J. Math. Soc. Japan
Vol. 22, No. 3, 1970

On stochastic differential equations associated with certain quasilinear parabolic equations

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(Received March 15, 1969)
(Revised Nov. 24, 1969)

§1. Introduction.

Let $I=(-\infty, 0]$ and $dB=\{B(t)-B(s), t, s\in I\}$ be a Wiener random measure. Given functions $\alpha(t, x, v)$, $\beta(t, x, v)$ and $\gamma(t, x, v)$ on $I\times R^1\times R^1$, we consider the stochastic differential equation

$$(1.1.a)\qquad \begin{aligned} dX^{(s,a)}(t)&=\alpha(t, X^{(s,a)}(t), U(t, X^{(s,a)}(t)))dt\\ &\quad+\beta(t, X^{(s,a)}(t), U(t, X^{(s,a)}t(t)))dB(t), \qquad s\leqq t\leqq 0,\\ X^{(s,a)}(s)&=a, \end{aligned}$$

$$(1.1.b)\qquad U(s, a)=Ef(X^{(s,a)}(0))\exp\int_s^0\gamma(\tau, X^{(s,a)}(\tau), U(\tau, X^{(s,a)}(\tau)))d\tau, \qquad s\in I,$$

for a given data f on R^1. The stochastic differential equations of this kind were considered in the investigation of the Cauchy problems for degenerate quasilinear parabolic equations, since, if $U(s, a)$ is smooth enough, then it satisfies a backward quasilinear diffusion equation (for example, see [1]).

The purpose of this note is to show the existence of a global solution of (1.1) under some smooth conditions of α, β, γ and f. Concerning the same stochastic differential equation of d-space variables,

$$(1.1.a)'\qquad \begin{aligned} dX_i^{(s,a)}&=\sum_{i=1}^{d}\alpha_i(t, X^{(s,a)}(t), U(t, X^{(s,a)}(t)))dt\\ &\quad+\sum_{j=1}^{d}\beta_{ij}(t, X^{(s,a)}(t), U(t, X^{(s,a)}(t)))dB_j, \quad s\leqq t\leqq 0, \quad i=1, \cdots d\\ X^{(s,a)}(s)&=a, \qquad a\in R^d \end{aligned}$$

$$(1.1.b)'\qquad U(s, a)=Ef(X^{(s,a)}(0))\exp\int_s^0\gamma(\tau, X^{(s,a)}(\tau), U(\tau, X^{(s,a)}(\tau)))d\tau, \quad s\in I$$

H. Tanaka [6] proved the existence and uniqueness of local solution of (1.1)′, under the assumption of boundedness and the Lipschitz condition of α_i, β_{ij}, γ and f. As to the global solution of (1.1)′, N. I. Freidlin [2] showed the following result: Assume that (i) β_{ij} does not depend on u, (ii) there exists the system of bounded functions $\varphi_i(t, x, u)$, $i=1, \cdots d$, such that

$$\sum_{j=1}^{d} \beta_{ij}(t, x)\varphi_j(t, x, u)=\alpha_i(t, x, u), \qquad i=1, \cdots d.$$

(iii) f is bounded and continuous and (iv) $\beta_{ij}(t, x)$, $\alpha_i(t, x, u)$, $\varphi_j(t, x, u)$ are bounded, continuous and satisfy the Lipschitz condition in x and u. Then there exists a global solution of (1. 1)′ and U is unique. In the case of $d=1$, namely in the stochastic differential equation (1, 1), we can remove the assumption (i) and get the following theorem.

THEOREM. *Let β, γ and f be bounded functions, which satisfy the following conditions:*

$$|\beta(t, x, u)-\beta(s, y, v)| \leqq W(|t-s|)+W(|x-y|)+W(|u-v|)$$

$$|\gamma(t, x, u)-\gamma(s, y, v)| \leqq W(|t-s|)+M|x-y|^{\tilde{\delta}}+M|u-v|$$

and

$$|f(x)-f(y)| \leqq M|x-y|^{\tilde{\delta}},$$

where M and $\tilde{\delta}$ are positive constants and W is a continuous function with $W(0)=0$. Moreover, suppose that β is non-negative and there exists a bounded function $\phi(t, x, u)$ such that

$$\alpha(t, x, u)=\phi(t, x, u)\beta(t, x, u)$$

and

$$|\phi(t, x, u)-\phi(s, y, v)| \leqq \bar{W}(|t-s|)+\bar{M}|x-y|^{\bar{\delta}}+\bar{M}|u-v|$$

where $\bar{M}$ and $\bar{\delta}$ are positive constants and $\bar{W}$ is a continuous functions with $\bar{W}(0)=0$. Then we have a solution of (1.1).

A pair of a function U on $I\times R^1$ and a system of stochastic processes $\{X^{(s,a)}(t),\ s\leqq t\leqq 0\}$, $s\in I$, $a\in R^1$, on a probability space, is called a solution of (1.1) if it satisfies (1.1) and $\mathcal{B}_{(s,t)}(X^{(s,a)})\vee\mathcal{B}_{(-\infty,t)}(dB)$[1)] is independent of $\mathcal{B}_{(t,0)}(dB)$ for each $t\in[s, 0]$.

In § 2, we prepare some preliminary facts and construct approximate solutions, using the Cauchy's polygonal method as Itô-Nisio [**3**] and Skorokhod [**5**]. We estimate, in § 3, the dependence of approximate solutions on the initial position (s, a). In § 4, we show that the system of our approximate solutions is totally bounded in Prohorov topology [**4**] and find a global solution in § 5.

In conclusion, the author wish to express her sincerely thanks to Professor H. Tanaka for his valuable suggestions.

1) $\mathcal{B}_{(s,t)}(\zeta)$ denotes the least Borel algebra for which $\zeta(\tau)$ is measurable for each $\tau\in[s, t]$. $\mathcal{B}_1\vee\mathcal{B}_2$ denotes the least Borel algebra that contains $\mathcal{B}_1$ and $\mathcal{B}_2$.

