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On infinitesimal automorphisms of Siegel domains

Dedicated to Prof. Atuo Komatu for his 60th birthday

By Noboru TANAKA

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Introduction

Let D be the Siegel domain of the second kind in the space $\boldsymbol{C}^N$ of N $(=n+m)$ complex variables due to Pyatetski-Shapiro [6], associated with a convex cone V in the space $\boldsymbol{R}^n$ of n real variables and a V-hermitian form F on the space $\boldsymbol{C}^m$ of m complex variables. By an infinitesimal automorphism of the domain D, we mean a holomorphic vector field X on D which is complete, that is, generates a global one parameter group φ_t of transformations.

The main purpose of the present paper is to give the details of the results announced in the note [8], establishing some theorems on the Lie algebra $\mathfrak{g}$ of all infinitesimal automorphisms of a Siegel domain D of the second kind.

Assume that the domain D is affine homogeneous. At the outset we prove that the Lie algebra $\mathfrak{g}$ is endowed with the structure of a graded Lie algebra as follows: $\mathfrak{g}=\sum_{p=-\infty}^{\infty}\mathfrak{g}^p$ (direct sum); $[\mathfrak{g}^p, \mathfrak{g}^q]\subset\mathfrak{g}^{p+q}$; $\mathfrak{g}^p=\{0\}$ $(p<-2)$ and the subalgebra $\mathfrak{g}_a=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^0$ of $\mathfrak{g}$ is just the Lie algebra of all infinitesimal affine automorphisms of D (Theorem 3.1). Then we prove that the graded Lie algebra $\mathfrak{g}_a$ is prolonged to a graded Lie algebra $\hat{\mathfrak{g}}=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^0+\sum_{p=1}^{\infty}\hat{\mathfrak{g}}^p$ and that the graded Lie algebra $\mathfrak{g}$ is determined as a suitable graded subalgebra of $\hat{\mathfrak{g}}$ (Theorem 4.1). From a geometric point of view, the Lie algebra $\hat{\mathfrak{g}}$ may be described as a Lie algebra of polynomial vector fields X on $\boldsymbol{C}^N$ tangent to the Silov boundary S of the domain D (See § 4). In [6], Pyatetski-Shapiro has determined the graded Lie algebra $\mathfrak{g}_a$ in terms of the cone V and the V-hermitian form F. Theorem 4.1 in turn enables us to compute the Lie algebra $\mathfrak{g}$ on the basis of the Lie algebra $\mathfrak{g}_a$ (See § 5, Examples).

In our discussion, it is important that every infinitesimal automorphism X on the domain D is extended to a holomorphic vector field defined on the whole $\boldsymbol{C}^N$ and tangent to the real submanifold S of $\boldsymbol{C}^N$ (Proposition 3.1). Owing to this fact, our problems can be connected, to a great extent, with the geometry of real submanifolds of complex manifolds and hence with the geometry of differential systems as developed by Tanaka [9].

In § 1, we study the standard Lie algebra sheaves, which we owe essentially to Tanaka [**9**], §§ 5 and 6. In § 2, we recall several known results about a Siegel domain of the second kind. § 3 (resp. § 4) is devoted to the proof of Theorem 3.1 (resp. of Theorem 4.1). In § 5, we above all show that, in the case when $\mathfrak{g}^{-2}=[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$, the Lie algebra $\mathfrak{g}$ coincides with the prolongation $\hat{\mathfrak{g}}$ of $\mathfrak{g}_a$ and consists of all holomorphic vector fields X on $\boldsymbol{C}^N$ tangent to S (Proposition 5.3). Finally in Appendix, we prove a uniqueness theorem (Theorem A) for the graded Lie algebra $\mathfrak{g}=\sum_p \mathfrak{g}^p$ stated as follows: Let D and D' be two Siegel domains of the second kind in $\boldsymbol{C}^N$ which are both affine homogeneous. If the domains D and D' are mutually isomorphic, so are the corresponding graded Lie algebras $\mathfrak{g}=\sum_p \mathfrak{g}^p$ and $\mathfrak{g}'=\sum_p \mathfrak{g}'^p$. It should be noted that our uniqueness theorem reproduces the uniqueness theorem for the realization of homogeneous bounded domains as Siegel domains of the second kind which was first asserted by Pyatetski-Shapiro [7] and later rigorously proved by Kaneyuki [2].

Preliminary remark

$\boldsymbol{R}$ (resp. $\boldsymbol{C}$) denotes the field of real numbers (resp. of complex numbers). Let V be a real vector space. V_c denotes the complexification of V. For each $x \in V_c$, we denote by $\operatorname{Re} x$ (resp. by $\operatorname{Im} x$) the real part (resp. the imaginary part) of x with respect to the real form V of V_c and by $\bar{x}$ the vector in V_c conjugate to x.

Throughout the present paper, we always assume the differentiability of class $\boldsymbol{C}^\omega$.

§ 1. The algebraic prolongations and the standard Lie algebra sheaves.

1.1. The algebraic prolongations (cf. Tanaka [**9**], § 5). Let $\mathfrak{m}=\sum_{p<0} \mathfrak{g}^p$ be a graded (Lie) algebra with $\dim \mathfrak{m}<\infty$. (We do not necessarily require that $\mathfrak{m}$ is fundamental in the sense of [**9**], i.e., $\mathfrak{m}$ is generated by $\mathfrak{g}^{-1}$.) We show that with such a graded algebra $\mathfrak{m}$ there is associated a graded algebra $\mathfrak{g}(\mathfrak{m})=\sum_p \mathfrak{g}^p(\mathfrak{m})$, called the prolongation of $\mathfrak{m}$, satisfying the following conditions:

1) $\mathfrak{m}=\sum_{p<0} \mathfrak{g}^p(\mathfrak{m})$ as graded algebras;

2) For each $p \geqq 0$, the condition " $X \in \mathfrak{g}^p(\mathfrak{m})$, $[X, \mathfrak{m}]=\{0\}$" implies $X=0$;

3) $\mathfrak{g}(\mathfrak{m})$ is maximum among the graded algebras satisfying conditions 1) and 2). More precisely, let $\mathfrak{h}=\sum_p \mathfrak{h}^p$ be any graded algebra satisfying conditions 1) and 2). Then $\mathfrak{h}$ is imbedded in $\mathfrak{g}(\mathfrak{m})$ as a graded subalgebra.

We put $\mathfrak{g}^p(\mathfrak{m})=\mathfrak{g}^p$ $(p<0)$. Let us define vector spaces $\mathfrak{g}^p(\mathfrak{m})$ $(p \geqq 0)$ inductively as follows: First of all, we define $\mathfrak{g}^0(\mathfrak{m})$ to be the Lie algebra of all derivations of $\mathfrak{m}$ as graded algebra. Suppose now that we have defined $\mathfrak{g}^p(\mathfrak{m})$ $(0 \leqq p < k)$ in such a way that $\mathfrak{g}^p(\mathfrak{m})$ is a subspace of $\mathfrak{q}^p(\mathfrak{m}) = \sum_{r<0} \mathrm{Hom}(\mathfrak{g}^r, \mathfrak{g}^{r+p}(\mathfrak{m})) \subset \mathrm{Hom}(\mathfrak{m}, \sum_{r<0} \mathfrak{g}^{r+p}(\mathfrak{m}))$. Then we define $\mathfrak{g}^k(\mathfrak{m})$ to be the subspace of $\mathfrak{q}^k(\mathfrak{m}) = \sum_{r<0} \mathrm{Hom}(\mathfrak{g}^r, \mathfrak{g}^{r+k}(\mathfrak{m}))$ which consists of all $X^k \in \mathfrak{q}^k(\mathfrak{m})$ satisfying the following equalities:

$$X^k(Y^r)(Z^s) - X^k(Z^s)(Y^r) = X^k([Y^r, Z^s]) \quad \text{for all} \quad Y^r \in \mathfrak{g}^r,\ Z^s \in \mathfrak{g}^s \ (r, s < 0),$$

where we put

$$X^k(Y^r)(Z^s) = [X^k(Y^r), Z^s] \quad (\text{if } r+k < 0)$$

and

$$X^k(Z^s)(Y^r) = [X^k(Z^s), Y^r] \quad (\text{if } s+k < 0).$$

Thus we have completed our inductive definition. We put $\mathfrak{g}(\mathfrak{m}) = \sum_p \mathfrak{g}^p(\mathfrak{m})$. Then we see easily that there is a unique bracket operation $[\ ,\ ]$ in $\mathfrak{g}(\mathfrak{m})$ such that $\mathfrak{g}(\mathfrak{m})$ becomes a graded algebra satisfying conditions 1) and 2) with respect to this bracket operation and such that $[X^k, Y^r] = X^k(Y^r)$ for all $X^k \in \mathfrak{g}^k(\mathfrak{m})$ and $Y^r \in \mathfrak{g}^r$ $(k \geqq 0,\ r < 0)$. Moreover it is easy to see that the graded algebra $\mathfrak{g}(\mathfrak{m})$ thus defined satisfies condition 3).

We note that the Lie algebra $\mathfrak{g}^0(\mathfrak{m})$ contains a unique element E in its center such that $[E, X] = pX$ for all $X \in \mathfrak{g}^p$ and all p.

Let $\mathfrak{m}$ and $\mathfrak{g}(\mathfrak{m})$ as above. Suppose that we are given a sequence $(\mathfrak{g}^p)_{0 \leqq p \leqq k}$ satisfying the following conditions:

(1.1) 1) $\mathfrak{g}^p$ is a subspace of $\mathfrak{g}^p(\mathfrak{m})$;

2) The family $(\mathfrak{g}^p)_{-\infty < p \leqq k}$ satisfies $[\mathfrak{g}^r, \mathfrak{g}^s] \subset \mathfrak{g}^{r+s}$ $(r+s \leqq k)$.

Then we define a sequence $(\mathfrak{g}^p)_{p>k}$ inductively as follows: l being an integer $> k$, suppose that we have defined $\mathfrak{g}^{k+1}, \cdots, \mathfrak{g}^{l-1}$ as subspaces of $\mathfrak{g}^{k+1}(\mathfrak{m}), \cdots, \mathfrak{g}^{l-1}(\mathfrak{m})$, respectively. Then we define $\mathfrak{g}^l$ to be the subspace of $\mathfrak{g}^l(\mathfrak{m})$ consisting of all the elements X^k such that $[X^l, \mathfrak{g}^r] \subset \mathfrak{g}^{l+r}$ $(r < 0)$. If we put $\mathfrak{g} = \sum_p \mathfrak{g}^p$, then we can easily prove $\mathfrak{g}$ to be a graded subalgebra of $\mathfrak{g}(\mathfrak{m})$. The graded algebra $\mathfrak{g}$ is called the prolongation of $(\mathfrak{m}, \mathfrak{g}^0, \cdots, \mathfrak{g}^k)$.

Let $\mathfrak{g}^0$ be a subalgebra of $\mathfrak{g}^0(\mathfrak{m})$. Then we clearly have $[\mathfrak{g}^r, \mathfrak{g}^s] \subset \mathfrak{g}^{r+s}$ $(r+s \leqq 0)$. Therefore we may talk about the prolongation of $(\mathfrak{m}, \mathfrak{g}^0)$.

1.2. We shall use the following lemma in § 3.

LEMMA 1.1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra, L^0 a subalgebra of $\mathfrak{g}$ and $(\mathfrak{g}^p)_{p<0}$ a family of subspaces of $\mathfrak{g}$. As for these things, assume the following conditions:*

1) *$(\mathfrak{g}, L^0)$ is effective, i.e., L^0 contains no ideals $(\neq \{0\})$ of $\mathfrak{g}$;*

2) *$\mathfrak{m} = \sum_{p<0} \mathfrak{g}^p$ (direct sum) and it is a graded algebra;*

3) $\mathfrak{g}=\mathfrak{m}+L^0$ *(direct sum)*;

4) *If we put* $L^p=\sum_{r=p}^{-1}\mathfrak{g}^r+L^0$, *then* $[L^p, L^0]\subset L^p$;

5) L^0 *contains an element* E *such that* $[E, X]=pX$ *for all* $X\in\mathfrak{g}^p$ *and* $p<0$.

For any integer p, *let* $\mathfrak{n}^p$ *be the subspace of* $\mathfrak{g}$ *consisting of all* $X\in\mathfrak{g}$ *such that* $[E, X]=pX$. *Then we have:*

(1) $\mathfrak{g}=\sum_p\mathfrak{n}^p$ *(direct sum) and it is a graded algebra;*

(2) $\mathfrak{g}^p=\mathfrak{n}^p$ $(p<0)$ *and* $L^0=\sum_{p\geqq 0}\mathfrak{n}^p$;

(3) *For each* $p\geqq 0$, *the condition* "$X\in\mathfrak{n}^p$, $[X, m]=\{0\}$" *implies* $X=0$.

PROOF. Let us define a family $(L^p)_{p>0}$ of subspaces of L^0 inductively as follows: p being an integer >0, suppose that we have defined L^{p-1} Then we define L^p to be the subspace of L^{p-1} consisting of all $X\in L^{p-1}$ such that $[X, L^q]\subset L^{p+q}$ for all $q<0$, completing our inductive definition. By 1)—4), we see that the family $(L^p)_{p>0}$ combined with the family $(L^p)_{p\leqq 0}$ satisfies $L^p\supset L^{p+1}$, $\mathfrak{g}=\bigcup_p L^p$, $\bigcap_p L^p=\{0\}$ and $[L^p, L^q]\subset L^{p+q}$. We put $\bar{\mathfrak{n}}^p=L^p/L^{p+1}$ and $\bar{\mathfrak{g}}=\sum_p\bar{\mathfrak{n}}^p$. Then it is clear that $\bar{\mathfrak{g}}$ is a graded algebra and that, for each $p\geqq 0$, the condition "$\bar{X}\in\bar{\mathfrak{n}}^p$, $[\bar{X}, \bar{\mathfrak{n}}^q]=\{0\}$ for all $q<0$" implies $\bar{X}=0$. Denoting by $\bar{E}$ the image of E by the projection of L^0 onto $\bar{\mathfrak{n}}^0$, we prove

$$[\bar{E}, \bar{X}]=p\bar{X} \tag{1.2}$$

for all $\bar{X}\in\bar{\mathfrak{n}}^p$ and all p. Indeed, (1.2) is clearly the case if $p<0$ (by 5)). k being an integer $\geqq 0$, suppose that (1.2) holds for any $p<k$. Then for all $\bar{X}\in\bar{\mathfrak{n}}^k$ and $\bar{Y}\in\bar{\mathfrak{n}}^q$ $(q<0)$, we have

$$\begin{aligned}[][[\bar{E}, \bar{X}]-k\bar{X}, \bar{Y}]&=[[\bar{E}, \bar{Y}], \bar{X}]+[\bar{E}, [\bar{X}, \bar{Y}]]-k[\bar{X}, \bar{Y}]\\ &=q[\bar{Y}, \bar{X}]+(k+q)[\bar{X}, \bar{Y}]-k[\bar{X}, \bar{Y}]=0\,, \quad\text{whence}\end{aligned}$$

$[\bar{E}, \bar{X}]=k\bar{X}$, proving our assertion. Let us now denote by $\mathfrak{n}_0^p$ the subspace of L^p consisting of all $X\in L^p$ such that $(ad\,E-p)X=0$ and show that

$$L^p=\mathfrak{n}_0^p+L^{p+1}\quad\text{(direct sum)}. \tag{1.3}$$

Indeed, let $X\in\mathfrak{n}_0^p\cap L^{p+1}$. Since (1.2) may be interpreted as $(ad\,E-r)L^r\subset L^{r+1}$, we have $(ad\,E-(p+1))X=-X\in L^{p+2}$, $(ad\,E-(p+2))X=-2X\in L^{p+3}$ and so on. Therefore we have $X\in\bigcap_r L^r=\{0\}$ and hence $\mathfrak{n}_0^p\cap L^{p+1}=\{0\}$. Since $\mathfrak{n}_0^p$ is the kernel of $L^p\ni X\rightarrow(ad\,E-p)X\in L^{p+1}$, we have $\dim(L^p/\mathfrak{n}_0^p)\leqq\dim L^{p+1}$. Thus we get (1.3). By (1.3) it is clear that $\mathfrak{n}^p=\mathfrak{n}_0^p$ and $L^p=\sum_{r\geqq p}\mathfrak{n}^r$ (direct sum). In particular, $\mathfrak{g}=\sum_p\mathfrak{n}^p$ (direct sum), which is clearly a graded algebra. Furthermore we easily have assertions (2) and (3).

