds to the projection $\bigoplus H^{p,q} \to H^{0,n}$. Hence the surjectivity of $j_1^*$ is clear.

COROLLARY 3.4. *Let $X$ be a non-singular projective variety and let $\rho$ be the rank of the Néron-Severi group of $X$. If $\rho = h^{1,1}$, then $H^2(X, \mathcal{O})/j^*H^2(X\boldsymbol{Z})$ has a structure of a complex torus.*

For the elliptic surface $B$, we get

COROLLARY 3.5. *If $r' = \operatorname{rank} i^*H^1(\varDelta, G)$ is equal to $2p_g$, then $H^1(\varDelta, \mathcal{O}(\mathfrak{f}))/i^*H^1(\varDelta, G)$ is a complex torus. Hence the cohomology group $H^1(\varDelta, \Omega(B_0^\#))$ (or $H^1(\varDelta, \Omega(B^\#))$) is a product of a complex torus and a finite group.*

REMARK 3.6. Theorem 3.2 and Corollary 3.5 were observed in [**24**] in the special case where $B$ is an elliptic modular surface (cf. § 7 below).

REMARK 3.7. It is unknown whether or not every elliptic surface in the family $\mathcal{F}(J, G)$ is a Kähler surface. Using Theorem 3.2 this question can be reduced to a local one around the singular fibres as follows. For each point $u \in \varDelta' \subset \varDelta$, let $(\omega(u), 1)$ be a normalized period of the elliptic curve $C_u = \Phi^{-1}(u)$. $\omega(u)$ is a multivalued holomorphic function on $\varDelta'$ with $\operatorname{Im}(\omega(u)) > 0$. Let $U'$ be the universal covering of $\varDelta'$; we consider $\omega$ as a single-valued function on $U'$. Now $B' = \Phi^{-1}(\varDelta')$ is a quotient of the product $U' \times \boldsymbol{C}$ by a certain group $\mathcal{G}$ of holomorphic automorphisms (cf. [**K**] p. 580 (8.5)). For each point $(\tilde{u}, \zeta)$ of $U' \times \boldsymbol{C}$, we put

$$\zeta = \xi_1 \cdot \omega(\tilde{u}) + \xi_2, \qquad \xi_1, \xi_2 \in \boldsymbol{R}.$$

Consider the metric on $U' \times \boldsymbol{C}$:

$$\Phi^*(ds_0^2) + \frac{|d\xi_1 \cdot \omega(\tilde{u}) + d\xi_2|^2}{\operatorname{Im}(\omega(\tilde{u}))}, \tag{3.7}$$

where $ds_0^2$ is a Kähler metric on $\varDelta$. Then it is easy to see that (3.7) is a Kähler metric on $U' \times \boldsymbol{C}$, invariant under the group $\mathcal{G}$. Thus it defines a Kähler metric $ds^2$ on $B'$. Moreover it is invariant under "constant" translations:

$$(\tilde{u}, \zeta) \longmapsto (\tilde{u}, \zeta + a_1\omega(\tilde{u}) + a_2),$$

where $a_1$, $a_2$ are real constants mod 1. By Theorem 3.2, $H^1(\varDelta, \mathcal{O}(\mathfrak{f}))$ is spanned over $\boldsymbol{R}$ by $i^*H^1(\varDelta, G)$. Hence we see that the elliptic surface $B^{\eta(t)}$ with $t \in H^1(\varDelta, \mathcal{O}(\mathfrak{f}))$ is obtained by twisting $B$ with local "constant" translations. Therefore $B^{\eta(t)}$ will be a Kähler surface provided that the metric $ds^2$ on $B'$ can be extended to a Kähler metric on $B$ by modifying at singular fibres.

For instance, for the singular fibre of type $I_0^*$, this last extendability can be verified.

## Part II. Analytic theory of elliptic modular surfaces.

### § 4. Elliptic modular surfaces.

In this section we shall introduce a certain class of elliptic surfaces, called elliptic modular surfaces, connected with the theory of automorphic functions of one variable. For the theory of automorphic functions, see for example [**10**] or [**29**]. Let $\Gamma$ denote a subgroup of finite index of the homogeneous modular group $SL(2, \boldsymbol{Z})$. The group $\Gamma$ acts on the upper half plane $\mathfrak{H}$ in the usual manner: for $\gamma \in \Gamma$ and $z \in \mathfrak{H}$, we put

$$\gamma \cdot z = \frac{az+b}{cz+d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{4.1}$$

The quotient $\Gamma \backslash \mathfrak{H}$ of $\mathfrak{H}$ by $\Gamma$, together with a finite number of cusps, forms a compact Riemann surface, say $\Delta_\Gamma$. If $\Gamma \subset \Gamma_1$, then the canonical map of $\Gamma \backslash \mathfrak{H}$ onto $\Gamma_1 \backslash \mathfrak{H}$ extends to a holomorphic map of $\Delta_\Gamma$ onto $\Delta_{\Gamma_1}$. In particular, we have a holomorphic map $J_\Gamma$ of $\Delta_\Gamma$ onto the projective line $\boldsymbol{P}^1$:

$$J_\Gamma : \Delta_\Gamma \longrightarrow \boldsymbol{P}^1, \tag{4.2}$$

by taking $\Gamma_1 = SL(2, \boldsymbol{Z})$ and identifying $\Delta_{\Gamma_1}$ with $\boldsymbol{P}^1$ by means of the ordinary elliptic modular function $j$.

Now we make the following assumption on $\Gamma$:

(*) $\Gamma$ acts effectively on the upper half plane $\mathfrak{H}$ (i.e. $\Gamma \not\ni -1$).

We put

$$\begin{aligned} \mu &= \text{the index of } \Gamma \cdot \{\pm 1\} \text{ in } SL(2, \boldsymbol{Z}), \\ t' &= \text{the number of cusps in } \Delta_\Gamma \ (t' \geqq 1), \\ s &= \text{the number of elliptic points in } \Delta_\Gamma \ (s \geqq 0), \\ t &= t' + s. \end{aligned} \tag{4.3}$$

For an elliptic point $v \in \Delta_\Gamma$, take a point $z$ in $\mathfrak{H}$ representing $v$. Then the generator of the stabilizer of $z$ in $\Gamma$ is of order 3 by the assumption (*) and is conjugate in $SL(2, \boldsymbol{Z})$ to either

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}. \tag{4.4}$$

For a cusp $v \in \Delta_\Gamma$, the stabilizer of a representative in $\boldsymbol{Q} \cup \{\infty\}$ of $v$ has a generator which is conjugate in $SL(2, \boldsymbol{Z})$ to either

(4.5) $$\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}-1 & -b\\ 0 & -1\end{pmatrix}, \qquad b>0;$$

accordingly, the cusp $v$ is called of the first or second kind. The genus $g$ of $\Delta_\Gamma$ is given by

(4.6) $$2g-2+t'+\left(1-\frac{1}{3}\right)s=\frac{1}{6}\mu.$$

Let $\Sigma$ be the set of cusps and elliptic points in $\Delta_\Gamma$ and put $\Delta'=\Delta_\Gamma-\Sigma\subset\Gamma\backslash\mathfrak{H}$. If we denote by $U'$ the universal covering of $\Delta'$ with the projection $\pi$: $U'\to\Delta'$, there is a holomorphic map $\omega$ of $U'$ into $\mathfrak{H}$ such that

(4.7) $$J_\Gamma(\pi(\tilde{u}))=j(\omega(\tilde{u})) \qquad (\tilde{u}\in U'),$$

$j$ being the elliptic modular function on $\mathfrak{H}$ (cf. Remark 3.7). Moreover there is a unique representation $\varphi$ of the fundamental group $\pi_1(\Delta')$ of $\Delta'$ into $SL(2, \boldsymbol{Z})$

(4.8) $$\varphi:\pi_1(\Delta')\longrightarrow\Gamma\subset SL(2, \boldsymbol{Z}),$$

such that

(4.9) $$\omega(\gamma\cdot\tilde{u})=\varphi(\gamma)\cdot\omega(\tilde{u}), \qquad \tilde{u}\in U', \quad \gamma\in\pi_1(\Delta'),$$

where the right side is defined as in (4.1). The representation $\varphi$ determines a sheaf $G_\Gamma$ over $\Delta_\Gamma$, locally constant over $\Delta'$ with the general stalk $\boldsymbol{Z}\oplus\boldsymbol{Z}$.

We can apply to this situation Kodaira's construction of elliptic surfaces ([**K**] § 8). Namely there exists a non-singular algebraic elliptic surface, $B_\Gamma$, over $\Delta_\Gamma$ with a global section having $J_\Gamma$ and $G_\Gamma$ as its functional and homological invariants (i. e. the basic member of $\mathcal{F}(J, G)$ in [**K**]); it is unique up to a biregular fibre-preserving map over $\Delta$.

DEFINITION 4.1. The elliptic surface $B_\Gamma$ over $\Delta_\Gamma$ will be called the *elliptic modular surface* attached to the group $\Gamma$.

We shall examine the singular fibres of $B_\Gamma$ over $\Delta_\Gamma$; obviously they lie over the subset $\Sigma$ of $\Delta_\Gamma$ consisting of the elliptic points and the cusps. The type of the singular fibre $C_v$ ($v\in\Sigma$) is determined by $\varphi(\gamma_v)\in\Gamma$, where $\gamma_v$ is a positively oriented loop around $v$ on $\Delta_\Gamma$ (cf. [**K**] p. 604). If $v$ is an elliptic point, $\varphi(\gamma_v)$ is conjugate to either one of the normal form (4.4). Hence the singular fibre $C_v$ is of type $IV^*$ or $IV$. Let $s_1$ (or $s_2$) denote the number of singular fibres of type $IV^*$ (or $IV$); $s=s_1+s_2$. (We shall see below that $s_2=0$.) If $v$ is a cusp, $\varphi(\gamma_v)$ is conjugate to one of the normal forms (4.5). Hence the singular fibre $C_v$ is of type $I_b$ or $I_b^*$, according to whether the cusp $v$ is of the first or second kind. Let $t_1$ (or $t_2$) denote the number of cusps of the first (or second) kind, and put $t'=t_1+t_2$.

The numerical characters of $B_\Gamma$ are computed as follows. First the irregularity $q$ of $B_\Gamma$ is equal to the genus $g$ of $\Delta_\Gamma$, which is given by (4.6). The arithmetic genus $p_a$ is, by (2.10),

$$12(p_a+1)=\mu+6t_2+4s_2+8s_1\,. \tag{4.10}$$

Hence the geometric genus $p_g=p_a+q$ is given by

$$p_g=2g-2+t-t_1/2-s_2/3\,. \tag{4.11}$$

Comparing (4.11) with Proposition 2.3, we get ($h^1=\operatorname{rank} H^1(\Delta, G)$)

$$h^1-2p_g=\frac{2}{3}s_2\,.$$

On the other hand, Proposition 2.8 implies

$$h^1-2p_g\geqq \nu(IV)=s_2\,.$$

Hence we get

$$s_2=0 \quad\text{and}\quad h^1=2p_g\,. \tag{4.12}$$

Thus we have proved the following

PROPOSITION 4.2. *The elliptic modular surface $B_\Gamma$ has $t_1$ singular fibres of types $I_b$ ($b\geqq 1$), $t_2$ singular fibres of types $I_b^*$ ($b\geqq 1$) and $s$ singular fibres of type $IV^*$, where $t_1$, $t_2$ and $s$ are respectively the number of cusps of the first kind, the number of cusps of the second kind and the number of elliptic points for $\Gamma$.*

Recall the following table from [**K**] § 6 and § 9; as in § 1, $m_v$ (resp. $m_v^{(1)}$) denotes the number of components (resp. simple components) of $C_v$.

| Type of $C_v$ | $m_v$ | $m_v^{(1)}$ | $C_v^\#/\Theta_{v,0}^\#$ |
|---|---|---|---|
| $I_b$ | $b$ | $b$ | $\boldsymbol{Z}/(b)$ |
| $I_b^*$ | $b+5$ | 4 | $\boldsymbol{Z}/(4)$ or $\boldsymbol{Z}/(2)\times\boldsymbol{Z}/(2)$ |
| $IV^*$ | 7 | 3 | $\boldsymbol{Z}/(3)$ |

(Here $\boldsymbol{Z}/(n)$ denotes a cyclic group of order $n$.)