§ 2. Preliminaries.

First we list two simple propositions without proof. We call a process $\{\eta(t, \omega), t \in [s, 0]\}$, on a probability space $\Omega(\mathcal{B}, P)$ a non-anticipating Brownian functional if

(i) $\eta(t, \omega)$ is (t, ω)-measurable

and

(ii) $\eta(t, \cdot)$ is $\mathcal{B}_{(s,t)}(dB)$-measurable for each $t \in [s, 0]$.

PROPOSITION 1. *Let $\{\eta(t, \omega), t \in [s, 0]\}$ be a bounded*[2] *non-anticipating Brownian functional and ξ a $\mathcal{B}_{(-\infty,s)}(dB)$-measurable function. If a non-anticipating Brownian functional $\{Z(t), s \leqq t \leqq 0\}$, with $\int_s^0 EZ^2(t)dt < \infty$, satisfies the stochastic integral equation,*

$$Z(t) = \xi + \int_s^t \eta(\tau) Z(\tau) dB(\tau),$$

then

$$Z(t) = \xi \exp\left(\int_s^t \eta(\tau)dB(\tau) - \frac{1}{2}\int_s^t \eta^2(\tau)d\tau\right)$$

and

$$E|Z(t)| = E|\xi|.$$

PROPOSITION 2. *Suppose that a bounded continuous function $g(t, x, u)$ on $I \times R^1 \times R^1$ satisfies the following conditions:*

$$|g(t, x, u) - g(s, y, v)|^i \leqq W_1^i(|t-s|) + W_2^i(|x-y|) + W_3^i(|u-v|) \quad i=1, 2$$

where W_k is a continuous function with $W_k(0) = 0$. Define g_n by

$$g_n(t, x, u) \equiv (g * N_n)(t, x, u)$$

$$\equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{n}{2\pi} e^{-n\frac{(y-x)^2+(v-u)^2}{2}} g(t, y, v)dydv.$$

Then we have

(i) $|g_n(t, x, u)| \leqq \sup_{t \in I, x, u \in R^1} |g(t, x, u)|$

(ii) $|g_n(t, x, u) - g_n(s, y, v)| \leqq W_1(|t-s|) + D_n(|x-y| + |u-v|)$

(iii) $|g_n(t, x, u) - g_n(s, y, v)| \leqq W_1(|t-s|) + W_2(|x-y|) + W_3(|u-v|)$

(iv) $|g_n(t, x, u) - g_n(s, y, v)| \leqq W_1^2(|t-s|) + W_2^2(|x-y|) + W_3^2(|u-v|)$

(v) *$g_n(t, x, u)$ converges to $g(t, x, u)$ for each (t, x, u).*

In order to construct an approximate solution, we define β_n, γ_n, ϕ_n, α_n and f_n by

2) We mean the existence of a constant M such that, for any $t \in [s, 0]$, $|\eta(t, \omega)| \leqq M$ for almost all ω.

$$\beta_n=(\beta*N_n)+\frac{1}{n}, \qquad \gamma_n=\gamma*N_n, \qquad \phi_n=\phi*N_n,$$

$$\alpha_n=\beta_n\phi_n \quad \text{and} \quad f_n=f*N_n{}^{3)} \quad \text{respectively}.$$

By virtue of Proposition 2 and the assumption of Theorem, we may assume that there exist a constant L and a continuous functions θ with $\theta(0)=0$ such that, putting $\delta=\min(\tilde{\delta}, \bar{\delta})$,

$$|\xi_n|\leqq L, \qquad \xi=\alpha, \beta, \gamma, \phi, f, \quad n=1, 2, \cdots \tag{2.1}$$

$$|\xi_n(t, x, u)-\xi_n(s, y, v)|^p\leqq\theta^p(|t-s|)+\theta^p(|x-y|)+\theta^p(|u-v|) \tag{2.2}$$
$$\xi=\alpha, \beta, \quad p=1, 2, \quad n=1, 2, \cdots$$

$$|\xi_n(t, x, u)-\xi_n(s, y, v)|^p\leqq\theta^p(|t-s|)+L|x-y|^{\delta p}+L|u-v|^p, \tag{2.3}$$
$$\xi=\gamma, \phi, \quad p=1, 2, \quad n=1, 2, \cdots$$

and

$$|f_n(x)-f_n(y)|^p\leqq L|x-y|^{\delta p}, \qquad p=1, 2, \quad n=1, 2, \cdots. \tag{2.4}$$

We define an approximation solution U_n and $X_n^{(s,a)}$, for $\frac{-1}{n}\leqq s\leqq t\leqq 0$, as follows

$$U_n(t, a)=f_n(a). \tag{2.5}$$

$$X_n^{(s,a)}(t)=a+\int_s^t\alpha_n(\tau, X_n^{(s,a)}(\tau), U_n(\tau, X_n^{(s,a)}(\tau)))d\tau \tag{2.6}$$
$$+\int_s^t\beta_n(\tau, X_n^{(s,a)}(\tau), U_n(\tau, X_n^{(s,a)}(\tau)))dB(\tau).$$

By the Lipschitz condition of f_n, α_n and β_n, $X_n^{(s,a)}$ is determined uniquely as the continuous non-anticipating Brownian functional. Moreover,

$$E|X_n^{(s,a)}(t)-X_n^{(s,b)}(t)|^2\leqq|a-b|^2+Q\int_s^t E|X_n^{(s,a)}(\tau)-X_n^{(s,b)}(\tau)|^2d\tau$$

with a constant Q. Hence we have

$$E|X_n^{(s,a)}(t)-X_n^{(s,b)}(t)|^2\leqq|a-b|^2e^{Q(t-s)}. \tag{2.7}$$

After we have defined $U_n(s, a)$ and $X_n^{(s,a)}$ for $s\in\left[-\frac{k}{n}, -\frac{k-1}{n}\right)$, we define them for $s\in\left[-\frac{k+1}{n}, -\frac{k}{n}\right)$ by

$$U_n(s, a)=Ef_n(X_n^{(s+\frac{1}{n},a)}(0))\exp\int_{s+\frac{1}{n}}^0\gamma_n(\tau, X_n^{(s+\frac{1}{n},a)})d\tau, \tag{2.8}$$