1.3. The standard Lie algebra sheaf of type $\mathfrak{m}$ (cf. [**9**], § 6). Let $\mathfrak{m}=\sum_{p<0}\mathfrak{g}^p$ be a graded algebra with $\dim\mathfrak{m}<\infty$ and let $\mathfrak{g}(\mathfrak{m})=\sum_p\mathfrak{g}^p(\mathfrak{m})$ be the prolongation of it. Let $M(\mathfrak{m})$ be the simply connected Lie group whose Lie algebra (of all left invariant vector fields) is equal to the Lie algebra $\mathfrak{m}$. Then we see that, for each $p<0$, the subspace $\mathfrak{d}^p=\sum_{r=p}^{-1}\mathfrak{g}^r$ of $\mathfrak{m}$ defines a differential system Δ^p on $M(\mathfrak{m})$ which is invariant under the left translations. Let $\mathcal{A}$ denote the sheaf of local vector fields X on $M(\mathfrak{m})$ leaving every differential system Δ^p invariant. $\mathcal{A}$ is a transitive Lie algebra sheaf on $M(\mathfrak{m})$, which is called the standard Lie algebra sheaf of type $\mathfrak{m}$.

Let ξ be the Maurer-Cartan form of the Lie group $M(\mathfrak{m})$ that is an $\mathfrak{m}$-valued 1-form on $M(\mathfrak{m})$ defined by $\xi(X_x)=X$ ($X\in\mathfrak{m}$ and $x\in M(\mathfrak{m})$) and let ξ^p be the $\mathfrak{g}^p$-component of ξ in the decomposition $\mathfrak{m}=\sum_{p<0}\mathfrak{g}^p$. Then the differential system Δ^p is defined by the equations $\xi^r=0$ $(r<p)$ and we can prove the following fact:

A local vector fields X defined on an open set U of $M(\mathfrak{m})$ is a local cross-section of $\mathcal{A}$ if and only if there is a $\mathfrak{g}^0(\mathfrak{m})$-valued function f^0 on U such that

$$L_X\xi^p\equiv[f^0,\xi^p]\quad (\mathrm{mod}\ \xi^r\ (r<p)),$$

where L_X denotes the Lie derivation with respect to the vector field X. It should be noted that Lemmas 6.1—6.5 in [**9**] remain true without any modification for the sheaf $\mathcal{A}$. In particular we have: For any local cross-section X of $\mathcal{A}$ defined on an open set U of $M(\mathfrak{m})$, there is a unique family $(f_X^p)_{p\in Z}$ satisfying the following conditions:

(1.4)
1) f_X^p is a $\mathfrak{g}^p(\mathfrak{m})$-valued function on U;
2) $f_X^p=\xi^p(X)$;
3) $df_X^p=\sum_{r<0}[f_X^{p-r},\xi^r]$.

Let us now confine ourselves to the case where $\mathfrak{g}^p=\{0\}$ $(p<-2)$, i.e., $\mathfrak{m}=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}$. Let $e_1^{-2},\cdots,e_n^{-2}$ (resp. $e_1^{-1},\cdots,e_m^{-1}$) be a base of $\mathfrak{g}^{-2}$ (resp. of $\mathfrak{g}^{-1}$) and let $x_1,\cdots,x_n,y_1,\cdots,y_m$ be the normal coordinate system of the Lie group $M(\mathfrak{m})$ corresponding to the base $e_1^{-2},\cdots,e_n^{-2},e_1^{-1},\cdots,e_m^{-1}$ of $\mathfrak{m}$.

Since $\exp:\mathfrak{m}\to M(\mathfrak{m})$ is a real analytic homeomorphism of $\mathfrak{m}$ onto $M(\mathfrak{m})$, this coordinate system is defined on the whole $M(\mathfrak{m})$. We put $x=\sum_i x_ie_i^{-2}$ and $y=\sum_j y_je_j^{-1}$, which are $\mathfrak{g}^{-2}$ and $\mathfrak{g}^{-1}$-valued functions on $M(\mathfrak{m})$, respectively. Let $c\in\mathfrak{g}^l(\mathfrak{m})$. By [**9**], Lemma 6.3, there is a global cross-section X of $\mathcal{A}$ such that $f_X^p(e)=0$ $(p<l)$, $f_X^l(e)=c$ and $f_X^p=0$ $(p>l)$, e being the identity of the Lie group $M(\mathfrak{m})$.

LEMMA 1.2.

(1) $f_X^p=H_l^p(x,y)c$,

where
$$H_l^p(x,y)=\sum_{\substack{2r+s=l-p\\ r,s\geqq 0}}\frac{(-1)^{r+s}}{r!\,s!}(ad\,x)^r(ad\,y)^s.$$

(2) $L_X x = H_l^{-2}(x, y)c + \frac{1}{2}[y, H_l^{-1}(x, y)c]$,

$L_X y = H_l^{-1}(x, y)c$.

PROOF. We easily find $\xi^{-2} = dx - \frac{1}{2}[y, dy]$ and $\xi^{-1} = dy$. By (1.4), we have

$$df_X^p = [f_X^{p+2}, \xi^{-2}] + [f_X^{p+1}, \xi^{-1}] . \tag{1.5}$$

We put $g^p = H_l^p(x, y)c$, which is a $\mathfrak{g}^p(\mathfrak{m})$-valued function on $M(\mathfrak{m})$. We have $g^p(e) = 0$ $(p < l)$, $g^l(e) = c$ and $g^p = 0$ $(p > l)$.

By using the equalities

$$d(ad\ x)^r c' = r\ ad(dx)(ad\ x)^{r-1}c' ,$$

$$d(ad\ y)^s c' = s\ ad(dy)(ad\ y)^{s-1}c' + \frac{s(s-1)}{2} ad([y, dy])(ad\ y)^{s-2}c'$$

for all $c' \in \mathfrak{g}(\mathfrak{m})$, we find

$$dg^p = [g^{p+2}, \xi^{-2}] + [g^{p+1}, \xi^{-1}] . \tag{1.6}$$

We have $f_X^p = g^p = 0$ $(p > l)$. k being an integer $\leqq l$, suppose that $f_X^p = g^p$ for all $p > k$. Then by (1.5) and (1.6), we have $f_X^k - g^k =$ constant. Since $f_X^k(e) = g^k(e)$, it follows that $f_X^k = g^k$. Thus we have proved (1). (2) follows from (1) and the equalities

$$f_X^{-2} = \xi^{-2}(X) = L_X x - \frac{1}{2}[y, L_X y] \quad \text{and} \quad f_X^{-1} = \xi^{-1}(X) = L_X y .$$

Let us return to the general case and suppose that we are given a sequence $(\mathfrak{g}^p)_{0 \leqq p \leqq k}$ satisfying (1.1). Let $\mathfrak{g} = \sum_p \mathfrak{g}^p$ be the prolongation of $(\mathfrak{m}, \mathfrak{g}^0, \cdots, \mathfrak{g}^k)$. We denote by $\mathcal{L}$ the sheaf of local vector fields X on $M(\mathfrak{m})$ such that X is a local cross-section of $\mathcal{A}$ and such that the $\mathfrak{g}^p(\mathfrak{m})$-valued function f_X^p is reduced to a $\mathfrak{g}^p$-valued function for all p $(0 \leqq p \leqq k)$.

Then we see that Lemma 6.9 in [**9**] remains true for the sheaf $\mathcal{L}$. It follows that $\mathcal{L}$ is a transitive Lie algebra sheaf on $M(\mathfrak{m})$, which is called the standard Lie algebra sheaf of type $(\mathfrak{m}, \mathfrak{g}^0, \cdots, \mathfrak{g}^k)$. The formal algebra L of $\mathcal{L}$ may be identified with the formal algebra $\bar{\mathfrak{g}} = \prod_p \mathfrak{g}^p$ associated with the prolongation $\mathfrak{g}$ of $(\mathfrak{m}, \mathfrak{g}^0, \cdots, \mathfrak{g}^k)$.

§ 2. Siegel domains of the second kind.

Let W^{-2} (resp. W^{-1}) be a real (resp. complex) vector space of finite dimension. We say that an open set V of W^{-2} is a convex cone in W^{-2} if it satisfies the following conditions:

1) For any $x \in V$ and any positive number λ, $\lambda x \in V$;
2) For any $x, x' \in V$, $x + x' \in V$;

3) V contains no entire straight lines.

Given a convex cone V in W^{-2}, we say that a mapping F of $W^{-1}\times W^{-1}$ to W_c^{-2} (=the complexification of W^{-2}) is a V-hermitian form on W^{-1} if it satisfies the following conditions:

1) $F(y, y')$ is complex linear with respect to the variable y, and $\overline{F(y, y')} = F(y', y)$;
2) $F(y, y)\in \bar{V}$, where $\bar{V}$ denotes the closure of V in W^{-2};
3) $F(y, y)\neq 0$ if $y\neq 0$.

In what follows, we shall consider a fixed convex cone V in W^{-2} and a fixed V-hermitian form F on W^{-1}.

We put $\tilde{W}=W_c^{-2}+W^{-1}$ and denote by z (resp. by u) the projection of $\tilde{W}$ onto W_c^{-2} (resp. onto W^{-1}). We define a mapping Φ of $\tilde{W}$ to W^{-2} by

$$\Phi(p)=\mathrm{Im}\, z(p)-F(u(p), u(p)) \qquad (p\in\tilde{W}).$$

Then the domain $D=\Phi^{-1}(V)$ (=the inverse image of V by Φ) of $\tilde{W}$ is called the Siegel domain of second kind associated with the cone V and the V-hermitian form F (Pyatetski-Shapiro [**6**]). We have $D\cap\sqrt{-1}W^{-2}=\sqrt{-1}V$.

2.2. It is easy to see (cf. [**3**]) that the closure $\bar{D}$ of D in $\tilde{W}$ is given by $\Phi^{-1}(\bar{V})$ and the boundary ∂D of D by $\Phi^{-1}(\partial V)$, where ∂V denotes the boundary of V.

Let S denote the real submanifold of $\tilde{W}$ defined by the equation $\Phi=0$, i.e., $S=\Phi^{-1}(0)$. S is a subset of ∂D. Let $\mathcal{E}$ be the ring of all functions f satisfying the following conditions:

1) f is defined and holomorphic on a neighborhood of $\bar{D}$;
2) f is equal to zero at the infinity.

LEMMA 2.1 ([**6**]).

(1) *For any $f\in\mathcal{E}$, there is a $p\in S$ such that the maximum of $|f|$ on $\bar{D}$ is attained at p.*

(2) *At any $p\in S$, there is an $f\in\mathcal{E}$ such that the maximum of $|f|$ on $\bar{D}$ is attained at only one point p.*

We owe the present formulation of this lemma to Kaneyuki-Sudo [**3**]. It follows from Lemma 2.1 that S is just the Silov boundary of the domain D with respect to the ring $\mathcal{E}$ of holomorphic functions on $\bar{D}$.

2.3. We denote by $AF(D)$ the closed subgroup of the complex affine transformation group $AF(\tilde{W})$ of $\tilde{W}$ which consists of all affine transformations leaving D invariant, and denote by $GL(D)$ the closed subgroup of the general linear group $GL(\tilde{W})$ of $\tilde{W}$ which consists of all linear transformations a satisfying the following conditions:

1) $aW^p=W^p$ $(p=-2, -1)$;
2) $aF(y, y')=F(ay, ay')$;
3) $aV=V$.

It is clear that $GL(D)$ is a closed subgroup of $AF(D)$.

We put $W=W^{-2}+W^{-1}$. For any $w\in W$, we define an affine transformation $S(w)$ of $\widetilde{W}$ by

$$S(w)p=z(p)+w^{-2}+2\sqrt{-1}F(u(p),w^{-1})$$
$$+\sqrt{-1}F(w^{-1},w^{-1})+u(p)+w^{-1}(p\in\widetilde{W}),\quad \text{where}\quad w^{p}\in W^{p}$$

and $w=w^{-2}+w^{-1}$. Then it is easy to see that $S(w)$ is in $AF(D)$ and that the totality of $S(w)$ forms a closed subgroup $M(D)$ of $AF(D)$. Moreover we can see that the group $M(D)$ leaves the Silov boundary S of D invariant and acts simply transitively on it (cf. [**3**]).

LEMMA 2.2 ([**6**]).

$$AF(D)=M(D)\cdot GL(D)\ (\textit{semi-direct}),$$
$$GL(D)=GL(\widetilde{W})\cap AF(D).$$

The group $AF(D)$ (resp. $GL(D)$) is called the affine automorphism group (resp. the linear automorphism group) of the domain D and the group $M(D)$ is called the group of parallel translations of D. We say that the domain D is affine homogeneous if the group $AF(D)$ acts transitively on D. It is easy to see that D is affine homogeneous if and only if the group $GL(D)$ acts transitively on the cone V.

2.4. We denote by $H(D)$ the automorphism group of D, i.e., the group of all holomorphic transformations of D. It is known that D is holomorphically equivalent to a bounded domain of $\widetilde{W}$ ([**6**]). Therefore there exists a volume element Ω on D, due to Bergmann, that is invariant by $H(D)$. The volume element Ω may be expressed, with respect to any local coordinate system $w_1,\cdots,w_N$ of D, as:

$$(\sqrt{-1})^{N^2}Kdw_1\wedge\cdots\wedge dw_N\wedge d\bar{w}_1\wedge\cdots\wedge d\bar{w}_N,$$

where K is a positive function of $w_1,\cdots,w_N$. Then the hermitian differential form

$$\sum_{i,j=1}^{N}\frac{\partial^2\log K}{\partial w_i\partial\bar{w}_j}dw_i d\bar{w}_j$$

is positive definite and defines a global kählerian metric g on D, the Bergmann metric, which is invariant by $H(D)$. In particular it follows that $H(D)$ is a Lie group. We note that the Lie group $AF(D)$ is a closed subgroup of $H(D)$.

LEMMA 2.3. *Assume that D is affine homogeneous. Let X be a holomorphic vector field on D. Then X is an infinitesimal automorphism of D, i.e., generates a global one parameter group of automorphisms of D, if and only if X leaves the volume element Ω invariant, i.e., $L_X\Omega=0$.*

PROOF. If X is an infinitesimal automorphism of D, then we clearly have

$L_X\Omega=0$. Conversely, suppose that $L_X\Omega=0$. Then we can easily verify $L_Xg=0$. Since D is a Riemannian homogeneous space with respect to the Riemannian metric g, we see that g is a complete Riemannian metric. Therefore by Kobayashi [4], X is a complete vector field, that is, generates a global one parameter group φ_t of transformations. Since each φ_t is holomorphic, it follows that X is an infinitesimal automorphism of D.

Let $e_1^{-2}, \cdots, e_n^{-2}$ (resp. $e_1^{-1}, \cdots, e_m^{-1}$) be a base of W_c^{-2} (resp. of W^{-1}) and let $z_1, \cdots, z_n, u_1, \cdots, u_m$ be the coordinate system of $\tilde{W}$ corresponding to the base $e_1^{-2}, \cdots, e_n^{-2}, e_1^{-1}, \cdots, e_m^{-2}$ of $\tilde{W}$. Then the projection z (resp. u) of $\tilde{W}$ onto W_c^{-2} (resp. onto W^{-1}), considered as a W_c^{-2}-(resp. a W^{-1}-) valued function on $\tilde{W}$, may be expressed as $\sum_i z_ie_i^{-2}$ (resp. as $\sum_j u_je_j^{-1}$).