For the sake of later reference, we rewrite (4.11), (4.12):

PROPOSITION 4.3. *The geometric genus $p_g$ of the elliptic modular surface $B_\Gamma$ is given by the formula:*

$$p_g=2g-2+t-t_1/2\,, \qquad (t=t_1+t_2+s)\,. \tag{4.13}$$

REMARK 4.4. In [**24**], we defined elliptic modular surfaces $B_\Gamma$ only for torsion-free subgroups $\Gamma$ of finite index of $SL(2, \boldsymbol{Z})$. Thus the present definition 4.1 is slightly more general than [**24**] § 2. This generalization allows us to consider some examples of elliptic modular surfaces which may be of some arithmetic interest (§ 5. Example 5.8).

## § 5. The group of sections. Examples.

THEOREM 5.1. *An elliptic modular surface has only a finite number of global holomorphic sections over its base curve.*

PROOF. From the second relation of (4.12) and Corollary 2.7, we get

$$r=0 \qquad (\text{and } r'=2p_g), \tag{5.1}$$

where $r$ is the rank of the group of global sections. Hence the assertion follows.

We can give a more precise result. For brevity, we denote by $\mathcal{S}(B)$ the group of global holomorphic sections of $B$ over its base curve $\Delta: \mathcal{S}(B) = H^0(\Delta, \Omega(B^{\#}))$.

THEOREM 5.2. *Let $B_\Gamma$ be the elliptic modular surface attached to $\Gamma$.*

( i ) *If $\Gamma$ has torsion* (i. e. $s>0$), *then the group of sections $\mathcal{S}(B_\Gamma)$ is either trivial or a cyclic group of order* 3.

(ii) *If $\Gamma$ has a cusp of the second kind* (i. e. $t_2>0$), *then the group of sections $\mathcal{S}(B_\Gamma)$ is either trivial or isomorphic to one of the groups*

$$\boldsymbol{Z}/(2),\ \boldsymbol{Z}/(4) \quad or \quad \boldsymbol{Z}/(2)\times \boldsymbol{Z}/(2).$$

(iii) *If $\Gamma$ is torsion-free and all cusps are of the first kind, then the group of sections $\mathcal{S}(B_\Gamma)$ is isomorphic to a subgroup of $\boldsymbol{Z}/(m)\times\boldsymbol{Z}/(m)$, where $m$ denotes the least common multiple of $b_i'$s $(1\leqq i\leqq t_1)$. Here we suppose that the singular fibres of $B_\Gamma$ are of types $I_{b_i}$ $(1\leqq i\leqq t_1)$.*

PROOF. If $\Gamma$ satisfies the condition of (i) (or (ii)), then $B_\Gamma$ contains a singular fibre $C_v$ of type $IV^*$ (or $I_b^*$ $(b\geqq 1)$) by Proposition 4.2. Since the torsion subgroup of $C_v^{\#}$ is isomorphic to

$$\boldsymbol{Z}/(3) \qquad (\text{or } \boldsymbol{Z}/(4),\ \boldsymbol{Z}/(2)\times \boldsymbol{Z}/(2)),$$

the assertion (i) (or (ii)) follows from Remark 1.10 (and Theorem 5.1). The assertion (iii) is an immediate consequence of Proposition 1.6, i. e. the injectivity of the homomorphism

$$\mathcal{S}(B_\Gamma)\to \prod_v C_v^{\#}/\Theta_{v,0}^{\#}\cong \prod_{1\leqq i\leqq t_1} \boldsymbol{Z}/(b_i).$$

This completes the proof.

REMARK 5.3. For an elliptic modular surface $B=B_\Gamma$, we have $r'=2p_g$ by (5.1). Therefore, by Corollary 3.4 or 3.5, we see that both

$$H^1(\Delta, \mathcal{O}(\mathfrak{f}))/i^*H^1(\Delta, G) \quad \text{and} \quad H^2(B, \mathcal{O})/j^*H^2(B, \boldsymbol{Z})$$

are complex tori of dimension $p_g$, isogenous to each other by Theorem 3.1. We shall see in § 7 that these complex tori are essentially the same as Shimura's complex torus attached to cusp forms of weight 3 with respect to $\Gamma$.

We shall give a few examples of elliptic modular surfaces.

EXAMPLE 5.4. Let $\Gamma(N)$ denote the principal congruence subgroup of level $N$ in $SL(2, \boldsymbol{Z})$:

$$\Gamma(N)=\{\gamma\in SL(2, \boldsymbol{Z})\mid\gamma\equiv 1 \bmod N\}.$$

Assume that $N\geqq 3$. Then it is known (and easily seen) that the group $\Gamma(N)$ is torsion-free and all cusps for $\Gamma(N)$ are of the first kind. The numerical characters $\mu, t, g, \cdots$ for $\Gamma=\Gamma(N)$ will be denoted by $\mu(N), t(N), g(N), \cdots$. Then we have (e.g. [10])

$$\mu(N)=\frac{1}{2}N^3\prod_{p\mid N}(1-p^{-2}), \qquad t(N)=\mu(N)/N, \tag{5.2}$$

and

$$g(N)=1+(N-6)\mu(N)/12N. \tag{5.3}$$

Let $B(N)$ denote the elliptic modular surface attached to $\Gamma(N)$. We call $B(N)$ *the elliptic modular surface of level* $N$. All singular fibres of $B(N)$ are of type $I_N$ lying over the $t(N)$ cusps in $\Delta(N)$. In view of Corollary 1.5 and Theorem 5.1, the Picard number $\rho(N)$ of $B(N)$ is

$$\rho(N)=2+(N-1)\mu(N)/N. \tag{5.4}$$

The geometric genus $p_g(N)$ of $B(N)$ is, by (4.13) (note $t=t_1$):

$$p_g(N)=(N-3)\mu(N)/6N. \tag{5.5}$$

Note that the second Betti number $b_2(N)$ of $B(N)$ is equal to

$$b_2(N)=\rho(N)+2p_g(N). \tag{5.6}$$

Incidentally we note the asymptotic behavior:

$$\lim_{N\to\infty}\rho(N)/b_2(N)=3/4.$$

THEOREM 5.5. *For the elliptic modular surface $B(N)$ of level $N$ ($N\geqq 3$), the group of sections $\mathcal{S}(B(N))$ of $B(N)$ over the base curve $\Delta(N)$ consists of $N^2$ sections of order $N$.*

PROOF. By Theorem 5.2 (iii), $\mathcal{S}(B(N))$ is isomorphic to a subgroup of $(\boldsymbol{Z}/(N))^2$, since all singular fibres of $B(N)$ are of type $I_N$. Hence we have only to prove that $B(N)$ admits (at least) $N^2$ sections. For that purpose we recall the construction of an elliptic modular surface $B=B_\Gamma$ with torsion-free $\Gamma$. Let $\Phi$ denote the canonical projection of $B$ over $\Delta=\Delta_\Gamma$. Put $\Delta'=\Gamma\backslash\mathfrak{H}$ and $B'=\Phi^{-1}(\Delta')$. Then $B'$ is the quotient of $\mathfrak{H}\times\boldsymbol{C}$ by the group of automorphisms of the form:

$$(z, \zeta)\longmapsto(\gamma\cdot z, (cz+d)^{-1}(\zeta+n_1z+n_2)), \tag{5.7}$$

where $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $n_1, n_2$ are integers (cf. [K] p. 580). We denote by $((z, \zeta))$ the image of $(z, \zeta) \in \mathfrak{H} \times \boldsymbol{C}$ in $B'$. Note that $\mathfrak{H} \times \boldsymbol{C}$ (or $\mathfrak{H}$) is the universal covering of $B'$ (or $\Delta'$).

Now if $s'$ is a holomorphic section of $B'$ over $\Delta'$, then $s'$ is induced by a holomorphic map

$$f : \mathfrak{H} \longrightarrow \mathfrak{H} \times \boldsymbol{C}, \qquad f(z)=(z, \zeta(z)), \tag{5.8}$$

such that, for all $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we have

$$\zeta(\gamma \cdot z)=(cz+d)^{-1}(\zeta(z)+n_1 z+n_2) \tag{5.9}$$

with some integers $n_1, n_2$ depending on $\gamma$. Two functions $\zeta(z)$ and $\zeta'(z)$ satisfying (5.9) induce the same $s'$ if and only if

$$\zeta'(z)=\zeta(z)+m_1 z+m_2, \quad m_1, m_2 \in \boldsymbol{Z}.$$

In particular $s'$ is a section of finite order if and only if we have

$$\zeta(z)=a_1 z+a_2 \tag{5.10}$$

with rational numbers $a_1, a_2$ with the property:

$$(a_1, a_2)(\gamma-1) \in \boldsymbol{Z} \oplus \boldsymbol{Z} \qquad \text{for all } \gamma \in \Gamma. \tag{5.11}$$

Going back to the case where $\Gamma=\Gamma(N)$, we see that the condition (5.11) is equivalent to

$$a_1=\frac{m_1}{N}, \qquad a_2=\frac{m_2}{N}$$

with integers $m_1$ and $m_2$. Hence we get $N^2$ sections $s'_m$ of $B'$ over $\Delta'$:

$$s'_m : \Delta'=\Gamma \backslash \mathfrak{H} \ni (z) \longmapsto \left(\left(z, \frac{m_1 z+m_2}{N}\right)\right), \tag{5.12}$$

where $m=(m_1, m_2)$ runs over pairs of integers mod $N$. We shall show that each $s'_m$ can be extended to a holomorphic section $s_m$ of $B(N)$ over $\Delta(N)$. To examine the behavior of $s'_m$ at a cusp $v$ of $\Delta(N)$, we may assume that $v$ is the cusp at infinity $v_0$, because any cusp can be transformed to $v_0$ by a modular transformation. We put $v=v_0$ and

$$\tau=e^{2\pi i\, z/N}, \qquad w=e^{2\pi i \zeta}.$$

Let $E$ be a small neighborhood of $v$ with the local parameter $\tau$. With the notations of [K] p. 597–600, the part $C_v^{\#}$ of the singular fibre $C_v$ (of type $I_N$) is covered by $N$ open sets $W_i$ $(0 \leqq i \leqq N-1)$ of $B$ with coordinates $((\tau, w))_i$. The section $s'_m$ on $E-\{v\}$ can be expressed as

$$\tau \longmapsto ((\tau, e^{2\pi i(m_1 z+m_2)/N}))_0. \tag{5.13}$$

Since we have

$$((\tau, e^{2\pi i(m_1 z+m_2)/N}))_0 = ((\tau, e^{2\pi i\, m_2/N}\tau^{m_1}))_0$$
$$= ((\tau, e^{2\pi i\, m_2/N}))_{-m_1},$$

it is obvious that $s'_m$ can be extended to a holomorphic section over $E$; in particular, we have

$$s_m(v) = ((0, e^{2\pi i\, m_2/N}))_{-m_1} \in W_{-m_1}. \tag{5.14}$$

Thus we have proved the existence of $N^2$ sections of order $N$ of $B(N)$ over $\Delta(N)$. This completes the proof of Theorem 5.5.