3) $f_n(x)=\int_{-\infty}^{\infty}\frac{\sqrt{n}}{\sqrt{2\pi}}e^{\frac{-n(y-x)^2}{2}}f(y)dy.$

and

$$X_n^{(s,a)}(t)=a+\int_s^t \alpha_n(\tau, X_n^{(s,a)})d\tau+\int_s^t \beta_n(\tau, X_n^{(s,a)})dB(\tau), \qquad t\in[s, 0], \tag{2.9}$$

where $\gamma_n(\tau, X_n^{(s,a)})=\gamma_n(\tau, X_n^{(s,a)}(\tau), U_n(\tau, X_n^{(s,a)}(\tau)))$ and similar to α_n and β_n. The following lemma shows the Lipschitz condition of the coefficients in (2.9), so that we can determine $U_n(s, a)$ and $X_n^{(s,a)}$ for every $s\in I$ and $a\in R^1$.

LEMMA 2.1. *If, for* $-\frac{k}{n}\leqq s<t\leqq 0$,

$$|U_n(s, a)-U_n(s, b)|\leqq K|a-b|$$

and

$$E|X_n^{(s,a)}(t)-X_n^{(s,b)}(t)|^2\leqq K|a-b|^2,$$

then there exists a constant K', *(may depends on* n*) such that, for* $-\frac{k+1}{n}\leqq s<t\leqq 0$,

$$|U_n(s, a)-U_n(s, b)|\leqq K'|a-b|$$

and

$$E|X_n^{(s,a)}(t)-X_n^{(s,b)}(t)|^2\leqq K'|a-b|^2.$$

PROOF. Recalling the definition of U_n, we have

$$\begin{aligned}
&\left|U_n\left(s-\frac{1}{n}, a\right)-U_n\left(s-\frac{1}{n}, b\right)\right| \\
&\leqq E|f_n(X_n^{(s,a)}(0))-f_n(X_n^{(s,b)}(0))|\exp\int_s^0 \gamma_n(\tau, X_n^{(s,a)})d\tau \\
&\quad +E|f_n(X_n^{(s,b)}(0))|\left(\exp\int_s^0 \gamma_n(\tau, X_n^{(s,a)})d\tau\right) \\
&\quad \times|1-\exp\int_s^0\{\gamma_n(\tau, X_n^{(s,b)})-\gamma_n(\tau, X_n^{(s,a)})\}d\tau| \\
&\leqq D_n e^{-Ls}E|X_n^{(s,a)}(0)-X_n^{(s,b)}(0)| \\
&\quad +Le^{-3Ls}\int_s^0 E|\gamma_n(\tau, X^{(s,a)})-\gamma_n(\tau, X^{(s,b)})|d\tau
\end{aligned}$$

by virtue of the inequality, $|e^x-1|\leqq e^c|x|$ if $|x|\leqq c$. By the assumption of Lemma 2.1, $\gamma_n(\tau, x)$ satisfies the Lipschitz condition to x. So, we have the former half of Lemma. Repeating the similar evaluation as (2.7), we complete the proof of Lemma 2.1.

Taking (2.5) into account, we can choose a constant $Q=Q(n, T)$ so that

$$|U_n(s, a)-U_n(s, b)|\leqq Q|a-b|, \qquad s\in[T, 0]. \tag{2.10}$$

In order to evaluate, in § 3, the continuity of $U_n(s, a)$ we define an auxiliary martingale process $Y_n^{(s,a)}$, as follows

$$Y_n^{(s,a)}(t) = a + \int_s^t \beta_n(\tau, Y_n^{(s,a)}) dB(\tau).$$

Put $Z_n(t) = Y_n^{(s,a)}(t) - Y_n^{(s,b)}(t)$ and

$$\eta_n(\tau) = \frac{\beta_n(\tau, Y_n^{(s,a)}) - \beta_n(\tau, Y_n^{(s,b)})}{Y_n^{(s,a)}(\tau) - Y_n^{(s,b)}(\tau)}.$$

So, η_n is a bounded non-anticipating Brownian functional and

$$Z_n(t) = a - b + \int_s^t \eta_n(\tau) Z_n(\tau) dB(\tau).$$

Thus, by Proposition 1, we have $E|Z_n(t)| = |a-b|$. Hence concerning the dependence of $Y_n^{(s,a)}$ on the starting point a, we obtain

LEMMA 2.2.

$$E|Y_n^{(s,a)}(t) - Y_n^{(s,b)}(t)|^l \leqq |a-b|^l,$$
$$l \in (0, 1], \ -\infty < s \leqq t \leqq 0, \ n = 1, 2, \cdots.$$

As to the dependence of $Y_n^{(s,a)}$ on the starting time s, we have

(2.11) $$E|Y_n^{(s',a)}(t) - Y_n^{(s,a)}(t)| = E\left|\int_s^{s'} \beta_n(\tau, Y_n^{(s,a)}) dB(\tau)\right|.$$

The method of proof of (2.11) is almost the same as that of Lemma 2.2. Namely, we have

$$Z(t) \equiv Y_n^{(s,a)}(t) - Y_n^{(s',a)}(t)$$
$$= \int_s^{s'} \beta_n(\tau, Y_n^{(s,a)}) dB(\tau) + \int_{s'}^t \beta_n(\tau, Y_n^{(s,a)}) - \beta_n(\tau, Y_n^{(s',a)}) dB(\tau).$$

So,

$$Z(t) = \int_s^{s'} \beta_n(\tau, Y_n^{(s,a)}) dB(\tau) + \int_{s'}^t \zeta(\tau) Z(\tau) dB(\tau)$$

where

$$\zeta(\tau) = \frac{\beta_n(\tau, Y_n^{(s,a)}) - \beta_n(\tau, Y_n^{(s',a)})}{Y_n^{(s,a)}(\tau) - Y_n^{(s',a)}(\tau)}.$$

Therefore, by Proposition 2, we get (2.11). Consequently, by (2.1), we have

LEMMA 2.3.

$$E|Y_n^{(s',a)}(t) - Y_n^{(s,a)}(t)|^l \leqq L|s'-s|^{l/2}, \quad l \in (0, 1], \ a \in R^1, \ n = 1, 2, \cdots.$$

§ 3. Continuity of $U_n(s, a)$.