LEMMA 2.4 ([6]). *There is a unique positive function λ on the cone V such that*

$$\Omega=(\sqrt{-1})^{N^2}\lambda\circ\Phi\, dz\wedge du\wedge\overline{dz\wedge du}\,,$$

where

$$dz\wedge du=dz_1\wedge\cdots\wedge dz_n\wedge du_1\wedge\cdots\wedge du_m\,.$$

Moreover the function λ satisfies

$$\lambda(ax)\,|\det a|^2=\lambda(x)\qquad(a\in GL(D),\ x\in V)\,,$$

where $\det a$ *denotes the determinant of a as an endomorphism of $\tilde{W}$.*

PROOF. There is a positive function K on D such that

$$\Omega=(\sqrt{-1})^{N^2}Kdz\wedge du\wedge\overline{dz\wedge du}\,.$$

Let $w\in W$ and put $\varphi=S(w)$. Then we have

$$z\circ\varphi=z+w^{-2}+2\sqrt{-1}F(u,w^{-1})+\sqrt{-1}F(w^{-1},w^{-1})$$

and $u\circ\varphi=u+w^{-1}$. It follows that $\varphi^*(dz\wedge du)=dz\wedge du$ and hence $K\circ\varphi=K$. We have $p=S(w)\sqrt{-1}\Phi(p)$ $(p\in D)$, where $w=\operatorname{Re}z(p)+u(p)$. Taking account of the fact that $\sqrt{-1}V\subset D$, we define a positive function λ on V by $\lambda(x)=K(\sqrt{-1}x)$ $(x\in V)$. Then we find from the above argument that $K=\lambda\circ\Phi$. Let $a\in GL(D)$. Then we have $a^*(dz\wedge du)=\det a\, dz\wedge du$ and $a^*\Omega=\Omega$, from which follows that $\lambda\circ\Phi\circ a\cdot|\det a|^2=\lambda\circ\Phi$. Since $\Phi(\sqrt{-1}x)=x$ and $\Phi(a\sqrt{-1}x)=ax$ $(x\in V)$, we get $\lambda(ax)|\det a|^2=\lambda(x)$. Uniqueness of λ is clear.

2.5. We denote by G_a the connected component of the identity of $AF(D)$ and by $\mathfrak{g}_a$ the Lie algebra of G_a that is a Lie algebra of infinitesimal affine transformations of $\tilde{W}$. For any $w\in W$, define an infinitesimal affine transformation $s(w)$ of $\tilde{W}$ by

$$s(w)p=w^{-2}+2\sqrt{-1}F(u(p),w^{-1})+w^{-1}\qquad(p\in\tilde{W})\,.$$

Then we see that $s(w)$ is in $\mathfrak{g}_a$, $S(w)=\exp s(w)$ and that the following equalities hold:

(2.1) 1) $[s(w), s(w')]=0 \qquad (w\in W^{-2},\ w'\in W)$;

2) $[s(w), s(w')]=4\, s(\mathrm{Im}\, F(w, w')) \qquad (w,\ w'\in W^{-1})$;

3) $Ad\, a\, s(w)=s(aw) \qquad (a\in GL(D),\ w\in W)$.

We denote by G^0 the connected component of the identity of $GL(D)$, by $\mathfrak{g}^0$ the Lie algebra of G^0 and by $\mathfrak{g}^p (p=-2,-1)$ the subspace of $\mathfrak{g}_a$ consisting of all $s(w)$ $(w\in W^p)$. Then we have

$$\mathfrak{g}_a=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^0 \quad \text{(direct sum)},$$

which is a graded algebra by (2.1). By Lemma 2.2, we have

$$G_a=\exp(\mathfrak{g}^{-2}+\mathfrak{g}^{-1})\cdot G^0 \quad \text{(semi-direct)}.$$

The group G^0 contains the one parameter group $E_t^\#$ (resp. $I_t^\#$) defined by

$$E_t^\#(p)=e^{-2t}z(p)+e^{-t}u(p) \ \text{(resp. } I_t^\#(p)=z(p)+e^{\sqrt{-1}t}u(p)) \qquad (t\in \boldsymbol{R},\ p\in \widetilde{W}).$$

Let E (resp. I) denote the element of $\mathfrak{g}^0$ induced by $E_t^\#$ (resp. $I_t^\#$):

$$E(p)=-2z(p)-u(p) \ \text{(resp. } I(p)=\sqrt{-1}u(p)).$$

Then we have:

(2.2) 1) $[E, s(w)]=ps(w) \qquad (w\in W^p)$;

2) $[I, s(w)]=0 \qquad (w\in W^{-2})$,

$[I, s(w)]=s(\sqrt{-1}w) \qquad (w\in W^{-1})$.

Finally we note that the condition "$X\in\mathfrak{g}^{-1}$, $[[I, X], X]=0$" implies $X=0$, from which follows that the graded algebra $\mathfrak{m}=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}$ is non-degenerate, i.e., the condition "$X\in\mathfrak{g}^{-1}$, $[X, \mathfrak{g}^{-1}]=0$" implies $X=0$.

2.6. From now on we assume that the domain D is affine homogeneous and fix a point v of the cone V. Let K_a denote the isotropy group of G_a at $\sqrt{-1}v\in D$. Then we have $D=G_a/K_a$. It is easy to see that the isotropy group of G^0 at v coincides with K_a. Since G^0 acts transitively on V, we have $V=G^0/K_a$. Let $\mathfrak{k}_a$ denote the Lie algebra of K_a and let us consider a fixed complementary subspace $\mathfrak{t}^0$ of $\mathfrak{k}_a$ in $\mathfrak{g}^0$: $\mathfrak{g}^0=\mathfrak{t}^0+\mathfrak{k}_a$ (direct sum). We put

$$\mathfrak{t}=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{t}^0.$$

Then we have

$$\mathfrak{g}_a=\mathfrak{t}+\mathfrak{k}_a \quad \text{(direct sum)}.$$

We denote by G the connected component of the identity of $H(D)$ and by K the isotropy group of G at $\sqrt{-1}v$. Then we have $D=G/K$ and

$$K_a=G_a\cap K=G^0\cap K,$$

$$G=G_a\cdot K.$$

Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, respectively. Then we have

$$\mathfrak{k}_a=\mathfrak{g}_a\cap\mathfrak{k}=\mathfrak{g}^0\cap\mathfrak{k},$$

$$\mathfrak{g}=\mathfrak{t}+\mathfrak{k}\quad\text{(direct sum)}.$$

The tangent space $T_{\sqrt{-1}v}(D)$ to D at $\sqrt{-1}v$ is endowed with the structure of complex vector space, and it may be identified, as usual, with the factor space $\mathfrak{g}/\mathfrak{k}$. Hence we can find an endomorphism j of $\mathfrak{g}$ such that $j\mathfrak{k}\subset\mathfrak{k}$ and such that j induces the complex structure on $T_{\sqrt{-1}v}(D)=\mathfrak{g}/\mathfrak{k}$. Then we clearly have $j^2X\equiv -X \pmod{\mathfrak{k}}$ $(X\in\mathfrak{g})$. G acting holomorphically on D, we have:

(2.3) 1) $j[X, Y]\equiv[X, jY] \pmod{\mathfrak{k}}$ $\quad(X\in\mathfrak{k},\ Y\in\mathfrak{g})$;

2) $[jX, jY]-[X, Y]\equiv j([jX, Y]+[X, jY]) \pmod{\mathfrak{k}}$ $\quad(X, Y\in\mathfrak{g})$.

Since $\mathfrak{g}=\mathfrak{t}+\mathfrak{k}$ (direct sum), we may further assume that $j\mathfrak{t}=\mathfrak{t}$.

LEMMA 2.5 (cf. Vinberg, Gindikin and Pyatetski-Shapiro [**10**]).

(1) *If* $X\in\mathfrak{g}^{-2}$, *then* $jX\in\mathfrak{t}^0$ *and* $[jX, s(v)]=X$.

(2) *If* $X\in\mathfrak{g}^{-1}$, *then* $jX\in\mathfrak{g}^{-1}$ *and* $jX=[I, X]$.

(3) *If* $X\in\mathfrak{t}^0$, *then* $jX\in\mathfrak{g}^{-2}$ *and* $jX=-[X, s(v)]$.

PROOF. We first remark that the condition "$A\in\mathfrak{t}^0$, $Av=0$" implies $A=0$. Take any $X\in\mathfrak{t}$. Then $jX\in\mathfrak{t}$. Since $\mathfrak{t}=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{t}^0$ (direct sum), X and jX may be expressed, respectively, as:

$$s(w^{-2}+w^{-1})+A \qquad (w^p\in W^p,\ A\in\mathfrak{t}^0)$$

and

$$s(w'^{-2}+w'^{-1})+A' \qquad (w'^p\in W^p,\ A'\in\mathfrak{t}^0).$$

Since j is compatible with the complex structure on $T_{\sqrt{-1}v}(D)$, we see easily that $(jX)(\sqrt{-1}v)=\sqrt{-1}X(\sqrt{-1}v)$, whence $w'^{-2}=-Av$, $w'^{-1}=\sqrt{-1}w^{-1}$ and $A'v=w^{-2}$. If $X\in\mathfrak{g}^{-2}$, then it follows that $w'^{-2}=w'^{-1}=0$. Hence $jX=A'\in\mathfrak{t}^0$ and $[jX, s(v)]=s(A'v)=s(w^{-2})=X$. If $X\in\mathfrak{g}^{-1}$, then it follows that $w'^{-2}=A'=0$. Hence $jX=s(w'^{-1})=[I, s(w^{-1})]=[I, X]\in\mathfrak{g}^{-1}$. If $X\in\mathfrak{t}^0$, then it follows that $w'^{-1}=A'=0$. Hence $jX=s(w'^{-2})=-[A, s(v)]=-[X, s(v)]\in\mathfrak{g}^{-2}$.

Finally we state the following

LEMMA 2.6 ([**5**] or [**10**]). *The Lie algebra* $\mathfrak{g}$ *is centreless.*

§ 3. Infinitesimal automorphisms of a Siegel domain, I.

3.1. Let D be the Siegel domain of the second kind associated with a convex cone V (in W^{-2}) and a V-hermitian form F (on W^{-1}). We assume that the domain D is affine homogeneous and preserve the notations in § 2. (We fix, once for all, a complementary subspace $\mathfrak{t}^0$ of $\mathfrak{k}_a$ in $\mathfrak{g}^0$ and an endomorphism j of $\mathfrak{g}$ having the properties in 2.6.)

Let us consider the complexification $\mathfrak{g}_c$ of the Lie algebra $\mathfrak{g}$, and define operators P and $\bar{P}$ on $\mathfrak{g}_c$ by

$$P=\frac{1}{2}(1-\sqrt{-1}j), \qquad \bar{P}=\frac{1}{2}(1+\sqrt{-1}j),$$

where j should be confounded with its complexification. We have $j\mathfrak{t}=\mathfrak{t}$ and $j^2X=-X$ $(X\in\mathfrak{t})$, whence $P(jX)=\sqrt{-1}P(X)$, $\bar{P}(jX)=-\sqrt{-1}\bar{P}(X)$ $(X\in\mathfrak{t})$. Therefore we have $P(\mathfrak{t})=P(\mathfrak{t}_c)$, $\bar{P}(\mathfrak{t})=\bar{P}(\mathfrak{t}_c)$ and

$$\mathfrak{t}_c=P(\mathfrak{t})+\bar{P}(\mathfrak{t}) \quad \text{(direct sum)}. \tag{3.1}$$

By Lemma 2.5, we have $j\mathfrak{g}^{-2}=\mathfrak{t}^0$ and $j\mathfrak{g}^{-1}=\mathfrak{g}^{-1}$, whence $P(\mathfrak{g}^{-2}+\mathfrak{t}^0)=P(\mathfrak{g}_c^{-2})$, $P(\mathfrak{g}^{-1})=P(\mathfrak{g}_c^{-1})$, $\bar{P}(\mathfrak{g}^{-2}+\mathfrak{t}^0)=\bar{P}(\mathfrak{g}_c^{-2})$ and $\bar{P}(\mathfrak{g}^{-1})=\bar{P}(\mathfrak{g}_c^{-1})$. Since $\mathfrak{t}=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{t}^0$ (direct sum), it follows that

$$\begin{aligned} &1)\quad P(\mathfrak{t})=P(\mathfrak{g}_c^{-2})+P(\mathfrak{g}^{-1}) \quad \text{(direct sum)},\\ &2)\quad \bar{P}(\mathfrak{t})=\bar{P}(\mathfrak{g}_c^{-2})+\bar{P}(\mathfrak{g}^{-1}) \quad \text{(direct sum)}. \end{aligned} \tag{3.2}$$

By (2.3), 1), we have

$$\begin{aligned} &1)\quad [\mathfrak{k}_c, P(\mathfrak{t})]\subset P(\mathfrak{t})+\mathfrak{k}_c,\\ &2)\quad [\mathfrak{k}_c, \bar{P}(\mathfrak{t})]\subset \bar{P}(\mathfrak{t})+\mathfrak{k}_c, \end{aligned} \tag{3.3}$$

and by (2.3), 2),

$$\begin{aligned} &1)\quad [P(\mathfrak{t}), P(\mathfrak{t})]\subset P(\mathfrak{t})+\mathfrak{k}_c,\\ &2)\quad [\bar{P}(\mathfrak{t}), \bar{P}(\mathfrak{t})]\subset \bar{P}(\mathfrak{t})+\mathfrak{k}_c. \end{aligned} \tag{3.4}$$

By using the fixed element v in V, we define an automorphism Q of the Lie algebra $\mathfrak{g}_c$ by

$$Q=\exp ad(-\sqrt{-1}s(v))=\sum_k\frac{1}{k!}(ad(-\sqrt{-1}s(v)))^k.$$

By (3.3), 2) and (3.4), 2), we see that $\bar{P}(\mathfrak{t})+\mathfrak{k}_c$ is a subalgebra of $\mathfrak{g}_c$. It follows that[1] $\mathfrak{b}=Q\bar{P}(\mathfrak{t})+Q(\mathfrak{k}_c)$ is also a subalgebra of $\mathfrak{g}_c$.

LEMMA 3.1.

(1) $\mathfrak{b}=\bar{P}(\mathfrak{g}^{-1})+\mathfrak{t}_c^0+Q(\mathfrak{k}_c)$ (*direct sum*).

(2) $\mathfrak{g}_c=\mathfrak{g}_c^{-2}+P(\mathfrak{g}^{-1})+\mathfrak{b}$ (*direct sum*).

(3) $\mathfrak{g}^0\subset\mathfrak{b}$.

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1) The subalgebra $\bar{P}(\mathfrak{t})+\mathfrak{k}_c$ of $\mathfrak{g}_c$ does not depend on the choice of j and $\mathfrak{t}^0$, but depends on the choice of v. The subalgebra $\mathfrak{b}$ of $\mathfrak{g}_c$ does not depend on the choice of v (cf. §4, Remark 2).