REMARK 5.6. Let $K_N$ denote the function field of $\Delta(N)$; it is nothing but the field of modular functions of level $N$. Let $E_N$ be the generic fibre of $B(N)$. Then $E_N$ is an elliptic curve defined over $K_N$ and Theorem 5.5 implies that the group $E_N(K_N)$ of $K_N$-rational points of $E_N$ is exactly the group of points of order $N$ of $E_N$ ($N \geqq 3$). For $N=2$ and 3, we recall the following facts due to Igusa [5].

(i) Let $k$ be a field of characteristic $\neq 2$. Consider the elliptic curve

$$E_2: y^2 z = x(x-z)(x-\lambda z)$$

defined over $K_2 = k(\lambda)$, $\lambda$ being a variable over $k$. Then $E_2$ has exactly 4 $K_2$-rational points and they are points of order 2 (if we take one of them as an origin).

(ii) Let $k$ be a field of characteristic $\neq 3$ containing 3 cubic roots of unity. Consider the elliptic curve

$$E_3:\ x^3+y^3+z^3-3\mu xyz = 0$$

defined over $K_3 = k(\mu)$, $\mu$ being a variable over $k$. Then $E_3$ has exactly 9 $K_3$-rational points (i.e. base points of the pencil) and they are of order 3.

Thus our result may be viewed as a generalization of these facts to the case of higher level (in characteristic zero). The case of positive characteristic will be discussed in the appendix.

REMARK 5.7. It might be possible to extend Theorem 5.5 to the case of Siegel modular functions of higher degree. In fact, let $\mathfrak{S}_n$ denote the Siegel upper half plane of degree $n$ and let $\Gamma_n(N)$ denote the principal congruence subgroup of level $N$ of the Siegel modular group $S_p(n, \boldsymbol{Z})$. If $N \geqq 3$, the quotient $\Gamma_n(N)\backslash\mathfrak{S}_n$ is biholomorphic to a non-singular quasi-projective variety $U$. Igusa ([7] Lemma 5) constructed a fibre system $f: U^* \to U$ whose fibres are principally polarized abelian varieties of dimension $n$ and which has $N^{2n}$ rational sections of order $N$, by applying his theory of the desingularization of the Satake compactification.

EXAMPLE 5.8. Let $q$ be a prime number $\neq 2, 3$. Consider

$$\Gamma_0'(q)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \boldsymbol{Z}) \mid c \equiv 0 \bmod q, \left(\frac{a}{q}\right)=1\right\},$$

where $\left(\frac{a}{q}\right)$ denotes the Legendre symbol. We assume that $q \equiv 3 \bmod 4$ to ensure that $\Gamma_0'(q)$ does not contain $-1$. The numerical characters (cf. (4.3)) for the group $\Gamma_0'(q)$ are given as follows (cf. [4] No. 41, [**10**]):

$$\mu=q+1, \quad t'=t_1=2, \quad s=1+\left(\frac{-3}{q}\right).$$

Hence we have by (4.6)

$$g=\begin{cases} (q+1)/12 & \text{for } q \equiv -1 \bmod 12, \\ (q-7)/12 & \text{for } q \equiv 7 \bmod 12. \end{cases}$$

Let $B$ be the elliptic modular surface attached to $\Gamma_0'(q)$. Then $B$ has singular fibres of types $I_1$ and $I_q$ if $q \equiv -1$ (12) and two more fibres of type $IV^*$ if $q \equiv 7$ (12). The geometric genus of $B$ is computed by (4.13):

$$p_g=\begin{cases} (q-5)/6 & q \equiv -1 \ (12) \\ (q-1)/6 & q \equiv 7 \ (12). \end{cases}$$

As for the group $\mathcal{S}(B)$ of sections of $B$ over its base, we can see the following, using Theorem 5.2: If $q \equiv -1$ (12), then $\mathcal{S}(B)$ is either trivial or a cyclic group of order $q$. If $q \equiv 7$ (12), then $\mathcal{S}(B)$ is either trivial or a cyclic group of order 3.

EXAMPLE 5.9. To show that singular fibres of type $I_b^*$ actually occur, we give another example. Let $\Gamma$ denote the commutator subgroup of $SL(2, \boldsymbol{Z})$; $\Gamma$ does not contain $-1_2$ and $\Gamma \supset \Gamma(6)$. The numerical characters for $\Gamma$ are given as follows: (cf. [**10**])

$$\mu=6, \quad t'=1, \quad s=0, \quad g=1.$$

Since $t'=t_1+t_2$ and $t_1$ must be even (cf. (4.13)), we have $t_1=0$, $t_2=1$. Thus $B_\Gamma$ has only one singular fibre and it is of type $I_6^*$. The geometric genus $p_g$ is equal to 1 by (4.13).

## § 6. $\Gamma$-cusp forms and holomorphic forms on $B_\Gamma$.

Let $\Gamma$ be as before a subgroup of finite index of $SL(2, \boldsymbol{Z})$ with $\Gamma \not\ni -1$. Let $w$ be a positive integer. We recall that a holomorphic function $f$ defined on the upper half plane $\mathfrak{H}$ is called a $\Gamma$-cusp form of weight $w$ if it satisfies the functional equation (6.1) and the condition (6.5) below for all cusps:

$$f(\gamma \cdot z)=(cz+d)^w f(z) \quad \text{for all } \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{6.1}$$

For each cusp $v$ for $\Gamma$, we take a representative $x$ of $v$ in $\boldsymbol{Q}\cup\{i\infty\}$ and an element $\delta\in SL(2,\boldsymbol{Z})$ such that $\delta\cdot x=i\infty$. If we denote by $\Gamma_x$ the stabilizer of $x$ in $\Gamma$, then $\delta\Gamma_x\delta^{-1}$ stabilizes $i\infty$ and hence it is generated by an element of the form

$$\varepsilon\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\quad(b>0),\qquad \varepsilon=\pm 1, \tag{6.2}$$

where the sign $\varepsilon$ depends on whether the cusp $v$ is of the first or second kind (cf. (4.5)). If we put

$$g(z)=(c'z+d')^{-w}f(\delta^{-1}\cdot z),\qquad \delta^{-1}=\begin{pmatrix}\cdot & \cdot\\ c' & d'\end{pmatrix}, \tag{6.3}$$

we have

$$g(z+b)=\varepsilon^w\cdot g(z),\qquad \varepsilon^w=\pm 1. \tag{6.4}$$

Thus $g(z)$ (if $\varepsilon^w=1$) or $g(z)^2$ (if $\varepsilon^w=-1$) can be considered as a holomorphic function $h(\tau)$ of $\tau=e^{2\pi iz/b}$ for $|\tau|\neq 0$. With these notations, the second condition can be stated as follows:

(6.5) $h(\tau)$ is holomorphic and vanishes at $\tau=0$.

The vector space of $\Gamma$-cusp forms of weight $w$ will be denoted by $S_w(\Gamma)$. As is well-known, the space $S_2(\Gamma)$ is isomorphic to the space of holomorphic 1-forms on the curve $\Delta_\Gamma$ under the correspondence $f\mapsto f(z)dz$. In particular, $\dim S_2(\Gamma)$ is equal to the genus $g$ of $\Delta_\Gamma$. For $w\geqq 3$, the dimension of $S_w(\Gamma)$ can be calculated, for instance, with the aid of the Riemann-Roch theorem on the curve $\Delta_\Gamma$:

$$\dim S_w(\Gamma)=(w-1)(g-1)+s[w/3]+(w/2-1)t'+\delta(w)t_2/2, \tag{6.6}$$

where $s$, $t'$, $t_2$ have the same meaning as in §4 (4.3); $[w/3]$ denotes the largest integer $\leqq w/3$; and $\delta(w)=0$ or 1 according to whether the weight $w$ is even or odd. Note that, for $w=3$, we have

$$\dim S_3(\Gamma)=p_g, \tag{6.7}$$

$p_g$ being the geometric genus of the elliptic modular surface $B_\Gamma$ attached to $\Gamma$, cf. (4.13).

THEOREM 6.1. *The space $S_3(\Gamma)$ of $\Gamma$-cusp forms of weight* 3 *is canonically isomorphic* (*over* $\boldsymbol{C}$) *to the space of holomorphic* 2*-forms on $B_\Gamma$, i.e.*

$$S_3(\Gamma)\cong H^0(B_\Gamma,\Omega^2).$$

PROOF. We put $B=B_\Gamma$, $\Delta=\Delta_\Gamma$ and $\Delta'=\Delta-\Sigma(\subset\Gamma\backslash\mathfrak{H})$, $\Sigma$ being the set of elliptic points and cusps in $\Delta$. If we denote by $\mathfrak{H}'$ the inverse image of $\Delta'$ under the canonical map $\mathfrak{H}\to\Gamma\backslash\mathfrak{H}$, then $B'=B|_{\Delta'}$ is the quotient of $\mathfrak{H}'\times\boldsymbol{C}$ by antomorphisms:

$$(z, \zeta) \longmapsto (\gamma \cdot z, (cz+d)^{-1}(\zeta+n_1 z+n_2)), \tag{6.8}$$

where $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $n_1, n_2 \in \boldsymbol{Z}$. Now, taking a $\Gamma$-cusp form $f$ of weight 3, we consider the holomorphic 2-form on $\mathfrak{H} \times \boldsymbol{C}$:

$$\omega=\omega_f=f(z) dz \wedge d\zeta. \tag{6.9}$$

It is easily seen that $\omega$ is invariant under the automorphisms (6.8). Hence $\omega$ defines a holomorphic 2-form on $B'$, which we also denote by $\omega$. We shall see in the following that $\omega$ extends to a holomorphic form in a neighborhood of each singular fibre $C_v$ of $B$ $(v \in \Sigma)$.