In this paragraph, we use the following proposition [5, Chap. 4].

PROPOSITION 3. *Suppose that X_t is the continuous solution of the stochastic differential equation,*

$$X_i(t) = c + \int_T^t a_i(\tau, X_i(\tau)) d\tau + \int_T^t b(\tau, X(\tau)) dB(\tau), \qquad T \leqq t \leqq 0, \ i = 1, 2,$$

the coefficients of which satisfy the following conditions,

(i) $a_i(t, x)$ *and* $b(t, x)$ *are bounded continuous in* (t, x)

(ii) *there exists a constant* M *such that*

$$|\xi(t, x) - \xi(t, y)| \leqq M|x-y|, \qquad \xi = a_1, a_2, b$$

(iii) $$\inf_{(t,x)} b(t, x) > 0.$$

Then, for any bounded continuous functional g *on* $C[T, 0]$,

$$Eg(X_2) = Eg(X_1) \exp\left(\int_T^0 \varphi(\tau, X_1(\tau)) dB(\tau) - \frac{1}{2} \int_T^0 \varphi^2(\tau, X_1(\tau)) d\tau\right),$$

where $\varphi(\tau, x) = (a_2(\tau, x) - a_1(\tau, x))/b(\tau, x)$.

Applying this proposition to $U_n(s, a)$, we can express $U_n(s, a)$ in the following form,

$$U_n\left(s - \frac{1}{n}, a\right) = Ef_n(Y_n^{(s,a)}(0)) \exp\left(\int_s^0 \gamma_n(\tau, Y_n^{(s,a)}) d\tau \right.$$
$$\left. + \int_s^0 \varphi_n(\tau, Y_n^{(s,a)}) dB - \frac{1}{2} \int_s^0 \varphi_n^2(\tau, Y_n^{(s,a)}) d\tau\right).$$

Putting

$$J(a, s) = \int_s^0 \varphi_n(\tau, Y_n^{(s,a)}) dB - \frac{1}{2} \int_s^0 \varphi_n^2(\tau, Y_n^{(s,a)}) d\tau,$$

$$J_1 = E|\exp J(a, s) - \exp J(b, s)| \exp \int_s^0 \gamma_n(\tau, Y_n^{(s,a)}) d\tau,$$

$$J_2 = E\left|\exp \int_s^0 \gamma_n(\tau, Y_n^{(s,a)}) d\tau - \exp \int_s^0 \gamma_n(\tau, Y_n^{(s,b)}) d\tau\right| \exp J(b, s)$$

and

$$J_3 = E|f_n(Y_n^{(s,a)}(0)) - f_n(Y_n^{(s,b)}(0))| \exp\left(\int_s^0 \gamma_n(\tau, Y_n^{(s,b)}) d\tau + J(b, s)\right),$$

we can see

$$\left|U_n\left(s - \frac{1}{n}, a\right) - U_n\left(s - \frac{1}{n}, b\right)\right| \leqq L(J_1 + J_2) + J_3. \tag{3.1}$$

In the sequent calculation, $K_i(s)$ denotes a suitably chosen constant, which is increasing when s tends to $-\infty$ and does not depend on a, b and n. Using Hölder's inequality, we have

$$J_1 \leqq e^{-Ls}(E \exp 4J(a, s))^{1/4}(E|1 - \exp(J(b, s) - J(a, s))|^{4/3})^{3/4}. \tag{3.2}$$

Put $A = \{\omega; |J(b, s) - J(a, s)| > 1\}$ and let χ be the indicator function of A. By virtue of the inequality: $|e^x - 1| \leqq e|x|$ for $|x| \leqq 1$, we can see

$$E|1-\exp(J(b, s)-J(a, s))|^{4/3}$$
$$\leqq eE|J(b, s)-J(a, s)|^{4/3}+E|1-\exp(J(b, s)-J(a, s))|^{4/3}\chi$$
$$\leqq e(E|J(b, s)-J(a, s)|^{2})^{2/3}+(E|1-\exp(J(b, s)-J(a, s))|^{4})^{1/3}(E\chi)^{2/3}$$
$$\leqq K_1(s)(E|J(b, s)-J(a, s)|^{2})^{2/3}$$

because, for any bounded non-anticipating Brownian functional ξ and η, we can easily see

$$E\exp\Big(\int_s^0 \xi(\tau)dB(\tau)+\int_s^0 \eta(\tau)d\tau\Big)$$
$$=E\exp\Big(\int_s^0 \xi(\tau)dB(\tau)-\frac{1}{2}\int_s^0 \xi^2(\tau)d\tau+\frac{1}{2}\int_s^0\{\xi^2(\tau)+2\eta(\tau)\}d\tau\Big)$$
$$\leqq \exp\Big(-\frac{s}{2}\sup_{\tau,\omega}(|\xi^2(\tau,\omega)|+2|\eta(\tau,\omega)|)\Big).$$

Hence, taking (3.2) into account, we have

$$J_1\leqq K_2(s)\Big(\int_s^0 E|\varphi_n(\tau, Y_n^{(s,a)})-\varphi_n(\tau, Y_n^{(s,b)})|^2 d\tau\Big)^{1/2}. \tag{3.3}$$