PROOF. (1) By (3.2), 2), $\mathfrak{b}=Q\bar{P}(\mathfrak{g}_c^{-2})+Q\bar{P}(\mathfrak{g}^{-1})+Q(\mathfrak{k}_c)$ (direct sum). Since $s(v)\in\mathfrak{g}^{-2}$, $[\mathfrak{g}^0,\mathfrak{g}^{-2}]\subset\mathfrak{g}^{-2}$ and $[\mathfrak{g}^{-2},\mathfrak{g}^{-2}]=\{0\}$, it follows from Lemma 2.5, (1) that $QjX=jX-\sqrt{-1}[s(v),jX]=jX+\sqrt{-1}X$ $(X\in\mathfrak{g}^{-2})$. We clearly have $QX=X$. Therefore $Q\bar{P}(X)=\frac{\sqrt{-1}}{2}jX$, whence $Q\bar{P}(\mathfrak{g}_c^{-2})=\mathfrak{t}_c^0$. Since $[\mathfrak{g}^{-2},\mathfrak{g}^{-1}]=\{0\}$, we have $Q\bar{P}(Y)=\bar{P}(Y)$ $(Y\in\mathfrak{g}^{-1})$ and hence $Q\bar{P}(\mathfrak{g}^{-1})=\bar{P}(\mathfrak{g}^{-1})$.

(2) Since $\mathfrak{g}=\mathfrak{t}+\mathfrak{k}$ (direct sum), it follows from (3.1) that $\mathfrak{g}_c=P(\mathfrak{t})+\bar{P}(\mathfrak{t})+\mathfrak{k}_c$ (direct sum). Hence $\mathfrak{g}_c=QP(\mathfrak{t})+\mathfrak{b}$ (direct sum). Furthermore by (3.2), 1), $\mathfrak{g}_c=QP(\mathfrak{g}_c^{-2})+QP(\mathfrak{g}^{-1})+\mathfrak{b}$ (direct sum). We have $QP(\mathfrak{g}^{-1})=P(\mathfrak{g}^{-1})$ as above, and by using the equality $QjX=jX+\sqrt{-1}X$ $(X\in\mathfrak{g}^{-2})$, we find $QP(X)\equiv X$ (mod $\mathfrak{t}_c^0$). Thus we get (2).

(3) The Lie algebra $\mathfrak{k}_a$ consists of all $X\in\mathfrak{g}^0$ such that $[X,s(v)]=0$, whence $\mathfrak{k}_a\subset Q(\mathfrak{k}_c)$. Therefore $\mathfrak{g}^0=\mathfrak{t}^0+\mathfrak{k}_a\subset\mathfrak{b}$.

3.2. We denote by G_c the adjoint group of the complex Lie algebra $\mathfrak{g}_c$. Since $\mathfrak{g}_c$ is centreless (Lemma 2.6), G_c may be characterized as a connected (complex) Lie group whose Lie algebra is given by $\mathfrak{g}_c$ and whose adjoint representation on $\mathfrak{g}_c$ is faithful.

Let B denote the closed subgroup of G_c consisting of all $a\in G_c$ such that $Ad\,a\,\mathfrak{b}=\mathfrak{b}$. Since $\mathfrak{b}$ is a complex subalgebra of $\mathfrak{g}_c$, B is a complex Lie subgroup of G_c. We now assert that the Lie algebra of B coincides with $\mathfrak{b}$. For this purpose, it is sufficient to show that the condition "$X\in\mathfrak{g}_c$, $[X,\mathfrak{b}]\subset\mathfrak{b}$" implies $X\in\mathfrak{b}$. Indeed, let $X\in\mathfrak{g}_c$ be such that $[X,\mathfrak{b}]\subset\mathfrak{b}$. Then by Lemma 3.1, (2), we can find a $Y\in\mathfrak{g}_c^{-2}$ and a $Z\in P(\mathfrak{g}^{-1})$ such that $X\equiv Y+Z$ (mod $\mathfrak{b}$). Since $E\in\mathfrak{g}^0\subset\mathfrak{b}$ (Lemma 3.1, (3)), we have $[Y,E]+[Z,E]\equiv 0$ (mod $\mathfrak{b}$), and by (2.2), 1), we have $[Y,E]=2Y$ and $[Z,E]=Z$. It follows that $Y=Z=0$, i.e., $X\in\mathfrak{b}$.

Let us now consider the homogeneous space $\tilde{M}=G_c/B$ of the complex Lie group G_c by the closed complex Lie subgroup B, and denote by π the projection of G_c onto $\tilde{M}$. We put $\mathfrak{n}=\mathfrak{g}_c^{-2}+P(\mathfrak{g}^{-1})$ and define a holomorphic mapping h' of $\mathfrak{n}$ to $\tilde{M}$ by $h'(x)=\pi\circ\exp(x)$ $(x\in\mathfrak{n})$.

LEMMA 3.2. *The mapping h' is a holomorphic imbedding of $\mathfrak{n}$ onto an open set of $\tilde{M}$.*

PROOF. We first prove that $\mathfrak{n}$ is a (complex) abelian subalgebra of $\mathfrak{g}_c$. Indeed, we have $jX=[I,X]$ $(X\in\mathfrak{g}^{-1})$ by Lemma 2.5, (2), and $[[I,X],[I,Y]]=[X,Y]$ $(X,Y\in\mathfrak{g}^{-1})$ by (2.1), 2) and (2.2), 2), whence $[P(\mathfrak{g}^{-1}),P(\mathfrak{g}^{-1})]=\{0\}$. We clearly have $[\mathfrak{g}_c^{-2},\mathfrak{n}]=\{0\}$. Hence $[\mathfrak{n},\mathfrak{n}]=\{0\}$, proving our assertion. Let N be the connected (complex) Lie subgroup of G_c generated by $\mathfrak{n}$, i.e., $N=\exp\mathfrak{n}$. Then it is clear from Lemma 3.1, (2) that $h'(\mathfrak{n})=\pi(N)$ is an open set of $\tilde{M}$. Let us now show that the mapping h' is injective. Indeed, let $X\in\mathfrak{n}$ be such that $\exp X\in B$. If we express X as $Y+Z$ $(Y\in\mathfrak{g}_c^{-2}, Z\in P(\mathfrak{g}^{-1}))$, then we have $Ad(\exp X)E=E+[X,E]=E+2Y+Z\in\mathfrak{b}$. It follows that $Y=Z=0$, i.e., $X=0$, which proves our assertion. Finally it is easy to see that h'

is a holomorphic homeomorphism of $\mathfrak{n}$ onto $h'(\mathfrak{n})$.

3.3. G_c being the adjoint group of $\mathfrak{g}_c$, there is a unique homomorphism ρ of the Lie group G into the Lie group G_c such that $Ad\, a\, X = Ad\, \rho(a)X$ ($a \in G$, $X \in \mathfrak{g}$), where Ad in the left (resp. right) hand side means the adjoint representation of G (resp. of G_c) on $\mathfrak{g}$ (resp. on $\mathfrak{g}_c$). We clearly have $\rho(\exp X) = \exp X$ ($X \in \mathfrak{g}$), where exp in the left (resp. right) hand side means the exponential mapping of $\mathfrak{g}$ (resp. of $\mathfrak{g}_c$) to G (resp. G_c). Let us now define a mapping h'' of $\tilde{W}$ to $\mathfrak{n}$ by

$$h''(p) = s(z(p)) + P(s(u(p))) \qquad (p \in \tilde{W}),$$

where

$$s(z) = s(\mathrm{Re}\, z) + \sqrt{-1}\, s(\mathrm{Im}\, z) \qquad (z \in W_c^{-2}).$$

Since

$$P(s(\sqrt{-1}u)) = P([I, s(u)]) = \sqrt{-1}P(s(u)) \qquad (u \in W_c^{-1}),$$

we see that h'' is a (complex) linear isomorphism of $\tilde{W}$ onto $\mathfrak{n}$. By Lemma 3.2, the mapping $h = h' \circ h''$ is clearly a holomorphic imbedding of $\tilde{M}$ onto an open set of $\tilde{M}$.

LEMMA 3.3.

(1) $\rho(k)h(\sqrt{-1}v) = h(\sqrt{-1}v)$ ($k \in K$).

(2) $\rho(t)h(\sqrt{-1}v) = h(t(\sqrt{-1}v))$ ($t \in G_a$).

PROOF. (1) Since K is connected, (1) is clear from the fact that

$$Ad(\exp(-\sqrt{-1}s(v)))\mathfrak{k} = Q(\mathfrak{k}) \subset \mathfrak{b}.$$

(2) Every $t \in G_a$ may be expressed as $\exp(X^{-2} + X^{-1})t^0$ ($X^p \in \mathfrak{g}^p$, $t^0 \in G^0$). Since $\mathfrak{g}_a = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^0$ is a graded algebra,

$$\rho(t)\exp(\sqrt{-1}s(v)) = \exp(X^{-2} + \sqrt{-1}Adt^0 s(v) + X^{-1})\rho(t^0).$$

We have

$$X^{-1} = P(X^{-1}) + \bar{P}(X^{-1}) \quad \text{and} \quad \exp X^{-1} = \exp\left(\frac{1}{2}[\bar{P}(X^{-1}), P(X^{-1})]\right) \exp P(X^{-1}) \exp \bar{P}(X^{-1}).^{2)}$$

2) Let A' be the simply connected complex Lie group whose Lie algebra is equal to $\mathfrak{m}_c$ (=the complexification of $\mathfrak{m}$). Since $\mathfrak{m}_c = \mathfrak{g}_c^{-2} + \mathfrak{g}_c^{-1}$ is a graded Lie algebra, we can prove the following equality:

$$\exp'(Y+Z) = \exp' \frac{1}{2}[Z, Y] \cdot \exp' Y \cdot \exp' Z \qquad \text{for all} \quad Y, Z \in \mathfrak{g}_c^{-1},$$

where $\exp'$ denotes the exponential mapping of $\mathfrak{m}_c$ to A'. Let A be the connected complex Lie subgroup of G_c generated by $\mathfrak{m}_c$. A' being simply connected, there is a unique holomorphic homomorphism θ of A' onto A. We have $\theta(\exp' X) = \exp X$ for all $X \in \mathfrak{m}_c$. Hence we get

$$\exp(Y+Z) = \exp \frac{1}{2}[Z, Y] \cdot \exp Y \cdot \exp Z \qquad \text{for all} \quad Y, Z \in \mathfrak{g}_c^{-1}.$$

Since $\exp \bar{P}(X^{-1})\rho(t^0) \in B$ (Lemma 3.1, (1) and (3)), it follows that

$$(3.5) \qquad \rho(t)h(\sqrt{-1}v) = \pi \circ \exp\left(X^{-2}+\sqrt{-1}\,Ad t^0 s(v)+\frac{1}{2}[\bar{P}(X^{-1}), P(X^{-1})]+P(X^{-1})\right).$$

Now X^p may be expressed as $s(w^p)$ ($w^p \in W^p$). Since $\exp s(w^{-2}+w^{-1}) = S(w^{-2}+w^{-1})$, then we have

$$t(\sqrt{-1}v) = \sqrt{-1}t^0 v + w^{-2} + \sqrt{-1}F(w^{-1}, w^{-1}) + w^{-1}.$$

Hence

$$(3.6) \qquad h(t(\sqrt{-1}v)) = \pi \circ \exp(\sqrt{-1}s(t^0 v)+s(w^{-2})+\sqrt{-1}s(F(w^{-1}, w^{-1}))+P(s(w^{-1}))).$$

We have $Ad\, t^0 s(v) = s(t^0 v)$ and by (2.1), 2)

$$\frac{1}{2}[\bar{P}(X^{-1}), P(X^{-1})] = \frac{\sqrt{-1}}{4}[jX^{-1}, X^{-1}] = \frac{1}{4}[[I, s(w^{-1})], s(w^{-1})]$$

$$= \frac{\sqrt{-1}}{4}[s(\sqrt{-1}w^{-1}), s(w^{-1})] = \sqrt{-1}s(F(w^{-1}, w^{-1})).$$

Thus we get $\rho(t)h(\sqrt{-1}v) = h(t(\sqrt{-1}v))$ by (3.5) and (3.6).

LEMMA 3.4. *The holomorphic imbedding $h: \tilde{W} \to \tilde{M}$ is compatible with the respective actions of G on D and $\tilde{M}$. More precisely $h(ap) = \rho(a)h(p)$ ($a \in G$, $p \in D$).*

PROOF. Let $p \in D$. Since G_a acts transitively on D, there is a $t \in G_a$ such that $t(\sqrt{-1}v) = p$. Hence by Lemma 3.3, (2), $h(p) = \rho(t)h(\sqrt{-1}v)$. Let $a \in G$. Then there is a $t' \in G_a$ such that $t'^{-1}at \in K$. Hence $ap = t'(\sqrt{-1}v)$. It follows from Lemma 3.3 that

$$\rho(a)h(p) = \rho(a)\rho(t)h(\sqrt{-1}v) = \rho(t')h(\sqrt{-1}v) = h(t'(\sqrt{-1}v)) = h(ap).$$

As an immediate corollary of Lemma 3.4, we have

COROLLARY (Kaneyuki [2]). *The homomorphism $\rho: G \to G_c$ is injective or equivalently the group G is centreless.*

In what follows, we shall identify G with a Lie subgroup of G_c by the injective homomorphism ρ.

3.4. LEMMA 3.5. *Let $a \in G$ and $p \in S$, where S is the Silov boundary of the domain D. If $ah(p) \in h(\tilde{W})$, then we have $ah(p) \in h(S)$.*

PROOF. Let $\overline{h(D)}$ be the closure of $h(D)$ in $\tilde{M}$. Then we have $\overline{h(D)} \cap h(\tilde{W}) = h(\bar{D})$, $\bar{D}$ being the closure of D in $\tilde{W}$. Since G leaves $h(D)$ invariant (Lemma 3.4), it also leaves $\overline{h(D)}$ invariant. Let $a \in G$ and $p \in S \subset \bar{D}$ and suppose that $ah(p) \in h(\tilde{W})$. Then we find $ah(p) \in h(\bar{D})$. Hence there is a $q \in \bar{D}$ such that $ah(p) = h(q)$. Since $\bar{D} = \Phi^{-1}(\bar{V})$, we have

$$\Phi(q)=\operatorname{Im} z(q)-F(u(q), u(q)) \in \bar{V}.$$

It follows that

$$\alpha(t)=\operatorname{Re} z(q)+\sqrt{-1}F(u(q), u(q))+u(q)+t\Phi(q) \in \bar{D} \quad \text{for all} \quad t \in \boldsymbol{C}$$

with $\operatorname{Im} t \geqq 0$. Since $a^{-1}h(\alpha(\sqrt{-1}))=a^{-1}h(q)=h(p) \in h(\tilde{W})$, we can find a connected neighbourhood U of $\sqrt{-1}$ in the Gaussian plane $\boldsymbol{C}$ such that $\beta(t)=h^{-1}(a^{-1}h(\alpha(t))) \in \bar{D}$ for all $t \in U$. By Lemma 2.1, (2), there is a holomorphic function f on $\bar{D}$ such that the maximum of $|f|$ on $\bar{D}$ is attained at only one point p. The function $f_0(t)=f(\beta(t))$ $(t \in U)$ is holomorphic and satisfies $f_0(\sqrt{-1})=f(p)$. Hence $|f_0|$ takes its maximum on U at $t=\sqrt{-1}$. Therefore by the maximum principle, we find that $f_0(t)=$ constant and hence $\beta(t)=$ constant $(t \in U)$. It follows that $\alpha(t)=$ constant, whence $\Phi(q)=0$, i.e., $q \in S$. Thus we have proved $ah(p)=h(q) \in h(S)$.