Case i) The point $v$ is an elliptic point and $C_v$ is a singular fibre of type $IV^*$. Take a representative $z_0 \in \mathfrak{H}$ of $v$ and let $\Gamma_{z_0}$ be the stabilizer of $z_0$ in $\Gamma$; $\Gamma_{z_0}$ is a cyclic group of order 3. We may assume without loss of generality that $z_0=e^{2\pi i/3}$ and $\Gamma_{z_0}$ is generated by $\gamma_0=\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, since $z_0$ is transformed to $e^{2\pi i/3}$ by an element of $SL(2, \boldsymbol{Z})$. We put

$$\sigma=(z-z_0)/(z-z_0^2), \qquad \tau=\sigma^3. \tag{6.10}$$

We denote by $D$ a small neighborhood of $z_0$ defined by $|\sigma|<\delta$ $(\delta>0)$, and by $F$ the quotient of $D \times \boldsymbol{C}$ by the group of automorphisms (6.8) with $\gamma=1$. Then we are exactly in the situation of [**K**] p. 591-592, Case $(2_3)$. A cyclic group $C$ of order 3 corresponding to $\Gamma_{z_0}$ acts on $F$ and the quotient $F/C$ has three singular points $p_\nu$ $(\nu=0, 1, 2)$. Moreover the non-singular model of $F/C$ obtained by a reduction of singularities gives a neighborhood $B_v$ (denoted by $B_\rho$ in [**K**]) of the singular fibre $C_v$ under consideration. Now it is obvious that the form $\omega$ of (6.9) is holomorphic on $F/C-\{p_0, p_1, p_2\}$, since $\omega$ is invariant under all automorphisms (6.8). Using $\sigma$, $\tau$ in (6.10), we have

$$\omega \sim h(\tau) d\sigma \wedge d\zeta,$$

where $h(\tau)$ is holomorphic in $\tau$ and the symbol $\sim$ denotes the equality up to a (locally) non-vanishing holomorphic function. Then, using the local coordinates on $B_v$ given by [**K**] p. 592 (8.24), we see immediately that our form $\omega$ is holomorphic on the whole neighborhood $B_v$ of $C_v$.

Case ii) The point $v$ is a cusp of the first kind and the singular fibre $C_v$ is of type $I_b$. With the notations (6.2), $\cdots$, (6.5) (noting $\varepsilon=1$), we have

$$\delta^{-1} \cdot \omega=h(\tau) \frac{b}{2\pi i} \frac{d\tau}{\tau} \wedge d\zeta. \tag{6.11}$$

Since $h(\tau)$ vanishes at $\tau=0$ by (6.5), the right side of (6.11) is holomorphic in $\tau$ and $\zeta$. Therefore, by the structure of a neighborhood of $C_v$ (cf. [**K**] p. 599-600, Case $(1_2)$), our form $\omega$ is holomorphic at every point of $C_v^{\#}=C_v-\{b \text{ points}\}$. Hence $\omega$ must be holomorphic in a neighborhood of $C_v$ in $B$.

Case iii) The point $v$ is a cusp of the second kind and the singular fibre $C_v$ is of type $I_b^*$. With the notations (6.2), $\cdots$, (6.5) (noting $\varepsilon=-1$), we have

$$\delta^{-1}\cdot\omega=\sqrt{h(\tau)}\,\frac{b}{2\pi i}\,\frac{d\tau}{\tau}\wedge d\zeta\,. \tag{6.12}$$

Again, by [**K**] p. 600-602, Case (2), we know that a neighborhood of a singular fibre $C_v$ of type $I_b^*$ is obtained by a reduction of singularities from the quotient of a neighborhood of a singular fibre of type $I_{2b}$ by a group of order 2. Using the result of case ii), we can argue as in case i).

Thus we have seen that, for each $f\in S_3(\Gamma)$, the 2-form $\omega=\omega_f$ defined by (6.9) is holomorphic on the whole surface $B$. Obviously the map $f\mapsto\omega_f$ is injective. In view of (6.7), this completes the proof of Theorem 6.1.

Let $\mathfrak{f}$ be the line bundle over $\Delta$ as in (2.7), and let $\mathfrak{k}$ be the canonical bundle of $\Delta$. Then the canonical bundle of $B$ is induced from the line bundle $\mathfrak{k}-\mathfrak{f}$ over $\Delta$ by the canonical projection $B\to\Delta$ ([**K**] Theorem 12.1). Therefore we have a canonical isomorphism:

$$H^0(B,\,\Omega^2)\cong H^0(\Delta,\,\mathcal{O}(\mathfrak{k}-\mathfrak{f}))\,. \tag{6.13}$$

Hence we get

COROLLARY 6.2. *There is a canonical isomorphism* (*over* $\boldsymbol{C}$):

$$H^0(\Delta,\,\mathcal{O}(\mathfrak{k}-\mathfrak{f}))\cong S_3(\Gamma)\,.$$

We identify the two spaces by the canonical isomorphism. By the duality theorem on a curve, we have a natural ($\boldsymbol{C}$-bilinear) non-degenerate pairing:

$$\begin{aligned} H^0(\Delta,\,\mathcal{O}(\mathfrak{k}-\mathfrak{f}))\times H^1(\Delta,\,\mathcal{O}(\mathfrak{f}))&\longrightarrow\boldsymbol{C}\\ (f,\,\xi)&\longmapsto\langle f,\,\xi\rangle\,. \end{aligned} \tag{6.14}$$

The value $\langle f,\,\xi\rangle$ is given as follows. We take a sufficiently fine finite open covering $\mathfrak{U}=\{U_i\}$ of $\Delta$ and represent the cohomology class $\xi$ by a 1-cocycle $(\xi_{ij})$, where $\xi_{ij}$ is a holomorphic section of $\mathfrak{f}$ over $U_i\cap U_j\neq\emptyset$. The 1-cocycle $(f\xi_{ij})$ determines a cohomology class in $H^1(\Delta,\,\mathcal{O}(\mathfrak{k}))=H^1(\Delta,\,\Omega^1)$, $\Omega^1$ being the sheaf of germs of holomorphic 1-forms on $\Delta$. We have

$$H^1(\Delta,\,\Omega^1)\cong H^2(\Delta,\,\boldsymbol{C})\cong\boldsymbol{C}\,,\qquad(f\xi_{ij})\mapsto\langle f,\,\xi\rangle\,, \tag{6.15}$$

where the first isomorphism comes from the exact sequence $0\to\boldsymbol{C}\to\mathcal{O}\to\Omega^1\to0$, and the second comes from the evaluation of a 2-cocycle on the fundamental class $\Delta$. Then the value $\langle f,\,\xi\rangle$ is the complex number corresponding to the cohomology class of $(f\xi_{ij})$ under (6.15). In the next section, we shall explicitly compute $\langle f,\,\xi\rangle$ when $\xi$ is an element of the image $i^*H^1(\Delta,\,G)\subset H^1(\Delta,\,\mathcal{O}(\mathfrak{f}))$.

On the other hand, the space $S_3(\Gamma)$ of $\Gamma$-cusp forms of weight 3 is self-dual (over $\boldsymbol{R}$) with respect to the Petersson metric. Recall that, for any weight $w$, the Petersson metric on the space $S_w(\Gamma)$ of $\Gamma$-cusp forms of weight $w$ is a positive definite Hermitian scalar product defined by

$$(f, g) = \int_{\Gamma \backslash \mathfrak{H}} f(z)\overline{g(z)}y^{w-2}dxdy\,, \qquad z = x+iy \in \mathfrak{H}\,, \tag{6.16}$$

for $f,\ g \in S_w(\Gamma)$. Comparing (6.14) and (6.16), we get the following result.

PROPOSITION 6.3. *For each $\xi \in H^1(\Delta_\Gamma, \mathcal{O}(\mathfrak{f}))$, let $\phi(\xi)$ denote the unique element of $S_3(\Gamma)$ satisfying*

$$\langle f, \xi \rangle = 4(f, \phi(\xi)) \qquad \textit{for all } f \in S_3(\Gamma)\,. \tag{6.17}$$

*Then $\phi$ is a $\boldsymbol{C}$-antilinear isomorphism:*

$$\phi : H^1(\Delta_\Gamma, \mathcal{O}(\mathfrak{f})) \xrightarrow{\sim} S_3(\Gamma)\,.$$

Similarly the space $H^0(B_\Gamma, \Omega^2)$ of holomorphic 2-forms on $B_\Gamma$ is dual (over $\boldsymbol{C}$) to the space $H^2(B_\Gamma, \mathcal{O})$ by Serre duality. Hence Theorem 6.1 implies that $H^2(B_\Gamma, \mathcal{O})$ is canonically isomorphic to $\overline{S_3(\Gamma)}$, the space $S_3(\Gamma)$ together with the complex structure which is conjugate to the usual one. In view of (5.1) and (2.11), we get

PROPOSITION 6.4. *The Hodge decomposition of the two dimensional cohomology of the surface $B_\Gamma$ is given as follows:*

$$H^2(B_\Gamma, \boldsymbol{C}) \cong S_3(\Gamma) \oplus \overline{S_3(\Gamma)} \oplus (NS(B_\Gamma) \otimes \boldsymbol{C})\,. \tag{6.18}$$

## § 7. Shimura's complex torus for weight 3.

As we have noted in Remark 5.3, for the elliptic modular surface $B = B_\Gamma$ (over $\Delta = \Delta_\Gamma$), the quotients

$$H^1(\Delta, \mathcal{O}(\mathfrak{f}))/i^*H^1(\Delta, G) \quad \text{and} \quad H^2(B, \mathcal{O})/j^*H^2(B, \boldsymbol{Z})$$

are complex tori of dimension $p_g$. In this section we shall show that these complex tori have another interpretation as an analogue for weight 3 of Shimura's abelian varieties attached to cusp forms of even weights (cf. Shimura [22], cited as [S] in the following). Incidentally this will give another proof that $H^1(\Delta, \mathcal{O}(\mathfrak{f}))/i^*H^1(\Delta, G)$ is a complex torus, independent of the results in § 3 (cf. [24]).

We shall begin with the definition of the parabolic cohomology groups of $\Gamma$ following [S], restricting our attention to the weight 3 case. Let $R$ denote one of the rings $\boldsymbol{Z}$, $\boldsymbol{R}$ or $\boldsymbol{C}$, and let $R^2$ denote the module of column vectors with coefficients in $R$. By an $R$-valued parabolic cocycle of $\Gamma$, we mean a map

$$\mathfrak{x} : \Gamma \longrightarrow R^2$$

satisfying the two conditions:

$$\mathfrak{x}(\sigma\sigma')=\mathfrak{x}(\sigma)+\sigma\mathfrak{x}(\sigma') \qquad \text{for } \sigma,\ \sigma'\in\Gamma\,; \tag{7.1}$$

$$\mathfrak{x}(\gamma)\in(\gamma-1)R^2 \qquad \text{for parabolic } \gamma\in\Gamma\,, \tag{7.2}$$

where $\Gamma\subset SL(2,\boldsymbol{Z})$ naturally acts on $R^2$ from the left. A coboundary is a cocycle $\mathfrak{x}$ of the form

$$\mathfrak{x}(\sigma)=(\sigma-1)\mathfrak{x}_0 \qquad \text{for all } \sigma\in\Gamma\,,$$

where $\mathfrak{x}_0$ is an arbitrary (fixed) element of $R^2$. The parabolic cohomology group of $\Gamma$, denoted by $H^1_{\mathrm{par}}(\Gamma, R^2)$, is defined as the quotient of the group of all $R$-valued parabolic cocycles modulo the subgroup of coboundaries. The natural injection $\boldsymbol{Z}\to\boldsymbol{R}$ induces a canonical homomorphism:

$$c:\ H^1_{\mathrm{par}}(\Gamma,\boldsymbol{Z}^2)\longrightarrow H^1_{\mathrm{par}}(\Gamma,\boldsymbol{R}^2)\,.$$

The following is a special case of Proposition 1 of [S] § 3.