Putting $V_n(\tau)=\sup\limits_{x\neq y}\dfrac{|U_n(\tau, x)-U_n(\tau, y)|^2}{|x-y|^{1\wedge 2\delta}}$, we can see that $V_n(\tau)$ is finite, recalling Lemma (2.1) and the boundedness of U_n. On the other hand

$$|\varphi_n(\tau, x, u)-\varphi_n(\tau, y, v)|\leqq 2L(|x-y|^{\delta}\wedge 1)^{4)}+L|u-v|$$
$$\leqq 2L(|x-y|^{\delta\wedge 1/2}\wedge 1)+L|u-v|.$$

So,

$$|\varphi_n(\tau, Y_n^{(s,a)})-\varphi_n(\tau, Y_n^{(s,b)})|^2\leqq |Y_n^{(s,a)}(\tau)-Y_n^{(s,b)}(\tau)|^{1\wedge 2\delta}\cdot(8L^2+2L^2V_n(\tau))$$

and, by virtue of Lemma 2.2,

$$J_1\leqq K_3(s)|a-b|^{1/2\wedge\delta}\Big(\int_s^0\{1+V_n(\tau)\}d\tau\Big)^{1/2}. \tag{3.4}$$

Using the same technique, we have

$$J_2\leqq K_4(s)|a-b|^{1/2\wedge\delta}\Big(\int_s^0 1+V_n(\tau)d\tau\Big)^{1/2} \tag{3.5}$$

and

$$J_3\leqq (E|f_n(Y_n^{(s,a)}(0))-f_n(Y_n^{(s,b)}(0))|^2)^{1/2}e^{-Ls}(E\exp 2J(b, s))^{1/2} \tag{3.6}$$
$$\leqq K_5(s)|a-b|^{1/2\wedge\delta}.$$

Therefore, by (3.1), we get

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4) $c\wedge d=\min(c, d)$.

$$\left|U_n\left(s-\frac{1}{n}, a\right)-U_n\left(s-\frac{1}{n}, b\right)\right|^2 \leqq K_6(T)|a-b|^{1\wedge 2\delta}\left(1+\int_s^0 V_n(\tau)d\tau\right)$$

$$T\leqq s\leqq 0, \quad a, b\in R^1, \quad n=1, 2, \cdots.$$

This implies,

$$V_n\left(s-\frac{1}{n}\right)\leqq K_6(T)+K_6(T)\int_s^0 V_n(\tau)d\tau\,.$$

Hence, we have

$$V_n(s)\leqq K_6(T)e^{-K_6(T)s}\leqq K_7(T) \qquad T\leqq s\leqq 0,\ n=1, 2, \cdots. \tag{3.7}$$

As to the dependence of $U_n(s, a)$ on the starting time s, we can prove

$$\bar{V}_n(a, T)\leqq K_8(T) \qquad a\in R^1,\ n=1, 2, \cdots \tag{3.8}$$

where $\bar{V}_n(a, T)=\sup_{T\leqq s<s'\leqq 0}\dfrac{|U_n(s', a)-U_n(s, a)|^2}{|s-s'|^{1/2\wedge\delta}}$. In order to obtain (3.8), we devide $|U_n(s', a)-U_n(s, a)|$ into three parts $\bar{J}_1$, $\bar{J}_2$ and $\bar{J}_3$, like (3.1), and carıy out the same technique, using Lemma 2.3 instead of Lemma 2.2.

By (3.7) and (3.8), we have

LEMMA 3.1. *For $T\leqq 0$, there exists a constant $D(T)$ such that*

$$|U_n(s, a)-U_n(s', b)|\leqq D(T)(|a-b|^{1/2\wedge\delta}+|s-s'|^{1/4\wedge\delta/2})$$

$$s, s'\in[T, 0], \quad a, b\in R^1, \quad n=1, 2, \cdots.$$

As a special case of Lemma 3.1, we remark the following Lemma, which will be useful in § 5.

LEMMA 3.2. *There exists a constant $D'(T, t)$ such that*

$$E|t_g^{-1}X_n^{(s,a)}(t)-t_g^{-1}X_n^{(s',b)}(t)|\leqq D'(T, t)(|a-b|^{1/2\wedge\delta}+|s-s'|^{1/4\wedge\delta/2})$$

$$s, s'\in[T, t], \quad a, b\in R^1, \quad n=1, 2, \cdots.$$

§ 4. Totally boundedness.

First we review the topology of stochastic processes, introduced by Prohorov [3]. Let S be a separable complete metric space with the metric ρ, and $\mathscr{B}(S)$ the topological Borel field on S. Given two probability measures μ_1, μ_2 on $S(\mathscr{B}(S))$, the Prohorov distance $L(\mu_1, \mu_2)$ is defined as follows. Let ε_{12} be the infimum of ε such that, for any closed subset F of S

$$\mu_1(F)\leqq\mu_2(U_\varepsilon(F))+\varepsilon$$

where $U_\varepsilon(F)$ is the ε-neighborhood of F. Define ε_{21} by switching μ_1 and μ_2 in the definition of ε_{12} and set

$$L(\mu_1, \mu_2)=\max(\varepsilon_{12}, \varepsilon_{21})\,.$$

With this metric L the set of all probability measures on $S(\mathscr{B}(S))$ is a separable complete metric space.

A mapping $X(\omega)$ from a probability space $\Omega(\mathscr{B}, P)$ into S is called an S-valued random variable, if it is measurable in the sence that $X^{-1}(B) \in \mathscr{B}$ for every $B \in \mathscr{B}(S)$. The probability law μ_x of X is defined as the probability measure on $S(\mathscr{B}(S))$, i. e.,

$$\mu_x(B) = P(X^{-1}(B)) .$$

The Prohorov metric between two S-valued random variables X_1, X_2 (whether or not they are defined on the same probability space) is defined as the Prohorov distance between μ_{x_1} and μ_{x_2} and is denoted by $L(X_1, X_2)$. We recall the following two theorems.