By Lemmas 3.4 and 3.5, we easily have

PROPOSITION 3.1. *Every infinitesimal automorphism of D is extended to a (unique) holomorphic vector field which is defined on the whole $\tilde{W}$ and which is tangent to the Silov boundary S of the domain D.*

3.5. In the homogeneous space $\tilde{M}=G_c/B$, let us consider the orbit M of G through the origin $o=h(0)$ of $\tilde{M}: M=G/G \cap B$. M being a real submanifold of the complex manifold $\tilde{M}$, there is defined a pseudo-complex structure (Δ, I) on M in a natural manner ([**9**], § 10), which we shall explain from now on. At each $p \in M$, the tangent space $T_p(\tilde{M})$ to $\tilde{M}$ at p is a complex vector space and the tangent space $T_p(M)$ to M at p is a real subspace of $T_p(\tilde{M})$. Let $\Delta(p)$ denote the maximum complex subspace of $T_p(\tilde{M})$ contained in $T_p(M)$, i.e., $\Delta(p)=T_p(M) \cap \sqrt{-1}T_p(M)$. Since G acts transitively on M, $\dim \Delta(p)=$ constant and hence the assignment $p \rightarrow \Delta(p)$ defines a differential system Δ on M. Let I_p denote the complex structure on $\Delta(p)$. Then the differential system Δ together with the assignment $p \rightarrow I_p$ defines the pseudo-complex structure (Δ, I) on M. It is clear that (Δ, I) is invariant by G.

It is easy to see that S is the orbit of $AF(D)$ through the origin 0 of $\tilde{W}$ and its isotropy group at 0 is given by $GL(D)$. By Lemma 3.4, it follows that $h(S)$ is the orbit of G_a through the origin $o=h(0)$ of $\tilde{M}$ and its isotropy group at o is given by $G^0: h(S)=G_a/G^0$. Therefore by Lemma 3.5, we see that $h(S)$ is an open submanifold of M and hence $\mathfrak{g}=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+L^0$ (direct sum), where $L^0=\mathfrak{g} \cap \mathfrak{b}$. If we identify $\mathfrak{g}_c$ with $T_e(G_c)$, then we have $T_o(\tilde{M})=\pi_*(\mathfrak{g}_c^{-2}+P(\mathfrak{g}^{-1}))$ (Lemma 3.1, (2)). We have $T_o(M)=\pi_*(\mathfrak{g}^{-2}+\mathfrak{g}^{-1})$ and $Y-P(Y)=\bar{P}(Y) \in \mathfrak{b}$ $(Y \in \mathfrak{g}^{-1})$ (Lemma 3.1, (1)). Therefore it follows that $T_o(M)=\pi_*(\mathfrak{g}^{-2}+P(\mathfrak{g}^{-1}))$, $\Delta(o)=\pi_* P(\mathfrak{g}^{-1})=\pi_*\mathfrak{g}^{-1}$ and

$$\dim M=\dim_c \tilde{M}+\dim_c \Delta(o). \tag{3.7}$$

Since $P([I, Y])=\sqrt{-1}P(Y)$ $(Y\in\mathfrak{g}^{-1})$ and $I\in\mathfrak{g}^0\subset L^0$, we have $\pi_*[I, Y]=\sqrt{-1}\pi_*Y$ $(Y\in L^{-1}=\mathfrak{g}^{-1}+L^0)$.

LEMMA 3.6.

(1) *G acts effectively on M.*

(2) *Putting* $L^0=\mathfrak{g}\cap\mathfrak{b}$, *we have* $\mathfrak{g}=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+L^0$ *(direct sum).*

(3) *Putting* $L^{-1}=\mathfrak{g}^{-1}+L^0$, *we have* $AdaL^{-1}\subset L^{-1}$ $(a\in G\cap B)$.

(4) $Ada[I, Y]\equiv[I, AdaY]$ (mod L^0) $(a\in G\cap B,\ Y\in L^{-1})$.

PROOF. (2) has been already proved. (3) and (4) follow from the fact that G leaves invariant the pseudo-complex structure (Δ, I). It remains to prove (1). By Lemma 3.4, we know that G acts effectively on $\tilde{M}$. Since (Δ, I) satisfies (3.7), it follows from [9], Proposition 10.3 that G acts effectively on M.

By Lemma 3.6, we see that the Lie algebra $\mathfrak{g}$, the subalgebra L^0 and the family $(\mathfrak{g}^p)_{p<0}$, where $\mathfrak{g}^p=\{0\}$ $(p<-2)$, together satisfy the conditions in Lemma 1.1. Therefore by using the notations in Lemma 1.1, we have the followings: i) $\mathfrak{g}=\sum_p\mathfrak{n}^p$ (direct sum); ii) $\mathfrak{n}^p=\mathfrak{g}^p$ $(p<0)$ and $L^0=\sum_{p=0}^{\infty}\mathfrak{n}^p$; iii) For any $p\geqq 0$, the condition "$X\in\mathfrak{n}^p$, $[X, \mathfrak{g}^{-2}+\mathfrak{g}^{-1}]=\{0\}$" implies $X=0$. Let us now prove $\mathfrak{g}^0=\mathfrak{n}^0$. We clearly have $\mathfrak{g}^0\subset\mathfrak{n}^0$. Take any $X\in\mathfrak{n}^0$ and put $a_t=\exp tX$. Since $[X, \mathfrak{g}^{-1}]\subset\mathfrak{g}^{-1}$ and $[X, [I, Y]]=[I, [X, Y]]$ $(Y\in\mathfrak{g}^{-1})$ (Lemma 3.6, (4)), we have $Ada_t\mathfrak{g}^{-1}=\mathfrak{g}^{-1}$ and $Ada_t[I, Y]=[I, Ada_tY]$. It follows that there is a $\varphi_t\in GL(W^{-1})$ such that $Ada_ts(u)=s(\varphi_tu)$ $(u\in W^{-1})$. We clearly have $Ada_tP(s(u))=P(s(\varphi_tu))$. Since $Ada_t\mathfrak{g}^{-2}=\mathfrak{g}^{-2}$ as above, there is a $\varphi'_t\in GL(W^{-2})$ such that $Ada_ts(z)=s(\varphi'_tz)$ $(z\in W_c^{-2})$. Let $p\in\tilde{W}$. Since $a_t\in G\cap B$, we find $a_th(p)=\pi\circ\exp(s(\varphi'_tz(p))+P(s(\varphi_tu(p))))$. Since $a_th(p)=h(a_tp)$ (Lemma 3.4), it follows that $a_tp=\varphi'_tz(p)+\varphi_tu(p)$, whence $a_t\in GL(\tilde{W})\cap G$. Therefore by Lemma 2.1, we get $a_t\in GL(D)$ and hence $X\in\mathfrak{g}^0$.

We have thereby proved the following

THEOREM 3.1. *Let D be the Siegel domain of the second kind associated with a convex cone V (in* W^{-2}*) and a V-hermitian form F (on* W^{-1}*). Assume that the domain D is affine homogeneous. Let* $\mathfrak{g}$ *be the Lie algebra of the automorphism group H(D) of D and let E be the element of* $\mathfrak{g}$ *induced by the one parameter group* $E_t^\#$ *of linear automorphisms:*

$$E_t^\#(p)=e^{-2t}z(p)+e^{-t}u(p) \qquad (p\in D).$$

For any integer p, let $\mathfrak{g}^p$ *be the subspace of* $\mathfrak{g}$ *consisting of all* $X\in\mathfrak{g}$ *such that* $[E, X]=pX$. *Then we have:*

(1) $\mathfrak{g}=\sum_p\mathfrak{g}^p$ *(direct sum), and it is a graded algebra.*

(2) $\mathfrak{g}^p=\{0\}$ $(p<-2)$, *and* $\mathfrak{g}_a=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+\mathfrak{g}^0$ *is the Lie algebra of the affine automorphism group AF(D) of D; More precisely,* $\mathfrak{g}^0$ *is the Lie algebra of the linear automorphism group GL(D) of D and* $\mathfrak{m}=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}$ *is the Lie algebra of the group M(D) of all parallel translations of D.*

(3) $\mathfrak{g}$ *being identified with a Lie algebra of holomorphic vector fields on* $\widetilde{W}$ (Proposition 3.1), $L^0=\sum_{p=0}^{\infty}\mathfrak{g}^p$ *is characterized as the isotropy algebra of* $\mathfrak{g}$ *at the origin* 0 *of* $\widetilde{W}$.

(4) *For any* $p\geqq 0$, *the condition* "$X\in\mathfrak{g}^p$, $[X,\mathfrak{m}]=\{0\}$" *implies* $X=0$.

§ 4. Infinitesimal automorphisms of a Siegel domain, II.

4.1. Let $\mathfrak{g}=\sum_p\mathfrak{g}^p$ be the graded algebra given in Theorem 3.1. Since the condition "$X\in\mathfrak{g}^0$, $[X,\mathfrak{m}]=\{0\}$" implies $X=0$, $\mathfrak{g}^0$ may be regarded as a subalgebra of the Lie algebra $\mathfrak{g}^0(\mathfrak{m})$ of all derivations of $\mathfrak{m}=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}$ (as graded algebra). This being said, we denote by $\hat{\mathfrak{g}}=\sum_p\hat{\mathfrak{g}}^p$ the prolongation of $(\mathfrak{m},\mathfrak{g}^0)$. By Theorem 3.1, (4), $\mathfrak{g}$ may be identified with a graded subalgebra of $\hat{\mathfrak{g}}$. Now consider the element I in the centre of $\mathfrak{g}^0$ given in 2.5. Then by (2.2), 2), we clearly have:

$$I\mathfrak{g}^{-2}=\{0\},\qquad I^2X=-X\quad(X\in\mathfrak{g}^{-1}),$$
$$[IX,IY]=[X,Y]\quad(X,Y\in\mathfrak{g}^{-1}).\tag{4.1}$$

LEMMA 4.1.

(1) *For any* $p>0$, *the condition* "$X\in\hat{\mathfrak{g}}^p$, $[X,\mathfrak{g}^{-2}]=\{0\}$" *implies* $X=0$.

(2) $[I,\hat{\mathfrak{g}}^{2s}]=\{0\}$.

(3) $[I,[I,X]]=-X$ $(X\in\hat{\mathfrak{g}}^{2s-1})$.

(4) $[[I,X],[I,Y]]=[X,Y]$ $(X\in\hat{\mathfrak{g}}^{2s-1},\ Y\in\hat{\mathfrak{g}}^{2t-1})$.

PROOF. (1) For any $Y,Z\in\mathfrak{g}^{-1}$, we put $\langle Y,Z\rangle=[IY,Z]$. Then we have $\langle Y,Z\rangle=\langle Z,Y\rangle$ and $\langle IY,IZ\rangle=\langle Y,Z\rangle$. Since $\mathfrak{m}$ is non-degenerate (cf. 2.5), the condition "$Y\in\mathfrak{g}^{-1}$, $\langle Y,\mathfrak{g}^{-1}\rangle=\{0\}$" implies $Y=0$. Thus $\langle\ ,\ \rangle$, so to speak, is a $\mathfrak{g}^{-2}$-valued hermitian inner product on $\mathfrak{g}^{-1}$. For any $p\geqq 0$, let $\mathfrak{h}^p$ denote the subspace of $\hat{\mathfrak{g}}^p$ consisting of all $Y\in\hat{\mathfrak{g}}^p$ such that $[Y,\mathfrak{g}^{-2}]=\{0\}$. Then we have $[\mathfrak{h}^p,\mathfrak{g}^{-1}]\subset\mathfrak{h}^{p-1}$, and the condition "$Y\in\mathfrak{h}^p$, $[Y,\mathfrak{g}^{-1}]=\{0\}$" implies $Y=0$. Moreover from the definition of the prolongation of $(\mathfrak{m},\mathfrak{g}^0)$, we see that, $\mathfrak{h}^0$ being identified with a subalgebra of $\mathfrak{gl}(\mathfrak{g}^{-1})$, $\mathfrak{h}^p$ may be identified with the p-th prolongation of $\mathfrak{h}^0$ in the usual sense. Therefore we have only to prove that $\mathfrak{h}^1$ vanishes. But this can be verified by using the equalities $\langle AY,Z\rangle+\langle Y,AZ\rangle=0$, $[X,Y]Z=[X,Z]Y$ $(A\in\mathfrak{h}^0,\ X\in\mathfrak{h}^1)$ and by the method in the proof of the fact that the first prolongation $\mathfrak{o}(n)^{(1)}$ of the orthogonal algebra $\mathfrak{o}(n)$ vanishes.

(2) We have $[I,\mathfrak{g}^{-2}]=[I,\mathfrak{g}^0]=\{0\}$. s being an integer >0, suppose that $[I,\hat{\mathfrak{g}}^{2s-2}]=\{0\}$. Then we have $[[I,\hat{\mathfrak{g}}^{2s}],\mathfrak{g}^{-2}]\subset[I,[\hat{\mathfrak{g}}^{2s},\mathfrak{g}^{-2}]]=\{0\}$, whence $[I,\hat{\mathfrak{g}}^{2s}]=\{0\}$ by (1).

(3) We have $[I,[I,X]]=-X$ $(X\in\mathfrak{g}^{-1})$. s being an integer >0, suppose that $[I,[I,X]]=-X$ $(X\in\hat{\mathfrak{g}}^{2s-3})$. Then we have $[[I,[I,X]]+X,Y]=$

$[I,[I,[X,Y]]]+[X,Y]=0$ $(X\in\hat{\mathfrak{g}}^{2s-1}, Y\in\mathfrak{g}^{-2})$, whence $[I,[I,X]]+X=0$ by (1). Finally (4) is clear from (2) and (3).

Let us consider the complexification $\hat{\mathfrak{g}}_c=\sum_p \hat{\mathfrak{g}}_c^p$ of $\hat{\mathfrak{g}}$. For each integer s, define operators P and $\bar{P}$ on $\hat{\mathfrak{g}}_c^{2s-1}$, respectively, by

$$P(Y)=\frac{1}{2}(Y-\sqrt{-1}[I,Y]),\quad \bar{P}(Y)=\frac{1}{2}(Y+\sqrt{-1}[I,Y])\qquad (Y\in\hat{\mathfrak{g}}_c^{2s-1}).$$

(In the case when $s=0$, these operators coincide, respectively, with the operators P and $\bar{P}$, restricted to $\mathfrak{g}_c^{-1}$, which was defined in 3.1.) It is clear that $P(\hat{\mathfrak{g}}^{2s-1})$ and $\bar{P}(\hat{\mathfrak{g}}^{2s-1})$ are complex subspaces of $\hat{\mathfrak{g}}_c^{2s-1}$ (Lemma 4.1, (3)) and that $\hat{\mathfrak{g}}_c^{2s-1}=P(\hat{\mathfrak{g}}^{2s-1})+\bar{P}(\hat{\mathfrak{g}}^{2s-1})$ (direct sum).

LEMMA 4.2.

(1) $[P(\hat{\mathfrak{g}}^{2s-1}), P(\hat{\mathfrak{g}}^{2t-1})]=[\bar{P}(\hat{\mathfrak{g}}^{2s-1}), \bar{P}(\hat{\mathfrak{g}}^{2t-1})]=\{0\}$.

(2) $[P(\hat{\mathfrak{g}}^{2s-1}), \hat{\mathfrak{g}}_c^{2t}]\subset P(\hat{\mathfrak{g}}^{2(s+t)-1})$,
$[\bar{P}(\hat{\mathfrak{g}}^{2s-1}), \hat{\mathfrak{g}}_c^{2t}]\subset \bar{P}(\hat{\mathfrak{g}}^{2(s+t)-1})$.

(3) $[\bar{P}(\hat{\mathfrak{g}}^{2s-1}), P(\hat{\mathfrak{g}}^{2t-1})]\subset\hat{\mathfrak{g}}_c^{2(s+t-1)}$.

This is easy from Lemma 4.1.

REMARK 1. Put $\hat{\mathfrak{n}}^p=\bar{P}(\hat{\mathfrak{g}}^{2p-1})+\hat{\mathfrak{g}}_c^{2p}+P(\hat{\mathfrak{g}}^{2p+1})$. Then we can easily prove the followings:

(1) $\hat{\mathfrak{g}}_c=\sum_{p=-1}^{\infty}\hat{\mathfrak{n}}^p$ (direct sum) and it is a graded algebra;

(2) For any integer $p\geqq 0$, the condition "$X\in\hat{\mathfrak{n}}^p$, $[X,\hat{\mathfrak{n}}^{-1}]=\{0\}$" implies $X=0$.