PROPOSITION 7.1. *The image of $H^1_{\mathrm{par}}(\Gamma,\boldsymbol{Z}^2)$ under $c$ is a lattice in the real vector space $H^1_{\mathrm{par}}(\Gamma,\boldsymbol{R}^2)$.*

We shall next consider the relation of cusp forms (of weight 3) to parabolic cohomology. For each $f\in S_3(\Gamma)$, we put

$$F_f(z)=\int_{z_0}^{z}\binom{z}{1}f(z)dz \qquad \text{and} \qquad \mathfrak{x}_f(\sigma)=F_f(\sigma\cdot z_0) \quad (\sigma\in\Gamma)\,, \tag{7.3}$$

where $z_0$ is a fixed base point in $\mathfrak{H}$. Since we have

$$\binom{\sigma\cdot z}{1}f(\sigma\cdot z)d(\sigma\cdot z)=\sigma\binom{z}{1}f(z)dz\,,$$

we get

$$F_f(\sigma\cdot z)=\sigma F_f(z)+\mathfrak{x}_f(\sigma)\,, \tag{7.4}$$

and $\mathfrak{x}_f$ is a $\boldsymbol{C}$-valued parabolic cocycle; note that the cohomology class of $\mathfrak{x}_f$ is uniquely determined by $f$ and independent of the choice of $z_0\in\mathfrak{H}$. If we denote by $\varphi(f)$ the cohomology class in $H^1_{\mathrm{par}}(\Gamma,\boldsymbol{R}^2)$ containing the real cocycle $\mathrm{Re}(\mathfrak{x}_f)$, we get an $\boldsymbol{R}$-linear homomorphism:

$$\varphi:\ S_3(\Gamma)\longrightarrow H^1_{\mathrm{par}}(\Gamma,\boldsymbol{R}^2)\,.$$

PROPOSITION 7.2. *$\varphi$ is an isomorphism of $S_3(\Gamma)$ onto $H^1_{\mathrm{par}}(\Gamma,\boldsymbol{R}^2)$.*

We omit the proof, because this is a special case of a general result of Shimura ([**29**] Chapter 8).

The purpose of this section is to prove:

THEOREM 7.3. *There is an isomorphism $\eta$ of $H^1(\Delta, G)$ onto $H^1_{\mathrm{par}}(\Gamma,\boldsymbol{Z}^2)$, which makes the following diagram commute:*

$$\begin{tikzcd}
H^1(\Delta, G) \arrow[rr, "i^*", "(2.8)"'] \arrow[dd, "\eta"'] & & H^1(\Delta, \mathcal{O}(\mathfrak{f})) \arrow[d, "\phi", "\wr"'] & \text{(Prop. 6.3)} \\
& & S_3(\Gamma) \arrow[d, "\varphi", "\wr"'] & \text{(Prop. 7.2)} \\
H^1_{\mathrm{par}}(\Gamma, \boldsymbol{Z}^2) \arrow[rr, "c", "\text{(Prop. 7.1)}"'] & & H^1_{\mathrm{par}}(\Gamma, \boldsymbol{R}^2) .
\end{tikzcd}$$

In order to apply the results of § 2, we need to make explicit the relation of $H_1(\Delta, G)$ and $H^1(\Delta, G)$. We consider a sufficiently fine simplicial decomposition $\mathfrak{T}$ of the Riemann surface $\Delta=\Delta_\Gamma$, which is a subdivision of the decomposition, say $\mathfrak{T}_0$, of $\Delta$ considered in § 2 to compute $H^i(\Delta, G)$. We denote by $(\lambda)$ $(\lambda\in\Lambda)$ the vertices of $\mathfrak{T}$, by $(\lambda\mu)$ the 1-simplex connecting $(\lambda)$ and $(\mu)$, and by $(\lambda\mu\nu)$ the 2-simplex with the vertices $(\lambda)$, $(\mu)$, $(\nu)$. Let $\mathfrak{T}^*$ be the dual cell decomposition of $\mathfrak{T}$. We denote by $[\lambda]$, $[\lambda\mu]$ or $[\lambda\mu\nu]$ the dual cells in $\mathfrak{T}^*$ (of dimension 2, 1, 0 respectively) corresponding to the simplices $(\lambda)$, $(\lambda\mu)$ or $(\lambda\mu\nu)$. For an alternating 1-chain $c_1$ with respect to $\mathfrak{T}$ with coefficients in the sheaf $G$:

$$c_1=\sum g_{\lambda\mu}(\lambda\mu)\,, \tag{7.5}$$

we can define a 1-cochain $c^1$ with respect to $\mathfrak{T}^*$:

$$c^1:\ [\lambda\mu]\longmapsto g_{\lambda\mu}\,, \tag{7.6}$$

and, as is easily seen, the map $c_1\mapsto c^1$ induces an isomorphism:

$$H_1(\Delta, G)\xrightarrow{\sim} H^1(\Delta, G)\,. \tag{7.7}$$

Moreover, if we denote by $U_\lambda$ the interior of the union of 2-simplices having the vertex $(\lambda)$ in common, then

$$\mathfrak{U}=\{U_\lambda\}_{\lambda\in\Lambda} \tag{7.8}$$

forms a (sufficiently fine) open covering of $\Delta$ and the 1-cochain $c^1$ (7.6) may be considered as a 1-cochain on the nerve of the covering $\mathfrak{U}$. Hence the right side of (7.7) can be considered as the cohomology group in Čech's sense.

We take a suitable fundamental domain $\mathfrak{F}$ of $\Gamma$ in the upper half plane $\mathfrak{H}$ and consider the simplicial decomposition of $\overline{\mathfrak{F}}$ corresponding to $\mathfrak{T}$ on $\Delta$. If we denote by $\tilde{\alpha}_i$, $\tilde{\beta}_i$, $\cdots$ the sides of $\mathfrak{F}$ corresponding to the paths $\alpha_i$, $\beta_i$, $\cdots$ of the fundamental group $\pi_1(\Delta')$ of $\Delta'$ in (2.1), then the boundary of $\overline{\mathfrak{F}}$ consists of $4g+2t$ sides

$$\begin{aligned}
&\tilde{\alpha}_i,\ \tilde{\beta}_i,\ -\varphi(\alpha_i)(\tilde{\alpha}_i),\ -\varphi(\beta_i^{-1})(\tilde{\beta}_i) && (1\leqq i\leqq g)\,,\\
&\tilde{\gamma}_j,\ -\varphi(\gamma_j)(\tilde{\gamma}_j) && (1\leqq j\leqq t=s+t')\,,
\end{aligned} \tag{7.9}$$

where $\varphi$ is the representation $\pi_1(\Delta')\to\Gamma\subset SL(2, \boldsymbol{Z})$. Writing $\alpha_i$, $\beta_i$, $\cdots$ for $\varphi(\alpha_i)$, $\varphi(\beta_i)$, $\cdots$ to simplify the notation, we have obtained standard generators of $\Gamma$ with the relations (cf. (2.1)):

$$\begin{aligned} &\gamma_t \cdots \gamma_1 \cdots \beta_1^{-1}\alpha_1^{-1}\beta_1\alpha_1 = 1 \\ &\gamma_1^e = \cdots = \gamma_s^e = 1\,, \qquad (e=3)\,, \end{aligned} \tag{7.10}$$

assuming that $s$ points $v_i \in \Sigma$ $(1 \leqq i \leqq s)$ are the elliptic points of $\Delta_\Gamma$. Put (cf. (2.4))

$$\kappa_i = \beta_i^{-1}\alpha_i^{-1}\beta_i\alpha_i\,, \qquad \kappa^{(i)} = \kappa_i \cdots \kappa_1\,, \qquad \kappa = \kappa^{(g)}\,;$$
$$\gamma^{(j)} = \gamma_j \cdots \gamma_1\,, \qquad \gamma^{(0)} = \kappa^{(0)} = 1\,.$$

PROPOSITION 7.4. *The Petersson metric $(f, g)$ $(f, g \in S_3(\Gamma))$ can be expressed in terms of the parabolic cocycles $\mathfrak{x}_f$ and $\overline{\mathfrak{x}_g}$ (complex conjugate) as follows. Put $\mathfrak{x} = \mathfrak{x}_f$ and $\mathfrak{y} = \overline{\mathfrak{x}_g}$. Then*

$$\begin{aligned} 4(f, g) = &\sum_{i=1}^{g} {}^t\mathfrak{x}(\alpha_i^{-1})P[\mathfrak{y}(\alpha_i^{-1}\beta_i\alpha_i\kappa_{i-1}) - \mathfrak{y}(\kappa_{i-1})] \\ &+ \sum_{i=1}^{g} {}^t\mathfrak{x}(\beta_i)P[\mathfrak{y}(\beta_i\alpha_i\kappa_{i-1}) - \mathfrak{y}(\alpha_i^{-1}\beta_i\alpha_i\kappa_{i-1})] \\ &- \sum_{j=1}^{t} {}^t\mathfrak{x}(\gamma_j^{-1})P[\mathfrak{y}(\gamma^{(j-1)}\kappa) - \mathfrak{y}_j]\,, \end{aligned} \tag{7.11}$$

*where $P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathfrak{y}_j$ is the value of the function $\bar{F}_g$ at the representative of $v_j$ in $\mathfrak{F}$ $(1 \leqq j \leqq t)$.*

The proof can be found in [S] § 4 (especially the formula (19)) by suitably changing the notations. The idea of the proof is similar to that of the well-known Riemann bilinear relation on a compact Riemann surface. Note that the above $\mathfrak{y}_j$ satisfies the relation: $\mathfrak{y}(\gamma_j) = (1-\gamma_j)\mathfrak{y}_j$ (cf. [S] (14)).

