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On the boundedness of pseudo-differential operators

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In this note we show that a bounded symbol $p(x, \xi)$ with bounded derivatives $\partial_x^\beta \partial_\xi^\alpha p(x, \xi)$ defines a bounded pseudo-differential operator

$$(Pf)(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} p(x, \xi) \hat{f}(\xi) d\xi$$

in L^2. Symbols $p(x, \xi)$ of the class $S^0_{\rho,0}$, $\rho > 0$ defined in Hörmander [1] by the inequality

$$|\partial_x^\beta \partial_\xi^\alpha p(x, \xi)| \leqq C_{\alpha\beta}(1+|\xi|)^{-\rho|\alpha|}$$

are of this form. The result is new for $\rho = 0$. Our proof makes use of a modification of a lemma of Cotlar (see [2]) for almost orthogonal operators in a Hilbert space.

This problem was proposed to us by Hitoshi Kumano-go who will present shortly applications to parabolic and semi-elliptic[1)] operators.

THEOREM. *Let the symbol $p(x, \xi)$ be a matrix of functions $p_{ij}(x, \xi)$ defined on $R^n_x \times R^n_\xi$ such that*

$$|\partial_{x_n}^{\beta_n} \cdots \partial_{x_1}^{\beta_1} \partial_{\xi_n}^{\alpha_n} \cdots \partial_{\xi_1}^{\alpha_1} p_{ij}(x, \xi)| \leqq C_{\alpha,\beta}$$

for $\alpha_k, \beta_l = 0, 1, 2, 3$ and all x and ξ. Then the pseudo-differential operator

$$(Pf)(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} p(x, \xi) \hat{f}(\xi) d\xi\,, \qquad f \in \mathcal{S}\,,$$

can be extended to a bounded operator from L^2 into L^2.

We state immediately the auxiliary lemma which will be proved later.

LEMMA. *Let A_z be a z-weakly measurable and uniformly bounded family of operators in L^2, $\|A_z\| \leqq M_0$ for all z in a measure space Z with element of measure dz. If the inequalities*

$$\|A_z A_{z'}^*\| \leqq h^2(z, z') \quad \textit{and} \quad \|A_z^* A_{z'}\| \leqq h^2(z, z')$$

hold with a nonnegative function $h(z, z')$ which is the kernel of a bounded integral operator H in L^2 with norm M, then the operator

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1) For the larger class of λ-elliptic operators, see Nagase and Shinkai [3].

$$A=\int A_z\,dz$$

is bounded in L^2 *with norm*

$$\|A\|\leqq M.$$

PROOF OF THEOREM. It will be sufficient to prove the Theorem for a single function $p(x, \xi)$ of two real variables x, ξ, since each pair of variables x_k, ξ_k can be considered separately. We first express $p(x, \xi)$ as a convolution of the bounded function

$$g(x, \xi)=(1+\partial_x)^3(1+\partial_\xi)^3 p(x, \xi)$$

with the function

$$\varphi(x)=\begin{cases}\frac{1}{2}x^2e^{-x} & x\geqq 0\\ 0 & x<0\end{cases}$$

which is the fundamental solution of the differential equation

$$(1+\partial_x)^3\varphi(x)=\delta(x),$$

namely

$$p(x, \xi)=\iint g(s, t)\varphi(x-s)\varphi(\xi-t)ds\,dt.$$

The pseudo-differential operator P becomes

$$\begin{aligned}Pf(x)&=(2\pi)^{-1}\iint ds\,dt\,g(s, t)\int e^{ix\xi}\varphi(x-s)\varphi(\xi-t)\hat{f}(\xi)d\xi\\ &=(2\pi)^{-1}\iint ds\,dt\,g(s, t)A_{st}f(x).\end{aligned}$$