(3) Putting $E_*=\frac{1}{2}(E+\sqrt{-1}I)$, we have $[E_*,X]=pX$ $(X\in\hat{\mathfrak{n}}^p)$. (For the proof of this fact, we note that $[E,X]=pX$ $(X\in\hat{\mathfrak{g}}^p)$.)

REMARK 2. We have $[E_*,\mathfrak{g}_c]\subset\mathfrak{g}_c$, showing that $\mathfrak{g}_c$ is a graded subalgebra of $\hat{\mathfrak{g}}_c=\sum_{p=-1}^{\infty}\hat{\mathfrak{n}}^p$: $\mathfrak{g}_c=\sum_{p=-1}^{\infty}\mathfrak{n}^p$, where $\mathfrak{n}^p=\hat{\mathfrak{n}}^p\cap\mathfrak{g}_c$. It is easy to see that $\mathfrak{n}^p=\bar{P}(\mathfrak{g}^{2p-1})+\mathfrak{g}_c^{2p}+P(\mathfrak{g}^{2p+1})$. Let $\mathfrak{b}$ be the subalgebra of $\mathfrak{g}_c$ given in Lemma 3.1, (1). Then we have $\mathfrak{g}_c=\mathfrak{g}_c^{-2}+P(\mathfrak{g}^{-1})+\mathfrak{b}=\mathfrak{n}^{-1}+\mathfrak{b}$ (direct sum). Since $E_*\in\mathfrak{g}_c^0\subset\mathfrak{b}$, we have $[E_*,\mathfrak{b}]\subset\mathfrak{b}$. It follows that $\mathfrak{b}=\sum_{p=0}^{\infty}\mathfrak{n}^p$.

4.2. We define a mapping f of $M(D)$ to $\mathfrak{n}$ $(=\mathfrak{g}_c^{-2}+P(\mathfrak{g}^{-1})=\mathfrak{n}^{-1}=\hat{\mathfrak{n}}^{-1})$ by

$$f(a)=h''(a0)\qquad (a\in M(D)),$$

0 being the origin of $\tilde{W}$. Since the mapping $a\to a0$ is a real analytic homeomorphism of $M(D)$ onto S, f is an imbedding and $f(M(D))=h''(S)$. $M(D)$ is a simply connected Lie group whose Lie algebra is given by $\mathfrak{m}$, i.e., $M(D)=M(\mathfrak{m})$, and the mapping $X\to\exp X$ is a real analytic homeomorphism of $\mathfrak{m}$ onto $M(D)$. This being said, we have

$$f(\exp X)=X^{-2}+\frac{1}{2}[\bar{P}(X^{-1}),P(X^{-1})]+P(X^{-1}).\tag{4.2}$$

Indeed, this is clear from the equalities:

$$(\exp s(w))0 = w^{-2} + \sqrt{-1}F(w^{-1}, w^{-1}) + w^{-1},$$

$$s(F(w^{-1}, w^{-1})) = \frac{1}{2\sqrt{-1}}[\bar{P}(s(w^{-1})), P(s(w^{-1}))] \quad (w \in W).$$

Hereafter we identify $\widetilde{W}$ with $\mathfrak{n}$ by the isomorphism h''. Then we have $\mathfrak{g}^{-2} = W^{-2}$, $P(\mathfrak{g}^{-1}) = W^{-1}$ and

$$F(Y, Y) = \frac{1}{2\sqrt{-1}}[\bar{Y}, Y] \qquad (Y \in P(\mathfrak{g}^{-1})),$$

from which follows that the mapping Φ of $\mathfrak{n}$ to $\mathfrak{g}^{-2}$ may be described as

$$\mathrm{Im}\, z - \frac{1}{2\sqrt{-1}}[\bar{u}, u].$$

Let us now define a pseudo-complex structure (Δ, I) on the Lie group $M(D) = M(\mathfrak{m})$ in the following manner (cf. [**9**], § 10): The subspace $\mathfrak{g}^{-1}$ of $\mathfrak{m}$ yields a left-invariant differential system Δ on $M(\mathfrak{m})$, the standard differential system of type $\mathfrak{m}$, and then the mapping $\mathfrak{g}^{-1} \ni Y \to IY \in \mathfrak{g}^{-1}$ yields a left-invariant cross-section I of the vector bundle $\mathrm{Hom}(\Delta, \Delta)$. By (4.1), it is clear that the pair (Δ, I) defines a pseudo-complex structure on $M(\mathfrak{m})$. Now let $\mathcal{L}$ be the standard Lie algebra sheaf of type $(\mathfrak{m}, \mathfrak{g}^0)$. Then it can be shown[3] that every local cross-section X of $\mathcal{L}$ is necessarily a local infinitesimal automorphism of (Δ, I).

We show that the imbedding f is an imbedding of the pseudo-complex structure (Δ, I) in the complex manifold $\mathfrak{n}$, that is, (Δ, I) coincides with the pseudo-complex structure induced by f. Indeed, we identify $T_e(M(\mathfrak{m}))$ and $T_0(\mathfrak{n})$ with $\mathfrak{m}$ and $\mathfrak{n}$, respectively. Then we have $\Delta(e) = \mathfrak{g}^{-1}$, $I_e =$ the restriction of I to $\mathfrak{g}^{-1}$ and, by (4.2), $f_*X = X^{-2} + P(X^{-1})$ $(X \in \mathfrak{m})$. It follows that $f_*\Delta(e)$ is the maximum complex subspace of $T_0(\mathfrak{n})$ contained in $f_*T_e(M(\mathfrak{m}))$ and $f_*I_eY = \sqrt{-1}f_*Y$ $(Y \in \Delta(e))$. Since (Δ, I) is left-invariant and since $f(ax) = af(x)$ $(a, x \in M(\mathfrak{m}))$, we know that f is an imbedding of (Δ, I) in $\mathfrak{n}$, proving our assertion. Finally we remark that the following equality holds:

$$\dim M(\mathfrak{m}) = \dim_c \mathfrak{n} + \dim_c (\Delta(e), I_e). \tag{4.3}$$

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3) The differential system Δ is defined by the equation $\xi^{-2} = 0$ and we have $\xi^{-1}(IY) = I\xi^{-1}(Y)$ for all tangent vectors Y to $M(\mathfrak{m})$. (For the notation, see 1.3.) Let $\bar{\mathfrak{g}}^0$ be the subalgebra of $\mathfrak{g}^0(\mathfrak{m})$ consisting of all $A \in \mathfrak{g}^0(\mathfrak{m})$ such that $AIY = IAY$ for all $Y \in \mathfrak{g}^{-1}$. In view of the above remark, we can prove the following: A local vector field X defined on an open set U of $M(\mathfrak{m})$ is a local infinitesimal automorphism of (Δ, I) if and only if there is a $\bar{\mathfrak{g}}^0$-valued function f^0 on U such that $L_X\xi^{-2} = [f^0, \xi^{-2}]$ and $L_X\xi^{-1} \equiv [f^0, \xi^{-1}] \pmod{\xi^{-2}}$. Let X be a local cross-section of $\mathcal{L}$. We have $L_X\xi^{-2} = [f_X^0, \xi^{-2}]$ and $L_X\xi^{-1} \equiv [f_X^0, \xi^{-1}] \pmod{\xi^{-2}}$. Since $\mathfrak{g}^0 \subset \bar{\mathfrak{g}}^0$ and since f_X^0 is a $\mathfrak{g}^0$-valued function, it follows that X is a local infinitesimal automorphism of (Δ, I).

4.3. Let $c\in\hat{\mathfrak{g}}^{k}$. Then by [**9**], Lemma 6.3, there is a (unique) global cross-section X of $\mathcal{L}$ such that $f_X^p(e)=0$ $(p<k)$, $f_X^k(e)=c$ and $f_X^p=0$ $(p>k)$. Since X is an infinitesimal automorphism of $(\varDelta, I)$ and since $(\varDelta, I)$ satisfies (4.3), it follows from [**9**], Proposition 10.4 that there is a (unique) holomorphic vector field $\tilde{X}$ defined on a neighborhood of $f(M(\mathfrak{m}))$ such that X and $\tilde{X}$ are f-related.

LEMMA 4.3.

$$L_{\tilde{X}}z=H_k^{-2}(z, u)c\,,$$

$$L_{\tilde{X}}u=PH_k^{-1}(z, u)c\,,$$

where

$$H_k^l(z, u)=\sum_{\substack{2p+q=k-l\\ p,q\geqq 0}}\frac{(-1)^{p+q}}{p\,!\,q\,!}(adz)^p(adu)^q \qquad (l=-2, -1)\,.$$

PROOF. We take a base $\varepsilon_1^{-2}, \cdots, \varepsilon_n^{-2}$ (resp. $\varepsilon_1^{-1}, \cdots, \varepsilon_{2m}^{-1}$) of $\mathfrak{g}^{-2}$ (resp. of $\mathfrak{g}^{-1}$) and denote by $x_1, \cdots, x_n, y_1, \cdots, y_{2m}$ the normal coordinate system of the Lie group $M(\mathfrak{m})$ corresponding to the base $\varepsilon_1^{-2}, \cdots, \varepsilon_n^{-2}, \varepsilon_1^{-1}, \cdots, \varepsilon_{2m}^{-1}$ of $\mathfrak{m}$. We put $x=\sum_i x_i\varepsilon_i^{-2}$ and $y=\sum_j y_j\varepsilon_j^{-1}$. Then in terms of x and y, (4.2) may be described as

$$z\circ f=x+\frac{1}{2}[\bar{P}(y), P(y)]\,, \tag{4.4}$$

$$u\circ f=P(y)\,.$$

Let us now consider the simply connected complex Lie group $M(\mathfrak{m}_c)$ whose Lie algebra is given by the complexification $\mathfrak{m}_c$ of $\mathfrak{m}$. We denote by $x_1^c, \cdots, x_n^c, y_1^c, \cdots, y_{2m}^c$ the normal coordinate system of the complex Lie group $M(\mathfrak{m}_c)$ corresponding to the base $\varepsilon_1^{-2}, \cdots, \varepsilon_n^{-2}, \varepsilon_1^{-1}, \cdots, \varepsilon_{2m}^{-1}$ of $\mathfrak{m}_c$, and put $x_c=\sum_i x_i^c\varepsilon_i^{-2}$ and $y_c=\sum_j y_j^c\varepsilon_j^{-1}$. $M(\mathfrak{m})$ being considered as a real Lie subgroup of $M(\mathfrak{m}_c)$, the mapping f of $M(\mathfrak{m})$ to $\mathfrak{n}$ is extended to a holomorphic mapping f_c of $M(\mathfrak{m}_c)$ to $\mathfrak{n}$. Since x_c and y_c are the holomorphic extensions of x and y, respectively, it follows from (4.4) that

$$z\circ f_c=x_c+\frac{1}{2}[\bar{P}(y_c), P(y_c)]\,, \tag{4.5}$$

$$u\circ f_c=P(y_c)\,.$$

The vector field X on $M(\mathfrak{m})$ is extended to a holomorphic vector field X_c on $M(\mathfrak{m}_c)$. By Lemma 1.2, then we have

$$L_{X_c}x_c=H_k^{-2}(x_c, y_c)c+\frac{1}{2}[(y_c, H_k^{-1}(x_c, y_c)c]\,. \tag{4.6}$$

$$L_{X_c}y_c=H_k^{-1}(x_c, y_c)c\,.$$

We now define a holomorphic mapping ξ of $\mathfrak{n}$ to $M(\mathfrak{m}_c)$ by the requirement that $z=x_c\circ\xi$ and $u=y_c\circ\xi$. Since $P(u)=u$ and $\bar{P}(u)=0$, we have $f_c\circ\xi=$

identity. As is easily observed, the two vector fields X_c and $\tilde{X}$ are f_c-related (cf. [**9**], Proof of Proposition 10.3). Hence, $\tilde{X}_x = f_{c*}X_{c\xi(x)}$ $(x \in \mathfrak{n})$. Therefore by (4.5) and (4.6), we have

$$L_{\tilde{X}}u = (L_{X_c}(u \circ f_c)) \circ \xi = P((L_{X_c}y_c) \circ \xi) = PH_k^{-1}(z, u)c\,,$$

$$\begin{aligned} L_{\tilde{X}}z &= (L_{X_c}(z \circ f_c)) \circ \xi \\ &= (L_{X_c}x_c) \circ \xi + \frac{1}{2}[\bar{P}((L_{X_c}y_c) \circ \xi), y_c \circ \xi] + \frac{1}{2}[\bar{P}(y_c \circ \xi), (L_{X_c}y_c) \circ \xi] \\ &= H_k^{-2}(z, u)c + \frac{1}{2}[u, H_k^{-1}(z, u)c] \\ &\quad + \frac{1}{2}[\bar{P}H_k^{-1}(z, u)c, u] + \frac{1}{2}[\bar{P}(u), H_k^{-1}(z, u)c]\,. \end{aligned}$$

Since $\bar{P}(u) = 0$, $P + \bar{P} = 1$ and $[P(\mathfrak{g}^{-1}), P(\mathfrak{g}^{-1})] = \{0\}$, it follows that $L_{\tilde{X}}z = H_k^{-2}(z, u)c$. Thus we have proved Lemma 4.3.

LEMMA 4.4. *Let c, X and $\tilde{X}$ be as in Lemma* 4.3. *Then we have*

$$L_{\tilde{X}}z = F^{-2}(z, u)c\,,$$

$$L_{\tilde{X}}u = PF^{-1}(z, u)c\,,$$

where $F^{-2}(z, u)$ and $F^{-1}(z, u)$ are defined as follows:

i) *The case k is odd* $(k = 2s-1)$.

$$F^{-2}(z, u) = \frac{(-1)^{s+1}}{s!}(ad\ z)^s ad\ u\,,$$

$$F^{-1}(z, u) = \frac{(-1)^s}{s!}(ad\ z)^s + \frac{1}{2}\frac{(-1)^{s+1}}{(s-1)!}(ad\ z)^{s-1}(ad\ u)^2\,.$$

ii) *The case k is even* $(k = 2s)$.

$$F^{-2}(z, u) = \frac{(-1)^{s+1}}{(s+1)!}(ad\ z)^{s+1}\,,$$

$$F^{-1}(z, u) = \frac{(-1)^{s+1}}{s!}(ad\ z)^s ad\ u\,.$$

PROOF. This lemma follows easily from Lemma 4.2. Indeed, suppose that k is odd $(k = 2s-1)$. Then $(ad\ u)c$ is a $\hat{\mathfrak{g}}_c^{2s-2}$-valued function. u being a $P(\mathfrak{g}^{-1})$-valued function, it follows from Lemma 4.2 that $(ad\ u)^3 c = 0$. Suppose now that k is even $(k = 2s)$. Then we have $(ad\ u)^2 c = 0$ as above.

REMARK 3. $(ad\ z)^{s-1}(ad\ u)^2 c$ (for the case $k = 2s-1$) and $(ad\ z)^s(ad\ u)c$ (for the case $k = 2s$) are $P(\mathfrak{g}^{-1})$-valued functions. Hence we have:

i) The case k is odd $(k = 2s-1)$.

$$L_{\tilde{X}}u = \frac{(-1)^s}{s!}P(ad\ z)^s c + \frac{1}{2}\frac{(-1)^{s+1}}{(s-1)!}(ad\ z)^{s-1}(ad\ u)^2 c\,.$$

ii) The case k is even $(k = 2s)$.

$$L_{\tilde{X}}u=\frac{(-1)^{s+1}}{s\,!}(ad\ z)^s(ad\ u)c\,.$$

From Lemma 4.3 (or Lemma 4.4), we know that $\tilde{X}$ is a polynomial vector field with respect to the linear coordinate system $z_1, \cdots, z_n, u_1, \cdots, u_m$. Let $c\in\hat{\mathfrak{g}}$ and express it as $\sum_{k=-2}^{l} c^k$. Then there is a (unique) global cross-section X of $\mathcal{L}$ such that $f_X^k(e)=c^k$ $(-2\leqq k\leqq l)$ and $f_X^k=0$ $(k>l)$ ([**9**], Lemma 6.3), and there is a (unique) polynomial vector field $\iota(c)$ on $\mathfrak{n}$ such that X and $\iota(c)$ are f-related (Lemma 4.3 and [**9**], Proposition 10.4). By [**9**], Lemma 6.4, it is clear that the assignment $c\rightarrow -\iota(c)$ is an injective homomorphism of $\hat{\mathfrak{g}}$ into the Lie algebra of all polynomial vector fields on $\mathfrak{n}$.