Now we shall compute the value $\langle f, \xi \rangle$ for $f \in S_3(\Gamma)$ and $\xi = i^*(g) \in i^*H^1(\Delta, G) \subset H^1(\Delta, \mathcal{O}(\mathfrak{f}))$ (cf. (6.14)). We take a representative cocycle $(g_{\lambda\mu})$ of $g$ and put $\xi_{\lambda\mu} = i^*g_{\lambda\mu}$, where $g_{\lambda\mu}$ (or $\xi_{\lambda\mu}$) is a section of $G$ (or $\mathfrak{f}$) over the open set $U_{\lambda\mu} = U_\lambda \cap U_\mu \neq \emptyset$. We lift $U_{\lambda\mu}$ to an open set $\tilde{U}_{\lambda\mu}$ in the upper half plane so that either $\tilde{U}_{\lambda\mu}$ is contained in $\mathfrak{F}$ or $\tilde{U}_{\lambda\mu}$ meets one of the sides $\tilde{\alpha}_i$, $\tilde{\beta}_i$ or $\tilde{\gamma}_j$. Then $\xi_{\lambda\mu}$ can be identified with a holomorphic function on $\tilde{U}_{\lambda\mu}$ of the form

$$\xi_{\lambda\mu} = n_1 z + n_2\,, \qquad (n_1, n_2) = g_{\lambda\mu} \in \boldsymbol{Z} \oplus \boldsymbol{Z}\,.$$

If we put

$$\begin{aligned} &Y_{\lambda\mu}(u) = \int_{z_0}^{z} f(z)\xi_{\lambda\mu}dz = g_{\lambda\mu}F_f(z)\,, \qquad z \in \tilde{U}_{\lambda\mu} \\ &c_{\lambda\mu\nu} = Y_{\lambda\mu}(u) + Y_{\mu\nu}(u) + Y_{\nu\lambda}(u)\,, \qquad u \in U_{\lambda\mu\nu}\,, \end{aligned} \tag{7.12}$$

$u$ being the image in $\Delta_\Gamma$ of the point $z \in \mathfrak{H}$, then we have by (6.15)

$$\langle f, \xi \rangle = \sum c_{\lambda\mu\nu}\,, \tag{7.13}$$

where the summation is over all positively oriented 2-simplices $(\lambda\mu\nu)$ of $\mathfrak{T}$. Obviously $c_{\lambda\mu\nu}=0$ unless $(\lambda\mu\nu)$ has a side lying on the paths $\alpha_i$, $\beta_i$ or $\gamma_j$. Suppose that $(\lambda\mu)$ lies on $\alpha_i$. Let $(\lambda\mu\nu)$ and $(\lambda\mu\nu')$ be the 2-simplices having $(\lambda\mu)$ as a side; we take $\nu$ so that $\tilde{U}_{\lambda\mu}$ meets $\tilde{U}_{\lambda\nu}$. Then we have

$$c_{\lambda\mu\nu}=0,$$

and

$$c_{\lambda\mu\nu'}=g_{\lambda\mu}\mathfrak{x}_f(\alpha_i^{-1}),$$

since, for $u\in U_{\lambda\mu\nu'}$ and $z\in\alpha_i(\tilde{U}_{\lambda\mu})\cap\tilde{U}_{\lambda\nu'}$, we have by (7.4)

$$Y_{\lambda\mu}(u)=g_{\lambda\mu}F(\alpha_i^{-1}\cdot z)=g_{\lambda\mu}(\alpha_i^{-1}F_f(z)+\mathfrak{x}_f(\alpha_i^{-1}))$$

and $(g_{\lambda\mu})$ is a cocycle with coefficients in $G$. Therefore, if we put

$$a_i=\sum g_{\lambda\mu}(\in \boldsymbol{Z}\oplus\boldsymbol{Z})$$

with $(\lambda\mu)$ running over all positive 1-simplices contained in the path $\alpha_i$, and define $b_i$, $c_j\in\boldsymbol{Z}\oplus\boldsymbol{Z}$ similarly for $\beta_i$, $\gamma_j$, then we get

$$-\langle f,\xi\rangle=\sum_{i=1}^{g}[a_i\mathfrak{x}_f(\alpha_i^{-1})+b_i\mathfrak{x}_f(\beta_i)]+\sum_{j=1}^{t}c_j\mathfrak{x}_f(\gamma_j^{-1}), \tag{7.14}$$

with

$$\sum_{i=1}^{g}[a_i(1-\alpha_i^{-1})+b_i(1-\beta_i)]+\sum_{j=1}^{t}c_j(1-\gamma_j^{-1})=0. \tag{7.15}$$

By comparing (7.11) and (7.14), we want to define an integral parabolic cocycle $\mathfrak{y}=\eta(g)$ for $g=(g_{\lambda\mu})\in H^1(\varDelta, G)$ by the conditions:

$$\begin{aligned}
-{}^t a_i &= P[\mathfrak{y}(\alpha_i^{-1}\beta_i\alpha_i\kappa_{i-1})-\mathfrak{y}(\kappa_{i-1})],\\
-{}^t b_i &= P[\mathfrak{y}(\beta_i\alpha_i\kappa_{i-1})-\mathfrak{y}(\alpha_i^{-1}\beta_i\alpha_i\kappa_{i-1})],\\
{}^t c_j &= P[\mathfrak{y}(\gamma^{(j-1)}\kappa)-\mathfrak{y}_j],\\
\mathfrak{y}(\gamma_j) &= (1-\gamma_j)\mathfrak{y}_j, \qquad (1\leqq i\leqq g,\ 1\leqq j\leqq t).
\end{aligned} \tag{7.16}$$

By the cocycle condition (7.1), we can express the values $\mathfrak{y}(\alpha_i)$, $\mathfrak{y}(\beta_i)$, $\mathfrak{y}(\gamma_j)$ and $\mathfrak{y}_j$ as integral linear combinations of ${}^t a_i$, ${}^t b_i$ and ${}^t c_j$; hence they are integral. Moreover it follows from (7.15) that the map $\mathfrak{y}:\Gamma\to\boldsymbol{Z}^2$ thus defined is compatible with the relations (7.10). Hence $\mathfrak{y}=\eta(g)$ is really an integral parabolic cocycle of $\Gamma$ and we get a homomorphism:

$$\eta:\ H^1(\varDelta, G)\longrightarrow H^1_{\mathrm{par}}(\Gamma, \boldsymbol{Z}^2).$$

It is clear by the above definition of $\eta$ that $\eta$ satisfies the conditions of Theorem 7.3. This completes the proof.

REMARK 7.5. Let $D_w(\Gamma)$ denote the subgroup of $S_w(\Gamma)$ consisting of cusp forms $f$ of weight $w$ whose "period" $\mathfrak{x}_f$ has integral real part. Then $D_w(\Gamma)$ is a lattice of the complex vector space $S_w(\Gamma)$ by Propositions 7.1 and 7.2, and the quotient $S_w(\Gamma)/D_w(\Gamma)$ is called Shimura's complex torus attached to $\Gamma$-cusp forms of weight $w$. For $w=3$, Theorem 7.3 implies that

$$S_3(\Gamma)/D_3(\Gamma) \cong H^1(\Delta, \mathcal{O}(\mathfrak{f}))/i^*H^1(\Delta, G),$$

if we take the complex structure on $S_3(\Gamma)$ that is conjugate to the usual one. Therefore Shimura's complex torus for weight 3 is essentially the same as the group $H^1(\Delta, \Omega(B^*))$ (cf. (3.1)), which has the geometric significance as the parameter space of the family $\mathcal{F}(J, G)$ of elliptic surfaces. In particular, the subgroup of division points of $S_3(\Gamma)/D_3(\Gamma)$ has an algebro-geometric (or arithmetic) interpretation as essentially the group of locally trivial principal homogeneous spaces for $B$ over $\Delta$, $B$ being the elliptic modular surface attached to $\Gamma$.

REMARK 7.6. As to the question of whether or not the complex torus $S_w(\Gamma)/D_w(\Gamma)$ for an odd weight $w$ has a structure of abelian variety (as in even weight case), the following has been remarked by Prof. Shimura. In general, a complex torus of dimension $n$ has a structure of abelian variety if its endomorphism algebra (tensored by $\boldsymbol{Q}$) contains a totally real field of degree $n$. By this fact and the theory of Hecke operators, it can be shown that $S_3(\Gamma)/D_3(\Gamma)$ has a structure of abelian variety for a certain class of $\Gamma$, for instance for the groups $\Gamma_0'(q)$ of Example 5.8.

EXAMPLE 7.7. For $\Gamma=\Gamma(4)$, the congruence subgroup of level 4 of $SL(2, \boldsymbol{Z})$, the elliptic modular surface $B(4)$ for level 4 is a $K3$ surface (cf. [**12**]), since we have

$$g=0 \quad \text{and} \quad p_g=1$$

by (5.3), (5.5). The complex torus $H^1(\Delta, \mathcal{O}(\mathfrak{f}))/i^*H^1(\Delta, G)$ or $S_3(\Gamma)/D_3(\Gamma)$ is of dimension 1, i.e. an elliptic curve. The space $S_3(\Gamma)$ of $\Gamma$-cusp forms of weight 3 is spanned by one element $f$:

$$f(z)=\Delta(z)^{1/4},$$

where $\Delta(z)$ is the well-known cusp form of weight 12 for $SL(2, \boldsymbol{Z})$. By an argument similar to [**S**] p. 309, we see that $S_3(\Gamma)/D_3(\Gamma)$ is an elliptic curve with complex multiplication by $\boldsymbol{Q}(\sqrt{-1})$.

REMARK 7.8. Kuga-Satake [**13**] has attached to a polarized $K3$ surface $S$ an abelian variety, $A_S$, of dimension $2^{19}$, and has shown among others that, if $S$ is "singular" in the sense that the Picard number of $S$ is 20 $(=b_2-2p_g)$, then the abelian variety $A_S$ is isogenous to the self-product of $2^{19}$ copies of an elliptic curve with complex multiplication. Now the $K3$ surface $B(4)$ is singular by (5.4)—in fact, every elliptic modular surface is "singular", i.e. $\rho=b_2-2p_g$ by (5.1) and Corollary 2.6, and the elliptic curve $S_3(\Gamma(4))/D_3(\Gamma(4))$ is presumably isogenous to the simple component of the abelian variety $A_{B(4)}$.

## Appendix. Arithmetic applications.

### A. Algebraic reformulations.

We first recall Igusa's theory of elliptic modular functions in arbitrary characteristic not dividing the level, as reformulated by Deligne (see [5], [6], [2]). Fix a positive integer $n \geqq 3$. Let $M_n$ denote the moduli scheme for the elliptic curves with level $n$ structure; $M_n$ exists and is an affine curve over Spec $(\boldsymbol{Z}[1/n])$. We recall the following facts:

i) The scheme $M_n$ is compactified to a curve scheme $M_n^*$, projective and smooth over Spec$(\boldsymbol{Z}[1/n])$, and $M_n^* - M_n$ is an étale covering of Spec$(\boldsymbol{Z}[1/n])$ ([2] Theorem 4.1).

ii) The curve $M_n \otimes \boldsymbol{C}$ (or $M_n^* \otimes \boldsymbol{C}$) over $\boldsymbol{C}$ is analytically isomorphic to the Riemann surface $\Gamma(n)\backslash\mathfrak{H}$ (or its compactification $\Delta(n)$), cf. Example 5.4.

iii) The algebraic closure of $\boldsymbol{Q}$ in the function field $K_n$ of $M_n^* \otimes \boldsymbol{Q}$ is $\boldsymbol{Q}(\zeta_n)$, $\zeta_n$ being a primitive $n$-th root of unity (cf. [23]), and hence $M_n^* \otimes \boldsymbol{Q}$ can be considered as a non-singular projective curve defined over $\boldsymbol{Q}(\zeta_n)$. All points of $\Sigma_0 = M_n^* \otimes \boldsymbol{Q} - M_n \otimes \boldsymbol{Q}$ are rational over $\boldsymbol{Q}(\zeta_n)$.

iv) Let $p$ be an arbitrary prime number not dividing $n$, and let $p^f$ be the smallest $p$-power such that $p^f \equiv 1 \mod n$. Then $M_n^* \otimes \boldsymbol{F}_p$ can be considered as a non-singular projective curve defined over $\boldsymbol{F}_{p^f}$. All points of $\Sigma_p = M_n^* \otimes \boldsymbol{F}_p - M_n \otimes \boldsymbol{F}_p$ are rational over $\boldsymbol{F}_{p^f}$ (cf. [5]).