THEOREM (Skorokhod). *If X_n, $n=1, 2, \cdots$ is an L-Cauchy sequence, then we can construct a sequence Y_n, $n=1, 2, \cdots$ and Y on the Lebesgue interval* $[0, 1]$ *such that*

$$L(X_n, Y_n) = 0$$

and

$$P(\rho(Y_n, Y) \to 0) = 1 .$$

By the Lebesgue interval we mean, of course, the probability space $\Omega(\mathscr{B}, P)$ where $\mathscr{B}$ consists of the Lebesgue measurable subsets of $[0, 1]$ and P is Lebesgue measure on $[0, 1]$.

A family of X_n, $n=1, 2, \cdots$ is called totally L-bounded, if every infinite sequence $\{X_{n_i}\}$ has an L-Cauchy subsequence.

THEOREM (Prohorov). *In order, for X_n, $n=1, 2, \cdots$, to be totally L-bounded, it is necessary and sufficient that, for every $\varepsilon > 0$, there exists a compact subset K of S such that*

$$P(X_n \in K) > 1-\varepsilon , \qquad \text{for } n=1, 2, \cdots .$$

In order to construct a solution of (1.1), we are concerned with the metric space $C[s, 0]$, associated with the usual metric ρ_s; $\rho_s(f, g) = \sup_{t \in [s,0]} |f(t)-g(t)|$. In the case, we have the following useful criterion for the totally L-boundedness.

PROPOSITION 4. *ξ_n, $n=1, 2, \cdots$ is totally L-bounded, if there exists a positive constant c such that, for $n=1, 2, \cdots$,*

$$E\xi_n^4(s) \leqq c$$

and

$$E|\xi_n(t)-\xi_n(t')|^4 \leqq c|t-t'|^2 .$$

Let (s_i, a_i) $i=1, 2, \cdots$ be a dense set of $I \times R^1$. We denote by D_i the direct product space $C[s_i, 0] \times C[s_i, 0]$ which is also a separable complete metric space with the metric $\rho_i(f, g)=\rho_{s_i}(f_1, g_1)+\rho_{s_i}(f_2, g_2)$, $f=(f_1, f_2)$, $g=(g_1, g_2)$. Appealing to the above proposition 4, we can easily see that the family of approximate solution $X_n^{(s_i,a_i)}$, $n=1, 2, \cdots$ is totally L-bounded, by the boundedness of α_n and β_n. Therefore we have

LEMMA 4.1. *The family of D_i-valued random variables $(X_n^{(s_i,a_i)}, B_{s_i})$ $n=1, 2, \cdots$, where $B_s(t)=B(t)-B(s)$, $t\in[s, 0]$, is totally L-bounded.*

We are concerned with the separable complete metric space $S=D_1\times D_2\times\cdots$ with the metric $\rho(f, g)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\rho_n(f_n, g_n)}{1+\rho_n(f_n, g_n)}$, $f=(f_1, f_2, \cdots)$, $g=(g_1, g_2, \cdots)$. By Lemma 4.1, we have, for $\varepsilon>0$, a compact subset K_i of D_i such that

$$P((X_n^{(s_i,a_i)}, B_{s_i})\in K_i)>1-\frac{\varepsilon}{2^{i+1}}, \qquad n=1, 2, \cdots.$$

Since the product set $K_1\times K_2\times\cdots$ is also compact in S, we obtain

LEMMA 4.2. *The family of S-valued random variables $X_n=((X_n^{(s_1,a_1)}, B_{s_1}), (X_n^{(s_2,a_2)}, B_{s_2}), \cdots)$, $n=1, 2, \cdots$, is totally L-bounded.*

Therefore, recalling Lemma 3.1, we can find a subsequence n_j, so that U_{n_j} converges uniformly on every compact subset of $I\times R^1$ and X_{n_j} converges in Prohorov metric. By Skorokhod's theorem, we can construct S-valued random variables $Y_j=((Y_j^{(s_1,a_1)}, B_{j,s_1}), (Y_j^{(s_2,a_2)}, B_{j,s_2}), \cdots)$ $j=1, 2, \cdots\infty$, on a certain probability space so that

$$L(Y_j, X_{n_j})=0, \qquad j=1, 2, \cdots \tag{4.1}$$

and

$$P(\rho(Y_j, Y_\infty)\to 0)=1. \tag{4.2}$$

Hence, B_{j,s_i}, $i=1, 2, \cdots$ satisfy the following consistency condition, with probability 1,

$$B_{j,s_i}(t)-B_{j,s_i}(s)=B_{j,s_k}(t)-B_{j,s_k}(s), \qquad s_i\vee s_k\leqq s<t\leqq 0. \tag{4.3}$$

So, we have the Wiener random measure dB_j such that $B_j(t)-B_j(s_i)=B_{j,s_i}(t)$, $s_i\leqq t\leqq 0$, and $(dB_j, Y_j^{(s_1,a_1)}, Y_j^{(s_2,a_2)}, \cdots)$ has the same probability law as $(dB, X_{n_j}^{(s_1,a_1)}, X_{n_j}^{(s_2,a_2)}, \cdots)$. On the other hand, by (4.2), we have also the Wiener random measure dB_∞ such that $B_{\infty,s_i}(t)=B_\infty(t)-B_\infty(s_i)$.

§ 5. Existence of solutions.

In this paragraph, we shall construct a solution of (1.1), making use of Y_∞ and dB_∞ of § 4.

Put $U(s, a)=\lim_{j\to\infty} U_{n_j}(s, a)$ and we shall prove

LEMMA 5.1.