4.4. LEMMA 4.5. *Let $c\in\mathfrak{g}$. Then the vector field $\iota(c)$, restricted to D, coincides with the vector field X on D induced by the one parameter group $D\ni x\rightarrow(\exp t\,c)x\in D$.*

PROOF. We may assume that $c\in\mathfrak{g}^k$. Let us consider the holomorphic imbedding $h=h'$ of $\widetilde{W}=\mathfrak{n}$ in G_c/B. Then we have $h((\exp t\,c)x)=(\exp t\,c)h(x)$ $(t\in\boldsymbol{R},\ x\in D)$ (Lemma 3.4). If we set $\varphi(t)=(\exp t\,c)x$, it follows that

$$\pi(\exp(-x)\cdot\exp\varphi(t))=\pi(\exp(-x)\cdot\exp t\,c\cdot\exp x)\,.$$

We have $\exp(-x)\cdot\exp\varphi(t)=\exp(\varphi(t)-x)$, because $\mathfrak{n}$ is an abelian subalgebra of $\mathfrak{g}_c$, and we have $\exp(-x)\cdot\exp t\,c\cdot\exp x=\exp t\,Ad(\exp(-x))c$. Therefore we get

$$\exp(\varphi(t)-x)\equiv\exp t\,Ad(\exp(-x))c \qquad (\mathrm{mod}\ B)\,. \tag{4.7}$$

We have

$$Ad(\exp(-x))c=\exp(-ad\ x)c=\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!}(ad(z(x)+u(x)))^r c$$
$$=\sum_{l=-2}^{\infty}H_k^l(z(x),\,u(x))c\,.$$

Since $\mathfrak{b}=\sum_{p=0}^{\infty}\mathfrak{n}^p$ (Remark 2), this implies

$$Ad(\exp(-x))c\equiv H_k^{-2}(z(x),\,u(x))c+PH_k^{-1}(z(x),\,u(x))c \qquad (\mathrm{mod}\ \mathfrak{b})\,. \tag{4.8}$$

We have $\varphi(0)=x$ and $\varphi(t)=z(\varphi(t))+u(\varphi(t))$, whence $\frac{d\varphi}{dt}(0)=(L_Xz)(x)+(L_Xu)(x)$. Therefore it follows from (4.7), (4.8) that $L_Xz=H_k^{-2}(z,\,u)c$, $L_Xu=PH_k^{-1}(z,\,u)c$ and hence from Lemma 4.3 that $X=\iota(c)$.

LEMMA 4.6. *$\mathfrak{g}^k$ consists of all $c\in\hat{\mathfrak{g}}^k$ such that $L_{\iota(c)}\Omega=0$, where Ω is the invariant volume element on D.*

PROOF. This is clear from Lemmas 2.3 and 4.5.

LEMMA 4.7. *Let $c\in\hat{\mathfrak{g}}^k$ be such that $[c,\,\mathfrak{g}^{-2}]\subset\mathfrak{g}^{k-2}$ and $[c,\,\mathfrak{g}^{-1}]\subset\mathfrak{g}^{k-1}$. Then there is a unique function $\mu(c)$ on the cone V such that*

$$L_{\iota(c)}\Omega=\mu(c)\circ\Phi\cdot\Omega\,.$$

PROOF. There is a function f on D such that $L_X\Omega=f\Omega$. We show that $f\circ\varphi=f$ ($\varphi\in M(D)$) or equivalently $L_{\iota(y)}f=0$ ($y\in\mathfrak{m}$). Indeed, we put $X=\iota(c)$ and $Y=\iota(y)$. Then by the assumption, both Y and $[X, Y]=-\iota([c, y])$ are infinitesimal automorphisms of D, whence $L_Y\Omega=L_{[X,Y]}\Omega=0$. Since $L_{[X,Y]}=L_XL_Y-L_YL_X$, it follows that $L_Yf=0$, proving our assertion. Therefore we can find a function $\mu(c)$ on V such that $f=\mu(c)\circ\Phi$ (cf. Proof of Lemma 2.4). Thus we have proved $L_{\iota(c)}\Omega=\mu(c)\circ\Phi\cdot\Omega$. Uniqueness of $\mu(c)$ is clear.

4.5. LEMMA 4.8. *Let $c\in\hat{\mathfrak{g}}^k$. Then there is a $\mathfrak{g}^0_c$-valued function A on $\mathfrak{n}$ satisfying the following conditions:*

1) $L_{\iota(c)}dz\equiv[A, dz]$ (mod du),
   $L_{\iota(c)}du\equiv[A, du]$ (mod dz).
2) i) *The case k is odd.*

$$A(\sqrt{-1}x)=0\quad(x\in V).$$

ii) *The case k is even ($k=2s$).*

$$A(\sqrt{-1}x)=\frac{(-\sqrt{-1})^s}{s\,!}(ad\ x)^sc\quad(x\in V).$$

PROOF. This is easy from Lemma 4.4 and the following equality:

$$d((ad\ z)^p(ad\ u)^qc)=p[dz, (ad\ z)^{p-1}(ad\ u)^qc]+q[du, (ad\ z)^p(ad\ u)^{q-1}c]\,.$$

Indeed, suppose that k is odd ($k=2s-1$). Since $L_{\iota(c)}dz=d(L_{\iota(c)}z)$, it follows from Lemma 4.4 and the above equality that

$$L_{\iota(c)}dz\equiv\frac{(-1)^{s+1}}{(s-1)\,!}[dz, (ad\ z)^{s-1}(ad\ u)c]\qquad(\text{mod } du)\,.$$

Analogously,

$$L_{\iota(c)}du\equiv\frac{(-1)^{s+1}}{(s-1)\,!}[du, (ad\ z)^{s-1}(ad\ u)c]\qquad(\text{mod } dz)\,.$$

Therefore the function $A=\frac{(-1)^s}{(s-1)\,!}(ad\ z)^{s-1}(ad\ u)c$ satisfies 1). Furthermore it clearly satisfies 2). The case k is even ($k=2s$) can be similarly dealt with and it can be shown that the function $A=\frac{(-1)^s}{s\,!}(ad\ z)^sc$ satisfies 1) and 2).

LEMMA 4.9. *Let $c\in\hat{\mathfrak{g}}^k$. Then there is a $\mathfrak{g}^0$-valued function B on $\mathfrak{n}$ satisfying the following conditions:*

1) $L_{\iota(c)}\Phi=[B, \Phi]$.
2) i) *The case k is odd.*

$$B(\sqrt{-1}x)=0\quad(x\in V).$$

ii) *The case k is even ($k=2s$)*

$$B(\sqrt{-1}x)=\frac{1}{2}\frac{(-\sqrt{-1})^s+(\sqrt{-1})^s}{(s+1)\,!}(ad\ x)^sc\quad(x\in V)\,.$$

PROOF. We have $\Phi=\frac{1}{2\sqrt{-1}}(z-\bar{z}-[\bar{u}, u])$ and hence

$$L_{\iota(c)}\Phi=\frac{1}{2\sqrt{-1}}(L_{\iota(c)}z-\overline{L_{\iota(c)}z}-[\overline{L_{\iota(c)}u}, u]-[\bar{u}, L_{\iota(c)}u]).$$

Therefore by Lemma 4.4, we have $L_{\iota(c)}\Phi=T(z, u, \bar{z}, \bar{u})c$, where

$$T(z, u, \bar{z}, \bar{u})=\frac{1}{2\sqrt{-1}}\{F^{-2}(z, u)-F^{-2}(\bar{z}, \bar{u})+(ad\ u)F^{-1}(\bar{z}, \bar{u})-(ad\ \bar{u})F^{-1}(z, u)\}.$$

Setting $\Phi'=\frac{1}{2}(z+\bar{z}+[\bar{u}, u])$, we have $z=\sqrt{-1}\Phi+\Phi'$. Since the vector field $\iota(c)$ is tangent to the Silov boundary S of D and since S is defined by $\Phi=0$, it follows that $T(\Phi', u, \bar{\Phi}', \bar{u})c=0$. Thus we have proved

$$L_{\iota(c)}\Phi=\{T(z, u, \bar{z}, \bar{u})-T(\Phi', u, \bar{\Phi}', \bar{u})\}c. \tag{4.9}$$

Suppose first that k is odd ($k=2s-1$). By Lemma 4.4, then we have

$$F^{-2}(z, u)-F^{-2}(\Phi', u)=\frac{(-1)^{s+1}}{s!}ad\ u((ad\ z)^{s}-(ad\ \Phi')^{s}).$$

Since

$$(ad\ z)^{s}-(ad\ \Phi')^{s}=ad(z-\Phi')\sum_{p=0}^{s-1}(ad\ z)^{p}(ad\ \Phi')^{s-1-p},$$

it follows that

$$F^{-2}(z, u)-F^{-2}(\Phi', u)=-\sqrt{-1}ad\ \Phi\ F_{*}^{-2}(z, u, \Phi'), \tag{4.10}$$

where

$$F_{*}^{-2}(z, u, \Phi')=\frac{(-1)^{s}}{s!}ad\ u\sum_{p=0}^{s-1}(ad\ z)^{p}(ad\ \Phi')^{s-1-p}.$$

Analogously,

$$F^{-1}(z, u)-F^{-1}(\Phi', u)=-\sqrt{-1}ad\ \Phi\ F_{*}^{-1}(z, u, \Phi'), \tag{4.11}$$

where

$$F_{*}^{-1}(z, u, \Phi')=\frac{(-1)^{s-1}}{s!}\sum_{p=0}^{s-1}(ad\ z)^{p}(ad\ \Phi')^{s-1-p}$$
$$+\frac{1}{2}\frac{(-1)^{s}}{(s-1)!}(ad\ u)^{2}\sum_{p=0}^{s-2}(ad\ z)^{p}(ad\ \Phi')^{s-2-p}.$$

By (4.9), then we see that the function

$$B=\frac{1}{2}\{F_{*}^{-2}(z, u, \Phi')+F_{*}^{-2}(\bar{z}, \bar{u}, \bar{\Phi}')-ad\ u\ F_{*}^{-1}(\bar{z}, \bar{u}, \bar{\Phi}')-ad\ \bar{u}\ F_{*}^{-1}(z, u, \Phi')\}c$$

satisfies 1). Furthermore it is clear that B also satisfies 2). Suppose now that k is even ($k=2s$). We put

$$F_{*}^{-2}(z, u, \Phi') = \frac{(-1)^s}{(s+1)!}\sum_{p=0}^{s}(ad\ z)^p(ad\ \Phi')^{s-p},$$

$$F_{*}^{-1}(z, u, \Phi') = \frac{(-1)^s}{s!} ad\ u \sum_{p=0}^{s-1}(ad\ z)^p(ad\ \Phi')^{s-1-p},$$

and define a function B by the same equality as above. Then as above it can be shown that equalities (4.10) and (4.11) hold and that B satisfies 1) and 2).

4.6. For each $A \in \mathfrak{g}_c^0$, we denote by $Tr\,A$ the trace of the (complex) endomorphism $\mathfrak{n} \ni y \rightarrow [A, y] \in \mathfrak{n}$, i.e., $Tr\,A$ is the trace of A as an endomorphism of $\tilde{W}$.

LEMMA 4.10. *Let $c \in \hat{\mathfrak{g}}^k$ be such that $[c, \mathfrak{g}^{-2}] \subset \mathfrak{g}^{k-2}$ and $[c, \mathfrak{g}^{-1}] \subset \mathfrak{g}^{k-1}$. Then the function $\mu(c)$ in Lemma 4.7 is computed as follows:*

i) *The case k is odd.*

$$\mu(c) = 0.$$

ii) *The case k is even and $\frac{k}{2}$ is odd $\left(\frac{k}{2} = 2t-1\right)$.*

$$\mu(c)(x) = \frac{2(-1)^{t+1}}{(2t-1)!} \operatorname{Im} Tr((ad\ x)^{2t-1}c) \quad (x \in V).$$

iii) *The case k is even and $\frac{k}{2}$ is even $\left(\frac{k}{2} = 2t\right)$.*

$$\mu(c)(x) = \frac{4t(-1)^t}{(2t+1)!} \operatorname{Re} Tr((ad\ x)^{2t}c) \quad (x \in V).$$

PROOF. By Lemma 2.4, we have

$$(4.12) \qquad L_{\iota(c)}\Omega = (\sqrt{-1})^{N^2}(L_{\iota(c)}(\lambda \circ \Phi))dz \wedge du \wedge \overline{dz \wedge du}$$
$$+ (\sqrt{-1})^{N^2}\lambda \circ \Phi(L_{\iota(c)}(dz \wedge du)) \wedge \overline{dz \wedge du}$$
$$+ (\sqrt{-1})^{N^2}\lambda \circ \Phi\ dz \wedge du \wedge \overline{L_{\iota(c)}(dz \wedge du)}.$$

For each $x \in D$, let $u(t)$ $(-\varepsilon < t < \varepsilon)$ be an integral curve of $\iota(c)$ with $u(0) = x$. G^0 being (identified with) a Lie subgroup of $GL(\mathfrak{n})$, we see that there is a (unique) curve $b(t)$ $(-\varepsilon < t < \varepsilon)$ in G^0 such that $\frac{db(t)}{dt} b(t)^{-1} = B(u(t))$ and $b(0) = 1$, where B is the function given in Lemma 4.9. By Lemma 4.9, then we have

$$\frac{d\Phi(u(t))}{dt} = (L_{\iota(c)}\Phi)(u(t)) = [B(u(t)), \Phi(u(t))].$$

Furthermore we clearly have

$$\frac{d(b(t)\Phi(x))}{dt} = B(u(t))b(t)\Phi(x) = [B(u(t)), b(t)\Phi(x)].$$

Since $\Phi(u(0)) = \Phi(x) = b(0)\Phi(x)$, it follows that $\Phi(u(t)) = b(t)\Phi(x)$. Therefore,

$\lambda(\Phi(u(t))) = |\det b(t)|^{-2}\lambda(\Phi(x))$ by Lemma 2.4, which implies

$$L_{\iota(c)}(\lambda \circ \Phi) = -2 \operatorname{Re} TrB \cdot \lambda \circ \Phi . \tag{4.13}$$

By Lemma 4.8, we have

$$L_{\iota(c)}(dz \wedge du) = TrA \cdot dz \wedge du . \tag{4.14}$$

From (4.12), (4.13) and (4.14), it follows that $L_{\iota(c)}\Omega = 2 \operatorname{Re} Tr(A-B)\cdot \Omega$. Hence,

$$\mu(c)(x) = 2 \operatorname{Re} Tr(A(\sqrt{-1}\,x) - B(\sqrt{-1}\,x)) \qquad (x \in V) .$$

Lemma 4.10 now follows from this equality and Lemmas 4.8 and 4.9.