Now, let $E \rightarrow M_n$ denote the universal family of elliptic curves with level $n$ structure. Then

v) $E$ admits $n^2$ sections of order $n$ over $M_n$.

vi) Let $E_0^*$ denote the Néron model of $E \otimes \boldsymbol{Q}$ over $M_n^* \otimes \boldsymbol{Q}$. Then $E_0^*$ is a (non-singular projective) elliptic surface over $M_n^* \otimes \boldsymbol{Q}$ having $n^2$ sections of order $n$, all defined over $\boldsymbol{Q}(\zeta_n)$. It can be verified with the aid of Theorem 5.5 that $E_0^* \otimes \boldsymbol{C}$ is analytically isomorphic to the elliptic modular surface (of level $n$) $B(n)$ over $\Delta(n)$.

vii) Let $E_p^*$ denote the Néron model of $E \otimes \boldsymbol{F}_p$ over $M_n^* \otimes \boldsymbol{F}_p$. Then $E_p^*$ is a (non-singular projective) elliptic surface over $M_n^* \otimes \boldsymbol{F}_p$, having $n^2$ sections of order $n$ defined over $\boldsymbol{F}_{p^f}$, $f$ being as in iv). $E_p^*$ will be called the *elliptic modular surface of level n in characteristic p.*

viii) The singular fibres $C_v$ of $E_0^*$ (or $E_p^*$) lie over $\Sigma_0$ (or $\Sigma_p$) and they are of type $I_n$. Since each point $v$ of $\Sigma_0$ (or of $\Sigma_p$) is rational over $\boldsymbol{Q}(\zeta_n)$ (or $\boldsymbol{F}_{p^f}$), the divisor $C_v$ is rational over the same field. By the construction of Néron model ([17]), we see that the components $\Theta_{v,i}$ of $C_v$ are rational over an at most quadratic extension of $\boldsymbol{Q}(\zeta_n)$ (or $\boldsymbol{F}_{p^f}$).

Finally we note the following.

ix) The Betti number $b_2(n)$ of $E_0^*$ (i.e. of $B(n)$) satisfies the relation:

$$b_2(n) = 2p_g(n) + \rho(n),$$
$$p_g(n) = \dim S_3(\Gamma(n)):$$

cf. (5.4), (5.6) and (6.7).

**B. Arithmetic theory of surfaces over a finite field.**

In order to consider arithmetic questions on the elliptic modular surfaces of level $n$, we recall the known facts and conjectures for an algebraic surface over a finite field (cf. [26], [27]). Let $X$ denote a non-singular projective surface defined over a finite field $\boldsymbol{F}_q$ such that $\bar{X} = X \otimes \bar{\boldsymbol{F}}_q$ is connected ($\bar{\boldsymbol{F}}_q$ is an algebraic closure of $\boldsymbol{F}_q$). Then the zeta function of $X$ is of the form:

$$\zeta(X, T) = \frac{P_1(X, T)P_1(X, qT)}{(1-T)P_2(X, T)(1-q^2T)}, \qquad T = q^{-s},$$

where $P_i(X, T) = \det(1-\varphi_{i,l}T)$ is the characteristic polynomial of the endomorphism $\varphi_{i,l}$ of the $l$-adic cohomology group $H^i(\bar{X}, \boldsymbol{Q}_l)$ induced by the Frobenius endomorphism $\varphi$ of $X$, $l$ being a prime number different from the characteristic. It is known that $P_1$ (and hence also $P_2$) has integral coefficients and is independent of $l$. The degree $b_i$ of the polynomial $P_i$ is equal to $\dim_{\boldsymbol{Q}_l} H^i(\bar{X}, \boldsymbol{Q}_l)$. In particular, if $X$ is obtained as a reduction of a non-singular surface $\tilde{X}$ in characteristic zero, $b_i$ is equal to the $i$-th Betti number of $\tilde{X}$. We put

$$P_2(X, T) = \prod_{j=1}^{b_2} (1-\alpha_j T), \qquad \alpha_j \in \boldsymbol{C}.$$

CONJECTURE 1 (Weil). The algebraic integers $\alpha_j$ have absolute value $q$.

The corresponding fact for $P_1$ is known (Weil). Let $\rho'$ denote the number of $j$'s such that $\alpha_j = q$, and we write

$$P_2(X, T) = (1-qT)^{\rho'} R(T), \qquad R(q^{-1}) \neq 0.$$

CONJECTURE 2 (Tate). $\rho'$ is equal to the rank $\rho$ of Néron-Severi group $NS(X)$ of $X$.

The inequality $\rho \leqq \rho'$ is known; see [26] § 3.

CONJECTURE 3 (Artin-Tate). The Brauer group $Br(X)$ of $X$ is finite, and

$$R(q^{-1}) = \frac{|Br(X)|\,|\det((D_i D_j))|}{q^{\alpha(X)}\,|NS(X)_{\mathrm{tor}}|^2},$$

where $D_i$ $(1 \leqq i \leqq \rho)$ is a basis of $NS(X)$ mod torsion and $\alpha(X)$ is a suitably defined integer with $0 \leqq \alpha(X) \leqq p_g(X)$.

This is the conjecture $(C)$ of [27] § 4 and its non-$p$ part is known to be true if $\rho'=\rho$ ([27] Theorem 5.2). The order of Brauer group $|Br(X)|$ is conjectured to be a square or twice a square.

### C. Main results.

We fix a positive integer $n\geqq 3$ and a prime number $p$ not dividing $n$. Let $H_{w,p}(u)$ denote the Hecke polynomial:

$$H_{w,p}(u)=\det(1-T_p u+p^{w-1}R_p u^2),$$

defined with respect to the space $S_w(\Gamma(n))$ of $\Gamma(n)$-cusp forms of weight $w\geqq 2$ ([4] No. 36, § 5-8). Writing $H_{w,p}(u)=\prod_j(1-\beta_j u)$, we put

$$H_{w,p^f}(u)=\prod_j(1-\beta_j{}^f u) \qquad (f\geqq 1).$$

Now we let $X=E_p^*$, the elliptic modular surface for level $n$ in characteristic $p$, defined in A vii); put $q_0=p^f$. It follows from the results of Eichler-Shimura-Igusa ([23], [6]) that the zeta function of the base curve $\Delta=M_n^*\otimes \boldsymbol{F}_p$ (considered over $\boldsymbol{F}_{q_0}$) is given by

$$\zeta(\Delta, T)=H_{2,q_0}(T)/(1-T)(1-q_0T).$$

Hence we have

$$P_1(X, T)=H_{2,q_0}(T),$$

since the Picard variety of $X$ is isomorphic to the Jacobian variety of $\Delta$.

To consider $P_2(X, T)$, we denote by $NS^0(X)$ the subgroup of the Néron-Severi group $NS(\bar{X})$ of $\bar{X}$ generated by the curves

$$(o),\ C_{u_0},\ \Theta_{v,i} \qquad (v\in\Sigma_p,\ 1\leqq i\leqq m_v-1),$$

in the notation of Theorem 1.1. Note that the rank $\rho_0$ of $NS^0(X)$ is equal to the Picard number $\rho(n)$ of $B(n)$ in characteristic zero, cf. (5.4):

$$\rho_0=2+\sum_v(m_v-1)=2+(n-1)\mu(n)/n.$$

We also note that all the elements of $NS^0(X)$ are rational over $\boldsymbol{F}_{q_0^2}$ by A viii).

THEOREM. *Take $q=q_0{}^d$ so that all the elements in $NS^0(X)$ are defined over $\boldsymbol{F}_q$. Then*

$$P_2(X\otimes\boldsymbol{F}_q, T)=(1-qT)^{\rho_0}H_{3,q}(T).$$

We shall indicate two proofs for this theorem.

The first proof is based on the results of Deligne [2]. As a special case ($w=3$) of the construction of $l$-adic representations in [2], there exists a Galois submodule $W$ of $H^2(\bar{X}, \boldsymbol{Q}_l)$:

$$W={}_n^1W_l\subset H^2(\bar{X}, \boldsymbol{Q}_l),$$

such that

i) $\dim_{\boldsymbol{Q}_l}(W) = 2\dim_{\boldsymbol{C}} S_3(\Gamma(n))$;

ii) $\det(1-\varphi_{2,l}T)|_W = H_{3,q}(T)$,

where $\varphi_{2,l}$ is the endomorphism of $H^2(\bar{X}, \boldsymbol{Q}_l)$ induced by the Frobenius endomorphism $\varphi$ of $X \otimes \boldsymbol{F}_q$;

iii) $\psi_m^*$ acts on $W$ by multiplication by $m$, where $\psi_m$ denotes the endomorphism of $X$ over $\Delta$ inducing multiplication by $m$ on the generic fibre, $m$ being an integer with $m \equiv 1 \bmod n$ and $m \not\equiv 0 \bmod p$, and $\psi_m^*$ is the endomorphism of $H^2(\bar{X}, \boldsymbol{Q}_l)$ induced by $\psi_m$ (cf. the proof of Theorem 3.1 and [2] Lemme 5.3).

On the other hand, we can easily prove:

iv) $\psi_m^*(C_u) = C_u$, $\psi_m^*(\Theta_{v,i}) = \Theta_{v,i}$ $(v \in \Sigma_p;\ i \geqq 0)$;
$\psi_m^*(D_0) \approx m^2 D_0$, where $D_0 = 2(o) + (p_a+1)C_u$.

If we consider $NS(X) \otimes \boldsymbol{Q}_l$ as a subspace of $H^2(\bar{X}, \boldsymbol{Q}_l)$, iii) and iv) imply that

$$W \cap NS^0(X) \otimes \boldsymbol{Q}_l = \{0\}\,.$$

Hence, by comparing dimensions (cf. A ix)), we get

$$H^2(\bar{X}, \boldsymbol{Q}_l) = (NS^0(X) \otimes \boldsymbol{Q}_l) \oplus W\,;$$

this is analogous to the Hodge decomposition of Proposition 6.4. By ii) and the assumption on $q$, it is immediate that

$$P_2(X \otimes \boldsymbol{F}_q, T) = (1-qT)^{\rho_0} H_{3,q}(T)\,.$$

This completes the first proof.

The second proof is based on Ihara's theory of congruence monodromy problems [**9**]. This method has been used by Morita [**16**] following a suggestion of Ihara (cf. [8] Introduction) for establishing the relation between Hecke polynomials for even weights and the zeta functions of (incomplete) fibre varieties whose fibres are self-product of *even* number of elliptic curves. To apply the same method to our case, we note:

1) The trace formula of Hecke operators is also available for odd weight, cf. Shimizu [**21**].

2) For any closed point $u$ of $\Delta'$, the fibre $C_u$ of $X$ over $u$ is an elliptic curve such that all points of order $n$ on $C_u$ is rational over $\boldsymbol{F}_{q_0}(u)$. Therefore the reciprocal roots $\pi$, $\pi'$ of its zeta function satisfy the congruence:

$$\pi \equiv \pi' \equiv 1 \quad \bmod n\,.$$

Since $n \geqq 3$, this eliminates any ambiguity of sign of $\pi$, $\pi'$, which was the main reason why the *odd* weight case (or fibre varieties whose fibres are self-products of an *odd* number of elliptic curves) had to be excluded in [8] and [**16**]. The rest of the proof is similar to that of [**16**], at least for the case $q=p^2$, in which Ihara's theory [**9**] is directly applicable.

We shall consider the conjectures stated in B for our surface $X=E_p^*$.