(i) $\mathscr{B}_{(s_i,t)}(Y_\infty^{(s_i,a_i)}) \vee \mathscr{B}_{(-\infty,t)}(dB_\infty)$ *is independent of* $\mathscr{B}_{(t,0)}(dB_\infty)$ *for every* $t \in [s_i, 0]$,

(ii) $U(s_i, a_i) = Ef(Y_\infty^{(s_i,a_i)}(0)) \exp\int_{s_i}^0 \gamma(\tau, Y_\infty^{(s_i,a_i)}(\tau), U(\tau, Y_\infty^{(s_i,a_i)}(\tau)))d\tau$,

(iii) *with probability* 1,

$$Y_\infty^{(s_i,a_i)}(t) = a_i + \int_{s_i}^t \alpha(\tau, Y_\infty^{(s_i,a_i)}(t), U(\tau, Y_\infty^{(s_i,a_i)}(\tau)))d\tau$$
$$+\int_{s_i}^t \beta(\tau, Y_\infty^{(s_i,a_i)}(\tau), U(\tau, Y_\infty^{(s_i,a_i)}(\tau)))dB_\infty(\tau), \quad s_i \leqq t \leqq 0.$$

PROOF. $\mathscr{B}_{(s_i,t)}(X_{n_j}^{(s_i,a_i)}) \vee \mathscr{B}_{(-\infty,t)}(dB)$ is independent of $\mathscr{B}_{(t,0)}(dB)$, by the definition of $X_{n_j}^{(s_i,a_i)}$. Therefore $\mathscr{B}_{(s_i,t)}(Y_j^{(s_i,a_i)}) \vee \mathscr{B}_{(-\infty,t)}(dB_j)$ is independent of $\mathscr{B}_{(t,0)}(dB_j)$ and so (i) holds by (4.2).

Recalling Lemma 3.1, we have

$$|U_{n_j}(\tau, x_j) - U(\tau, x)| \leqq |U_{n_j}(\tau, x_j) - U_{n_j}(\tau, x)| + |U_{n_j}(\tau, x) - U(\tau, x)|$$
$$\leqq D(T)|x_j - x|^{1/2\wedge\delta} + |U_{n_j}(\tau, x) - U(\tau, x)|.$$

So, with probability 1,

$$U_{n_j}(\tau, Y_j^{(s_i,a_i)}(\tau)) \to U(\tau, Y_\infty^{(s_i,a_i)}(\tau)), \quad s_i \leqq \tau \leqq 0. \tag{5.1}$$

Therefore, with probability 1,

$$\xi_{n_j}(\tau, Y_j^{(s_i,a_i)}(\tau)) \to \xi(\tau, Y_\infty^{(s_i,a_i)})^{5)} \quad s_i \leqq \tau \leqq 0,\ \xi = \alpha, \beta, \gamma. \tag{5.2}$$

Hence, by Lebesgue's convergence theorem, we have

$$Ef(Y_\infty^{(s_i,a_i)}(0)) \exp\int_{s_i}^0 \gamma(\tau, Y_\infty^{(s_i,a_i)})d\tau$$
$$= \lim_j Ef_{n_j}(Y_j^{(s_i,a_i)}(0)) \exp\int_{s_i}^0 \gamma_{n_j}(\tau, Y_j^{(s_i,a_i)})d\tau$$
$$= \lim_j U_{n_j}\Big(s_i - \frac{1}{n_j}, a_i\Big) = U(s_i, a_i).$$

Since $Y_\infty^{(s_i,a_i)}$ are continuous with probability 1, it is enough, for the proof of (ii), to prove for each t that

$$Y_\infty^{(s_i,a_i)}(t) = a_i + \int \alpha(\tau, Y_\infty^{(s_i,a_i)})d\tau + \int \beta(\tau, Y_\infty^{(s_i,a_i)})dB_\infty(\tau) \tag{iii)'}$$

holds with probability 1. Again by Lebesgue's convergence theorem and (5.2), we have

$$\int_{s_i}^t \alpha_{n_j}(\tau, Y_j^{(s_i,a_i)})d\tau \to \int_{s_i}^t \alpha(\tau, Y_\infty^{(s_i,a_i)})d\tau. \tag{5.3}$$

5) $\alpha(\tau, x) = \alpha(\tau, x(\tau), u(\tau, x(\tau)))$, etc.

Put $I_j=\int_{s_i}^t \beta_{n_j}(\tau, Y_j^{(s_i,a_i)})dB_j$, $j=1, 2, \cdots \infty$,[6] and let $L_j(\Delta)$ be the approximate sum of I_j for a division Δ; $s_i=v_0<v_1<\cdots<v_l=t$. On the other hand, recalling Proposition 2 and Lemma 3.1 we can choose a bounded and continuous function $\tilde{\theta}$ with $\tilde{\theta}(0)=0$, so that

$$|\beta_{n_j}(\tau, Y_j^{(s_i,a_i)})-\beta_{n_j}(t, Y_j^{(s_i,a_i)})|\leqq\tilde{\theta}(|\tau-t|)+\tilde{\theta}(|Y_j^{(s_i,a_i)}(\tau)-Y_j^{(s_i,a_i)}(t)|)$$
$$s_i\in[T, 0],\ a\in R^1,\ j=1, 2, \cdots, \infty .$$

Therefore, for $\varepsilon>0$, there exists a division Δ such that

$$E|I_j-L_j(\Delta)|^2<\varepsilon, \qquad j=1, 2, \cdots, \infty . \tag{5.4}$$

On the other hand, $L_j(\Delta)$ tends to $L_\infty(\Delta)$ with probability 1. So, I_j tends to I_∞ in probability. Hence, taking (5.3) into account, we have (iii)$'$.

In order to complete the proof of Theorem, we shall evaluate the dependence of $X_n^{(s,a)}$ on the initial point (s, a) using Lemma 3.2. Since α_n and β_n are bounded, we have a positive $d=d(\varepsilon)$, for $\varepsilon>0$, such that

$$P(|X_n^{(s,a)}(t)|>d)<\varepsilon, \quad |a|<\frac{1}{\varepsilon}, \quad s\in\left[-\frac{1}{\varepsilon}, 0\right], \qquad n=1, 2, \cdots .$$