Since $g(s, t)$ is a bounded function, it is enough to show that the family A_{st}, with $(s, t)=z$, satisfies the hypothesis of the lemma. From $\max_x \varphi(x)=2e^{-2}$, we have immediately

$$\|A_z\|\leqq 4e^{-4}=M_0.$$

We shall show that

$$(1)\qquad \begin{aligned}\|A_zA_{z'}^*\|&\leqq c_1^2(1+|s-s'|)^{-3}e^{-|t-t'|/2}\\ &\leqq c^2(1+|s-s'|)^{-3}(1+|t-t'|)^{-3}\\ &\equiv h^2(s-s', t-t')\end{aligned}$$

and, by a simple interchange of s and t,

$$(2)\qquad \begin{aligned}\|A_z^*A_{z'}\|&\leqq c_1^2(1+|t-t'|)^{-3}e^{-|s-s'|/2}\\ &\leqq c^2(1+|t-t'|)^{-3}(1+|s-s'|)^{-3}\\ &=h^2(s-s', t-t').\end{aligned}$$

Since $h(s, t)$ is (s, t)-integrable, Theorem will follow from the lemma with

$$M=c\Big[\int(1+|t|)^{-3/2}dt\Big]^2.$$

To derive (1) we shall estimate the L^2-norm $\|A_{st}A^*_{s't'}\|$ of $A_{st}A^*_{s't'}$ by the Hilbert-Schmidt norm $\iint|k(x, y)|^2dx\,dy$ of its kernel

$$k(x, y)=\int e^{i(x-y)\xi}\varphi(x-s)\varphi(\xi-t)\varphi(y-s')\varphi(\xi-t')d\xi$$
$$=\varphi(x-s)\varphi(y-s')\int e^{i(x-y)\xi}\varphi(\xi-t)\varphi(\xi-t')d\xi\,.$$

With

$$w=x-y\,,\qquad u=\xi-t\,,\qquad \Delta_t=t-t'\geqq 0\,,$$

the last integral becomes

$$e^{iwt}\int e^{iwu}\varphi(u)\varphi(u+\Delta_t)du$$
$$=\frac{1}{4}e^{iwt-\Delta_t}\int_0^\infty e^{iwu}u^2(u+\Delta_t)^2e^{-2u}du$$
$$=\frac{1}{4}e^{iwt-\Delta_t}\Big(\frac{1}{i}\frac{d}{dw}\Big)^2\Big(\frac{1}{i}\frac{d}{dw}+\Delta_t\Big)^2\frac{1}{2+iw}\,.$$

Since a similar expression holds with $-\Delta_t$ when $\Delta_t<0$ we thus have

$$|k(x, y)|\leqq ce^{-|\Delta_t|/2}(1+|x-y|)^{-3}\varphi(x-s)\varphi(y-s')\,.$$

Squaring and integrating give

$$\int|k(x, y)|^2dx\,dy\leqq ce^{-|\Delta_t|}\iint(1+|x-y|)^{-6}\varphi^2(x-s)\varphi^2(y-s')dx\,dy\,.$$

Now setting

$$u+v=x-s\,,\qquad u-v=y-s'\,,\qquad \Delta_s=s-s'\geqq 0\,,$$

the last integral becomes

$$2\iint(1+|2v+\Delta_s|)^{-6}\varphi^2(u+v)\varphi^2(u-v)du\,dv$$
$$=\frac{1}{8}\int_v(1+|2v+\Delta_s|)^{-6}\int_{u\geqq|v|}(u^2-v^2)^2e^{-4u}du\,dv$$
$$\leqq c\int_v(1+|2v+\Delta_s|)^{-6}\int_{u\geqq|v|}e^{-u}du\,dv$$
$$\leqq c\int(1+|2v+\Delta_s|)^{-6}e^{-|v|}dv$$

and by the inequality $(1+|a+b|)^r\leqq C_r(1+|a|)^{|r|}(1+|b|)^{-r}$,

$$\leqq c(1+|\Delta_s|)^{-6}\int(1+|2v|)^6 e^{-|v|}dv$$
$$\leqq c(1+|\Delta_s|)^{-6}.$$