COROLLARY 1. *Conjecture* 1 (*Weil*) *for the surface* $X=E_p^*$ *is equivalent to the Petersson conjecture for the eigenvalues of Hecke operator* $T_p$ *acting on* $S_3(\Gamma(n))$.

COROLLARY 2. *If* $H_{3,q}(q^{-1})\neq 0$, *then the rank* $\rho$ *of* $NS(X)$ *is equal to* $\rho_0$, *and Conjecture* 2 (*Tate*) *is true for the surface* $X=E_p^*$ *over* $\boldsymbol{F}_q$.

PROOF. Let $\rho'$ be the multiplicity of $q$ as the reciprocal roots in $P_2(X, T)$. Then $\rho'\geqq\rho$. On the other hand, the above theorem implies

$$\rho'=\rho_0\leqq\rho\,.$$

Hence $\rho=\rho_0=\rho'$.

COROLLARY 3. *If* $H_{3,q}(q^{-1})\neq 0$, *then the group of sections of* $E_p^*$ *over* $M_p^*\otimes\boldsymbol{F}_p$ *consists of* $n^2$ *sections of order* $n$.

This follows from the above Corollary 2 and Corollary 1.5 of § 1 (cf. Theorem 5.5). Moreover we can give explicit values for quantities in Conjecture 3 (Artin-Tate) under the assumption that $H_{3,q}(q^{-1})\neq 0$. By Corollary 1.7 of § 1, we have

$$\frac{|\det((D_iD_j))|}{|NS(X)_{\mathrm{tor}}|^2}=\frac{n^{t(n)}}{(n^2)^2}=n^{t(n)-4}\,,$$

since

$$E(K)\cong(\boldsymbol{Z}/n)^2\,,\qquad m_v^{(1)}=n\,,\qquad |\Sigma|=t(n)=\mu(n)/n\,,$$

in the notations used there. Hence

COROLLARY 4. *If Conjecture* 3 (*Artin-Tate*) *is true, then the order of the Brauer group* $Br(X\otimes\boldsymbol{F}_q)$ *of* $X\otimes\boldsymbol{F}_q$ *is given by the formula:*

$$|Br(X\otimes\boldsymbol{F}_q)|=q^{\alpha(X)}H_{3,q}(q^{-1})/n^{t(n)-4}\,,$$

*provided that* $H_{3,q}(q^{-1})\neq 0$.

By [**27**] Theorem 5.2 and Corollary 2 above, this formula is true up to a factor of a $p$-power. Therefore, using [27] Theorem 5.1, we can restate Corollary 4 as follows:

COROLLARY 5. *If* $H_{3,q}(q^{-1})\neq 0$, *then the integer* $q^{p_g}H_{3,q}(q^{-1})$ *is of the form:*

$$q^{p_g}H_{3,q}(q^{-1})=\pm a\cdot b^2\cdot n^{t(n)-4}\,,$$

*where* $a$, $b$ *are integers and* $a$ *is a* $p$-*power or twice a* $p$-*power.*

REMARK 6. The values of $t(n)=\mu(n)/n$ for small $n\geqq 3$ are given as follows:

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $t(n)$ | 4 | 6 | 12 | 12 | 24 | 24 | 36 | 36 | 60 | $\cdots$ |

.

Thus Corollary 5 implies rather remarkable divisibility properties of the integer $q^{p_g}H_{3,q}(q^{-1})$ associated with the Hecke polynomials for weight 3.

In the above discussions, we assumed that $H_{3,q}(q^{-1})\neq 0$. In the general case, we put

$$H_{3,q}(u)=(1-qu)^{r_p}R(u)\,,\qquad R(q^{-1})\neq 0\,,$$

with a non-negative integer $r_p$. Then we have

COROLLARY 7. *The following statements are equivalent:*

1) *Conjecture 2 (Tate) is true for* $X\otimes \boldsymbol{F}_q$.
2) *The Picard number* $\rho$ *of* $X\otimes \boldsymbol{F}_q$ *is equal to*

$$\rho=\rho_0+r_p\,.$$

3) *The rank of the group of sections of* $X\otimes\boldsymbol{F}_q$ *over* $M_n^*\otimes\boldsymbol{F}_q$ *(or equivalently, the rank of the group of rational points of the generic fibre over the function field of* $M_n^*\otimes\boldsymbol{F}_q$*) is equal to* $r_p$.

The case $r_p>0$ actually occurs, as is shown by the example below.

REMARK 8. Recall that our $X$ is a compactification of $E\otimes\boldsymbol{F}_p$, which is the universal family of elliptic curves with level $n$ structure in characteristic $p$ (over the moduli scheme $M_n\otimes\boldsymbol{F}_p$). Thus, if the statements in Corollary 7 are true in the case $r_p>0$, then $E\otimes\boldsymbol{F}_p$ will admit sections *of infinite order* over $M_n\otimes\boldsymbol{F}_p$.

EXAMPLE 9 (level 3 case). Let $n=3$. Then we have (cf. Example 5.4)

$$g=p_g=0\,,\qquad \rho_0=b_2=10\,,\qquad t(3)=4\,.$$

Let $p$ be a prime number $\neq 3$ and put $q=p$ or $p^2$ according as $p\equiv 1$ or $-1$ mod 3. Then the zeta function of $X=E_p^*$ (over $\boldsymbol{F}_q$) is given by

$$\zeta(X,T)=1/(1-T)(1-qT)^{10}(1-q^2T)\,.$$

Hence Conjectures 1 and 2 are trivial and Conjecture 3 implies $|Br(X)|=1$, which is compatible with the fact that $X$ is a rational surface (cf. Remark 5.6 (ii)).

EXAMPLE 10 (level 4 case). Let $n=4$. In this case, $p_g=\dim S_3(\Gamma(4))=1$ and a non-trivial element of $S_3(\Gamma(4))$ is given by $\Delta(z)^{1/4}$, $\Delta(z)$ being the cusp form of weight 12 for $SL(2,\boldsymbol{Z})$ (cf. Example 7.7). By Schoeneberg [**25**] p. 181, we have

$$H_{3,p}(u)=\begin{cases}1-(\pi^2+\pi'^2)u+p^2u^2\,, & p\equiv 1\ (4)\,,\\ 1-p^2u^2\,, & p\equiv -1\ (4)\,,\end{cases}$$

where $\pi$, $\pi'$ are integers in $\boldsymbol{Z}[\sqrt{-1}]$ such that $p=\pi\pi'$, $\pi\equiv 1 \bmod 2\sqrt{-1}$. Let $q_0=p$ or $p^2$ according as $p\equiv 1$ or $-1 \bmod 4$; we can take $q=q_0$ in the theorem, since the elliptic modular surface of level 4 has a model, classically known as Jacobi quartic:

$$A:\ y^2=(1-\sigma^2x^2)(1-\sigma^{-2}x^2)\,,$$

over $K=\boldsymbol{F}_q(\sigma)$, $\sigma$ being a variable over $\boldsymbol{F}_q$. The zeta function of $X=E_p^*$ (for $n=4$) is given by

$$\zeta(X,T)=1/(1-T)(1-qT)^{20}H_{3,q}(T)(1-q^2T)\,.$$

Therefore we get the following (cf. [**24**] I. Introduction):

i) Conjecture 1 (Weil) is true, since $|\pi^2|=|\pi'^2|=p$.

ii) If $p\equiv 1 \bmod 4$, then Conjecture 2 (Tate) is true.

iii) If $p\equiv -1 \bmod 4$, then Conjecture 2 is true if and only if the rank of group of $K$-rational points of the elliptic curve $A$ is equal to $r_p=2$. (We do not know whether or not $A$ has a $K$-rational point of infinite order in the case $p\equiv -1 \bmod 4$.)

iv) If $p\equiv 1 \bmod 4$, we put $\pi=a+2b\sqrt{-1}$, $a,\ b\in\boldsymbol{Z}$. Then

$$pH_{3,p}(p^{-1})=-(\pi-\pi')^2=(4b)^2\,.$$

By Corollary 4, the conjectured value of $|Br(X)|$ is equal to

$$pH_{3,p}(p^{-1})/4^{6-4}=b^2 \qquad (t(4)=6)\,;$$

a square integer!

v) Let $E_0^*$ $(n=4)$ be as in the part A vi). The Hasse-Weil zeta function of $E_0^*$ over $k=\boldsymbol{Q}(\sqrt{-1})$ is given as follows:

$$\prod_{\mathfrak{p}\neq 2}\zeta(E_{\mathfrak{p}}^*,\ N\mathfrak{p}^{-s})\sim\zeta_k(s)\zeta_k(s-1)^{20}\zeta_k(s-2)D_4(s)^2\,.$$

Here $\zeta_k(s)$ is the Dedekind zeta function of $\boldsymbol{Q}(\sqrt{-1})$, and

$$D_4(s)=\prod_{\mathfrak{p}\neq 2}H_{3,\mathfrak{p}}(p^{-s})^{-1}$$

is a zeta function of $\boldsymbol{Q}(\sqrt{-1})$ with a Grössencharacter, cf. [**25**] Formula (10); $\sim$ indicates equality up to a factor of a rational function of $2^{-s}$.

vi) The Picard number $\rho_0$ of the $K3$ surface $E_0^*$ is equal to 20. On the other hand, we have

$$D_4(2)\neq 0\,,$$

e.g. by [**28**] p. 288, Theorem 11. Hence the Picard number of $E_0^*$ is equal to the order of the pole at $s=2$ of its Hasse-Weil zeta function (cf. Tate [**26**] § 4, Conjecture 2).

REMARK 11. The Hasse-Weil zeta function of $E_0^*$ for an arbitrary level $n$ can be obtained in the same way.

REMARK 12. There is no doubt that the arithmetic theory of elliptic modular surfaces $B_\Gamma$ is meaningful also for certain groups $\Gamma$ other than $\Gamma(n)$. For example, let $\Gamma=\Gamma_0'(q)$ be the group considered in Example 5.8. For $q=7$, 11 or 19, Hecke ([**4**] No. 41, pp. 906-910) constructs a basis of $S_3(\Gamma)$ consisting of eigenfunctions of the Hecke operators. For $q=7$ or 11, we have $\dim S_3(\Gamma)=1$ and the Hecke polynomial $H_{3,p}(u)$ is similar to that of $\Gamma(4)$. For $q=19$, $p_g=\dim S_3(\Gamma)=3$. Using the result of Hecke, we see that

$$p^3 H_{3,p}(p^{-1})=19\cdot(\text{square}) \qquad \text{if } \left(\frac{p}{19}\right)=1,$$

$$p^3[H_{3,p}(u)/(1-pu)]_{u=p^{-1}}=2p\cdot 13\cdot(\text{square}) \qquad \text{if } \left(\frac{p}{19}\right)=-1.$$

In view of Conjecture 3 (Artin-Tate), this seems to suggest that there might be some connection between the field of eigenvalues of Hecke operator for weight 3 (e.g. $\boldsymbol{Q}(\sqrt{-13})$) and the discriminant of the intersection matrix of the Néron-Severi group $NS(X)$ of $X=B_\Gamma \bmod p$, ($B_\Gamma$ being assumed to be defined over $\boldsymbol{Q}$,) or the order of the Brauer group $Br(X)$ of $X$.

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