Since $|t_g^{-1}x-t_g^{-1}y|\geqq\dfrac{|x-y|}{1+d^2}$ for $|x|, |y|\leqq d$, we get

$$\begin{aligned}
(5.5)\qquad & P(|X_n^{(s,a)}(t)-X_n^{(s',b)}(t)|>\varepsilon)\\
&\leqq P(|X_n^{(s,a)}(t)|>d)+P(|X_n^{(s',b)}(t)|>d)\\
&\quad+P\left(|t_g^{-1}X_n^{(s,a)}(t)-t_g^{-1}X_n^{(s',b)}(t)|>\frac{\varepsilon}{1+d^2}\right)\\
&\leqq 2\varepsilon+\frac{1+d^2}{\varepsilon}E|t_g^{-1}X_n^{(s,a)}(t)-t_g^{-1}X^{(s',b)}(t)|\\
&\leqq 2\varepsilon+\frac{1+d^2}{\varepsilon}D'\left(\frac{1}{\varepsilon}, t\right)(|a-b|^{1/2\wedge\delta}+|s-s'|^{1/4\wedge\delta/2})\\
&\qquad s, s'\in\left[-\frac{1}{\varepsilon}, t\right],\quad a, b\in R^1,\quad n=1, 2, \cdots .
\end{aligned}$$

Therefore, by (4.2), we have a positive $h=h(\varepsilon)$ such that, for $|a_i|, |a_k|\leqq\frac{1}{\varepsilon}$, $s_i, s_k\in\left[-\frac{1}{\varepsilon}, t\right]$, $|a_i-a_k|<h$ and $|s_i-s_k|<h$,

$$P(|Y_j^{(s_i,a_i)}(t)-Y_j^{(s_k,a_k)}(t)|>\varepsilon)<\varepsilon, \qquad j=1, 2, \cdots, \infty .$$

So, when (s_i, a_i) tends to (s, a), $Y_j^{(s_i,a_i)}$ converges in probability. Setting

$$Y_j^{(s,a)}(t)=p-\lim_{\substack{s_i\to s\\ a_i\to a}} Y_j^{(s_i,a_i)}(t), \qquad j=1, 2, \cdots, \infty ,$$

6) $\beta_\infty\equiv\beta$.

we can see that

(5.6) $$\mathscr{B}_{(s,t)}(Y_j^{(s,a)}) \vee \mathscr{B}_{(-\infty,t)}(dB_j) \text{ is independent of } \mathscr{B}_{(t,0)}(dB_j), \quad j=1, 2, \cdots, \infty,$$

and

(5.7) $$P(|Y_j^{(s,a)}(t)-Y_j^{(s',b)}(t)|>\varepsilon)<\varepsilon, \qquad j=1, 2, \cdots, \infty,$$

for $|a|, |b| \leqq \frac{1}{\varepsilon}$, $s, s' \in \left[-\frac{1}{\varepsilon}, t\right]$, $|a-b|<h$ and $|s-s'|<h$.

Hence,

(5.8) $$\begin{aligned} &P(|Y_j^{(s,a)}(t)-Y_\infty^{(s,a)}(t)|>3\varepsilon) \\ &\leqq P(|Y_j^{(s,a)}(t)-Y_j^{(s_k,a_k)}(t)|>\varepsilon)+P(|Y_j^{(s_k,a_k)}(t)-Y_\infty^{(s_k,a_k)}(t)|>\varepsilon) \\ &\quad +P(|Y_\infty^{(s_k,a_k)}(t)-Y_\infty^{(s,a)}(t)|>\varepsilon) \to 0, \qquad \text{as } j\to\infty. \end{aligned}$$

On the other hand, $(Y_j^{(s,a)}, dB_j)$ has the same probability law as $(X_{n_j}^{(s,a)}, dB)$, since $X_n^{(s,a)}(t)=p-\lim_{\substack{s_i\to s\\ a_i\to a}} X_n^{(s_i,a_i)}$, by (5.5). So, $Y_j^{(s,a)}$ are continuous and satisfy the same stochastic differential equation as $X_{n_j}^{(s,a)}$. Thus the family of $Y_j^{(s,a)}$, $j=1, 2, \cdots$ is totally L-bounded. Therefore, by (5.8), $Y_j^{(s,a)}$ is itself an L-Cauchy sequence, whose L-limit has the same probability as $Y_\infty^{(s,a)}$. Hence $Y_\infty^{(s,a)}$ are continuous.

Since $\alpha(\tau, Y_\infty^{(s,a)})=p-\lim_{\substack{s_i\to s\\ a_i\to a}} \alpha(\tau, Y_\infty^{(s_i,a_i)})$ and $\beta(\tau, Y_\infty^{(s,a)})=p-\lim_{\substack{s_i\to s\\ a_i\to a}}\beta(\tau, Y_\infty^{(s_i,a_i)})$, we have, for each t,

(5.9) $$\begin{aligned} &\int_s^t \alpha(\tau, Y_\infty^{(s,a)})d\tau+\int_s^t \beta(\tau, Y_\infty^{(s,a)})dB_\infty \\ &= \operatorname*{l.i.m}_{\substack{s_i\to s\\ a_i\to a}} \int_{s_i}^t \alpha(\tau, Y_\infty^{(s_i,a_i)})d\tau+\int_{s_i}^t \beta(\tau, Y_\infty^{(s_i,a_i)})dB_\infty \end{aligned}$$

by the boundedness of α and β. Recalling Lemma 5.1, we have, with probability 1,

$$Y_\infty^{(s,a)}(t)=a+\int_s^t \alpha(\tau, Y_\infty^{(s,a)})d\tau+\int_s^t \beta(\tau, Y_\infty^{(s,a)})dB_\infty, \quad \text{for every } t\in[s, 0],$$

by virtue of the continuity of $Y_\infty^{(s,a)}$.

As to $U(s, a)$, we get

$$\begin{aligned} U(s, a) &= \lim_{\substack{s_i\to s\\ a_i\to a}} U(s_i, a_i) = \lim_{\substack{s_i\to s\\ a_i\to a}} Ef(Y_\infty^{(s_i,a_i)}(0)) \exp \int_{s_i}^0 \gamma(\tau, Y_\infty^{(s_i,a_i)})d\tau \\ &= Ef(Y_\infty^{(s,a)}(0)) \exp \int_s^0 \gamma(\tau, Y_\infty^{(s,a)})d\tau. \end{aligned}$$

This completes the proof of Theorem.

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