Collecting the terms we get (1) in the form

$$\|A_{st}A^*_{s't'}\|^2 \leqq \int |k(x, y)|^2 dx\, dy$$
$$\leqq c^4(1+|s-s'|)^{-6}e^{-2|t-t'|}.$$

Inequality (2) follows in the same way by noting that the operator $(A^*_{st}A_{s't'})^\wedge$ mapping $\hat{f}$ into $\widehat{A^*_{st}A_{s't'}f}$ has kernel

$$k_1(\xi, \eta) = \varphi(\xi-t)\varphi(\eta-t')\int e^{-ix(\xi-\eta)}\varphi(x-s)\varphi(x-s')dx,$$

hence k_1 is the complex conjugate of k with s and t interchanged.

This completes the proof of Theorem.

COROLLARY.

$$\|P\| \leqq c \sup_{x,\xi} |(1+\partial_{x_n})^3 \cdots (1+\partial_{x_1})^3(1+\partial_{\xi_n})^3 \cdots (1+\partial_{\xi_1})^3 p(x, \xi)|$$

where the constant c is independent of p but depends only on the dimension n of the x and ξ spaces.

PROOF OF LEMMA. We multiply the two inequalities

$$T_m \equiv \|A_{z_1}A^*_{z_2}A_{z_3} \cdots A^*_{z_{2m}}\| \leqq \|A_{z_1}A^*_{z_2}\| \cdots \|A_{z_{2m-1}}A^*_{z_{2m}}\|$$

and

$$T_m \leqq \|A_{z_1}\|\|A^*_{z_2}A_{z_3}\| \cdots \|A^*_{z_{2m-2}}A_{z_{2m-1}}\|\|A^*_{z_{2m}}\|$$

to obtain the estimate

$$T^2_m \leqq \|A_{z_1}\|\|A_{z_1}A^*_{z_2}\|\|A^*_{z_2}A_{z_3}\| \cdots \|A_{z_{2m-1}}A^*_{z_{2m}}\|\|A^*_{z_{2m}}\|$$

which, after taking square roots and using the hypotheses of the lemma, becomes

$$T_m \leqq M_0 h(z_1, z_2)h(z_2, z_3) \cdots h(z_{2m-1}, z_{2m}).$$

Thus, if N is a set of finite measure with characteristic function $\chi_N(z)$, $S_N = \int_N dz = \int \chi_N(z)dz$, we have

$$\left\|\left[\left(\int_N A_z dz\right)\left(\int_N A_z dz\right)^*\right]^m\right\|^{1/m}$$
$$\leqq \left[\int \cdots \int_{z_i \in N} \|A_{z_1}A^*_{z_2} \cdots A^*_{z_{2m}}\| dz_1 \cdots dz_{2m}\right]^{1/m}$$
$$\leqq \left| M_0 \iint_{z_1, z_{2m} \in N} dz_1\, dz_{2m} \int \cdots \int_Z h(z_1, z_2) \cdots h(z_{2m-1}, z_{2m}) dz_2 \cdots dz_{2m-1}\right|^{1/m}$$

$$= \left| M_0 \iint_{z_1, z_{2m} \in N} h^{(2m-1)}(z_1, z_{2m})\, dz_1 dz_{2m} \right|^{1/m}$$

$$= \left| M_0 \iint \chi_N(z_1) h^{(2m-1)}(z_1, z_{2m}) \chi_N(z_{2m})\, dz_{2m}\, dz_1 \right|^{1/m}$$

$$\leqq [M_0 S_N M^{2m-1}]^{1/m}.$$

The last estimate follows from the fact that $h^{(2m-1)}(z, z')$ is the kernel of the operator H^{2m-1} and $\|H^{2m-1}\| \leqq M^{2m-1}$.

Letting m go to infinity, the above inequalities give

$$\left\| \int_N A_z\, dz \right\|^2 \leqq M^2.$$

Since M is independent of N, we finally obtain the desired estimate

$$\left\| \int A_z\, dz \right\| \leqq M.$$

This completes the proof of the lemma.

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