We are now in a position to prove the following

THEOREM 4.1. *Let D be the Siegel domain of the second kind associated with a convex cone V (in W^{-2}) and a V-hermitian form F (on W^{-1}). Assume that the domain D is affine homogeneous. Let $\mathfrak{g} = \sum_{p=-2}^{\infty} \mathfrak{g}^p$ be the graded algebra given in Theorem 3.1 and let $\hat{\mathfrak{g}} = \sum_{p=-2}^{\infty} \hat{\mathfrak{g}}^p$ be the prolongation of $(\mathfrak{g}^{-2}+\mathfrak{g}^{-1}, \mathfrak{g}^0)$. For each $X \in \mathfrak{g}^0$, denote by $Tr(X)$ the trace of X as an endomorphism of $\widetilde{W}$. Then $\mathfrak{g}$ is a graded subalgebra of $\hat{\mathfrak{g}}$ and the subspaces $\mathfrak{g}^p \subset \hat{\mathfrak{g}}^p$ $(p>0)$ are inductively determined as follows:*

(1) $\mathfrak{g}^1 = \hat{\mathfrak{g}}^1$.

(2) *$\mathfrak{g}^2$ consists of all $X \in \hat{\mathfrak{g}}^2$ such that $\operatorname{Im} Tr([X, Y]) = 0$ for all $Y \in \mathfrak{g}^{-2}$.*

(3) *$\mathfrak{g}^3$ consists of all $X \in \hat{\mathfrak{g}}^3$ such that $[X, \mathfrak{g}^{-1}] \subset \mathfrak{g}^2$.*

(4) *$\mathfrak{g}^4$ consists of all $X \in \hat{\mathfrak{g}}^4$ such that $[X, \mathfrak{g}^{-1}] \subset \mathfrak{g}^3$ and $Tr([[X, Y], Y]) = 0$ for all $Y \in \mathfrak{g}^{-2}$.*

(5) *For each $k > 4$, $\mathfrak{g}^k$ consists of all $X \in \hat{\mathfrak{g}}^k$ such that $[X, \mathfrak{g}^{-2}] \subset \mathfrak{g}^{k-2}$ and $[X, \mathfrak{g}^{-1}] \subset \mathfrak{g}^{k-1}$.*

PROOF. From Lemmas 4.6 and 4.7, we see that $\mathfrak{g}^k$ consists of all $X \in \hat{\mathfrak{g}}^k$ such that $[X, \mathfrak{g}^{-2}] \subset \mathfrak{g}^{k-2}$, $[X, \mathfrak{g}^{-1}] \subset \mathfrak{g}^{k-1}$ and $\mu(X) = 0$. Theorem 4.1 follows easily from this fact and Lemma 4.10.

§ 5. Infinitesimal automorphisms of a Siegel domain, III.

5.1. In this paragraph, we shall investigate the special case where $W^{-1} = \{0\}$, i.e., the domain D is a Siegel domain of the first kind. In this case, we clearly have the followings:

(1) The group $GL(D)$ is identical with the linear automorphism group of the cone V, i.e., consists of all $a \in GL(W^{-2}) \subset GL(W_c^{-2})$ such that $aV = V$.

(2) $\hat{\mathfrak{g}}^{2s-1} = \{0\}$ and, for each $s > 0$, $\hat{\mathfrak{g}}^{2s}$ is the s-th prolongation of the linear Lie algebra $\mathfrak{g}^0 \subset \mathfrak{gl}(W^{-2})$.

PROPOSITION 5.1. *$\mathfrak{g}^2 = \hat{\mathfrak{g}}^2$, i.e., $\mathfrak{g}^2$ is the first prolongation of the linear Lie algebra $\mathfrak{g}^0 \subset \mathfrak{gl}(W^{-2})$.*

This is clear from Theorem 4.1.

5.2. Let us now investigate the case where the vector space W^{-2} is generated by the vectors of the form $F(y, y)$ $(y \in W^{-1})$, i.e., $\mathfrak{g}^{-2}=[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$.

PROPOSITION 5.2 ([3]). *The group $GL(D)$ consists of all $a \in GL(\widetilde{W})$ satisfying the following conditions:*

1) $aW^p=W^p$ $(p=-2, -1)$;

2) $aF(y, y')=F(ay, ay')$.

PROPOSITION 5.3. (1) *The graded algebra* $\mathfrak{g}$ *coincides with the prolongation* $\hat{\mathfrak{g}}$ *of* $(\mathfrak{m}, \mathfrak{g}^0)$.

(2) *The Lie algebra* $\mathfrak{g}$ *may be identified with the Lie algebra of all holomorphic vector fields X on $\widetilde{W}$ which are tangent to the Silov boundary S of the domain D.*

The proof of this proposition is preceded by the following two lemmas.

LEMMA 5.1. $\hat{\mathfrak{g}}$ *is finite dimensional.*

PROOF. We have $I^2X=-X$ $(X \in \mathfrak{g}^{-1})$ and $[IX, IY]=[X, Y]$ $(X, Y \in \mathfrak{g}^{-1})$. Furthermore Proposition 5.2 implies that $\mathfrak{g}^0$ is identical with the subalgebra $\mathfrak{g}^0(\mathfrak{m}, I)$ of $\mathfrak{g}^0(\mathfrak{m})$ consisting of all $A \in \mathfrak{g}^0(\mathfrak{m})$ such that $IAX=AIX$ $(X \in \mathfrak{g}^{-1})$. Since $\mathfrak{g}^{-2}=[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$ and since $\mathfrak{m}=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}$ is non-degenerate, it follows from [**9**], Corollary 3 to Theorem 11.1 that $(\mathfrak{m}, \mathfrak{g}^0)$ is of finite type, i.e., $\dim \hat{\mathfrak{g}} < \infty$. (For each $p \geqq 0$, let $\mathfrak{h}^p$ be the subspace of $\hat{\mathfrak{g}}^p$ defined in the proof of Lemma 4.1, (1). Then $\mathfrak{h}^p$ is the p-th prolongation of $\mathfrak{h}^0 \subset \mathfrak{gl}(\mathfrak{g}^{-1})$ and $\mathfrak{h}^1=\{0\}$. Hence by [**9**], Corollary 2 to Theorem 11.1 we find that $(\mathfrak{m}, \mathfrak{g}^0)$ is of finite type.)

LEMMA 5.2. *Let $\mathfrak{g}=\sum_p \mathfrak{g}^p$ be a graded algebra, where* $\dim \mathfrak{g} < \infty$. *We assume that $\mathfrak{g}^p=\{0\}$ $(p<-2)$ and that $\mathfrak{g}$ is semi-simple. Then we have:*

(1) $\dim \mathfrak{g}^p=\dim \mathfrak{g}^{-p}$. *In particular,* $\mathfrak{g}^p=\{0\}$ $(p>2)$.

(2) *If* $\mathfrak{g}^{-2}=[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$, *then* $\mathfrak{g}^2=[\mathfrak{g}^1, \mathfrak{g}^1]$.

PROOF. (1) Let B be the Killing form of $\mathfrak{g}$. Then it is easy to see that $B(\mathfrak{g}^p, \mathfrak{g}^q)=\{0\}$ if $p+q \neq 0$. Since B is non-degenerate, it follows that, for each $p>0$, the bilinear mapping $\mathfrak{g}^p \times \mathfrak{g}^{-p} \ni (X, Y) \rightarrow B(X, Y) \in \boldsymbol{R}$ gives a duality between the two vector spaces $\mathfrak{g}^p$ and $\mathfrak{g}^{-p}$. Hence $\dim \mathfrak{g}^p=\dim \mathfrak{g}^{-p}$.

(2) Assume that $\mathfrak{g}^{-2}=[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$. Then it can be easily verified that $\mathfrak{a}=\mathfrak{g}^{-2}+\mathfrak{g}^{-1}+[\mathfrak{g}^{-1}, \mathfrak{g}^1]+\mathfrak{g}^1+[\mathfrak{g}^1, \mathfrak{g}^1]$ is a (graded) ideal of $\mathfrak{g}$. Since $\mathfrak{a}$ is semi-simple, it follows from (1) that $\dim [\mathfrak{g}^1, \mathfrak{g}^1]=\dim \mathfrak{g}^{-2}=\dim \mathfrak{g}^2$. Hence $\mathfrak{g}^2=[\mathfrak{g}^1, \mathfrak{g}^1]$.

PROOF OF PROPOSITION 5.3. (1) $\hat{\mathfrak{g}}$ is finite dimensional by Lemma 5.1. This being said, let $\mathfrak{r}$ denote the radical of $\hat{\mathfrak{g}}$. Since $[E, \mathfrak{r}] \subset \mathfrak{r}$, we see that $\mathfrak{r}$ is a graded ideal of $\hat{\mathfrak{g}}: \mathfrak{r}=\sum_{p=-2}^{\infty} \mathfrak{r}^p$, where $\mathfrak{r}^p=\mathfrak{r} \cap \mathfrak{g}^p$. If we put $\dot{\mathfrak{g}}^p=\mathfrak{g}^p/\mathfrak{r}^p$, then $\mathfrak{g}/\mathfrak{r}=\sum_{p=-2}^{\infty} \dot{\mathfrak{g}}^p$ is endowed with the structure of graded algebra. $\mathfrak{g}/\mathfrak{r}$ is semi-simple and $\dot{\mathfrak{g}}^{-2}=[\dot{\mathfrak{g}}^{-1}, \dot{\mathfrak{g}}^{-1}]$. Hence by Lemma 5.2, $\dot{\mathfrak{g}}^p=\{0\}$, i.e., $\hat{\mathfrak{g}}^p=\mathfrak{r}^p$ $(p>2)$ and $\dot{\mathfrak{g}}^2=[\dot{\mathfrak{g}}^1, \dot{\mathfrak{g}}^1]$, i.e., $\hat{\mathfrak{g}}^2=[\hat{\mathfrak{g}}^1, \hat{\mathfrak{g}}^1]+\mathfrak{r}^2$. $\mathfrak{r}$ being the radical of $\hat{\mathfrak{g}}$, it is known that $ad\,A$ is a nilpotent endomorphism of $\hat{\mathfrak{g}}$ for all $A \in [\mathfrak{r}, \hat{\mathfrak{g}}]$ (Bourbaki [**1**]). It follows that $Tr(A)$ (= the trace of $\mathfrak{n} \ni Z \to [A, Z] \in \mathfrak{n}) = 0$ for all $A \in \mathfrak{g}^0 \cap [\mathfrak{r}, \hat{\mathfrak{g}}]$. Therefore by Theorem 4.1, we find $\mathfrak{r}^2 \subset \mathfrak{g}^2$, because $[\mathfrak{r}^2, \mathfrak{g}^{-2}] \subset \mathfrak{g}^0 \cap [\mathfrak{r}, \hat{\mathfrak{g}}]$. Since $\mathfrak{g}^1 = \hat{\mathfrak{g}}^1$ (Theorem 4.1) and $\hat{\mathfrak{g}}^2 = [\hat{\mathfrak{g}}^1, \hat{\mathfrak{g}}^1] + \mathfrak{r}^2$, we have $\mathfrak{g}^2 = \hat{\mathfrak{g}}^2$. Hence we also have $\mathfrak{g}^3 = \hat{\mathfrak{g}}^3$ by Theorem 4.1. Furthermore $\mathfrak{g}^4 = \mathfrak{r}^4$ and $[[\mathfrak{r}^4, \mathfrak{g}^{-2}], \mathfrak{g}^{-2}] \subset \mathfrak{g}^0 \cap [\mathfrak{r}, \hat{\mathfrak{g}}]$. Therefore by Theorem 4.1, it follows that $\mathfrak{g}^4 = \hat{\mathfrak{g}}^4$ and hence $\mathfrak{g}^k = \hat{\mathfrak{g}}^k$ $(k > 4)$.

(2) Let us consider the pseudo-complex structure (Δ, I) on $M(\mathfrak{m})$ defined in 4.2. Since $\mathfrak{g}^0 = \mathfrak{g}^0(\mathfrak{m}, I)$, the standard Lie algebra sheaf $\mathcal{L}$ of type $(\mathfrak{m}, \mathfrak{g}^0)$ coincides with the sheaf of all local infinitesimal automorphisms of (Δ, I) ([**9**], § 10). By [**9**], Theorem 6.2, the formal algebra of $\mathcal{L}$ may be identified with the prolongation $\hat{\mathfrak{g}}$ of $(\mathfrak{m}, \mathfrak{g}^0)$. Since $\mathfrak{g} = \hat{\mathfrak{g}}$ (by (1)), it follows that the mapping $c \to -\iota(c)$ gives an isomorphism of the Lie algebra $\mathfrak{g}$ onto the Lie algebra of all holomorphic vector fields on $\mathfrak{n} = \widetilde{W}$ which are tangent to S. We have thereby proved (2) and hence Proposition 5.3.

5.3. EXAMPLES. Let K be a field. We denote by $M(m, n, K)$ the space of all $m \times n$ matrices over K and by $H(m, K)$ the space of all symmetric matrices over K of degree m. Furthermore we denote by $H^+(m, \boldsymbol{R})$ the set of all $X \in H(m, \boldsymbol{R})$ which are positive definite. It is well known that $H^+(m, \boldsymbol{R})$ is a homogeneous convex cone in $H(m, \boldsymbol{R})$ and that the linear automorphism group of the cone $H^+(m, \boldsymbol{R})$ consists of all linear transformations of the form $H^+(m, \boldsymbol{R}) \ni X \to AX^tA \in H^+(m, \boldsymbol{R})$, where $A \in GL(m, \boldsymbol{R})$ ([**6**]).

Let us consider the case where $W^{-2} = H(m+n, \boldsymbol{R})$, $W^{-1} = \boldsymbol{C}^m = M(m, 1, \boldsymbol{C})$, $V = H^+(m+n, \boldsymbol{R})$ and the V-hermitian form $F : \boldsymbol{C}^m \times \boldsymbol{C}^m \to H(m+n, \boldsymbol{C}) = W_c^{-2}$ is defined by

$$F(Y, Y') = \frac{1}{2}(Y^t\bar{Y}' + \bar{Y}'^tY) \qquad (Y, Y' \in \boldsymbol{C}^m),$$

where $A = F(Y, Y')$ should be identified with an element of $H(m+n, \boldsymbol{C})$ as follows:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.$$

Indeed, it can be verified that the mapping F thus defined is a $H^+(m+n, \boldsymbol{R})$-hermitian form on $\boldsymbol{C}^m$ and that the corresponding Siegel domain D is affine homogeneous ([**6**]).

(A) The case $m = 0$. In this case, the Siegel domain D is of first kind and is a symmetric domain. And we have the natural isomorphisms as follows: $\mathfrak{g}^{-2} \cong H(n, \boldsymbol{R})$; $\mathfrak{g}^0 \cong \mathfrak{gl}(n, \boldsymbol{R})$. If $n = 1$, then we have: $\dim \hat{\mathfrak{g}}^{2s} = 1$, $\mathfrak{g} = \mathfrak{g}^{-2} + \mathfrak{g}^0 + \hat{\mathfrak{g}}^2$ and $\mathfrak{g} \cong \mathfrak{sp}(1, \boldsymbol{R}) = \mathfrak{sl}(2, \boldsymbol{R})$. If $n > 1$, then we can prove the followings:

(1) $\hat{\mathfrak{g}}^2 \cong H(n, \boldsymbol{R})$ and $\hat{\mathfrak{g}}^{2s} = \{0\}$ $(s > 1)$.

(2) $\mathfrak{g} = \hat{\mathfrak{g}}$.

(3) $\mathfrak{g}\cong\mathfrak{sp}(n, \boldsymbol{R})$.

(B) The case $m>0$. In this case, the group $GL(D)$ consists of all linear transformations φ of $\widetilde{W}=H(m+n, \boldsymbol{C})+\boldsymbol{C}^{m}$ defined by

$$\varphi(X+Y)=AX^{t}A+BY \qquad (X\in H(m+n, \boldsymbol{C}),\ Y\in \boldsymbol{C}^{m}),$$

where

$$A=\begin{pmatrix} a & 0\\ c & b\end{pmatrix}, \qquad B=\lambda b$$

and

$$a\in GL(n, \boldsymbol{R}), \quad b\in GL(m, \boldsymbol{R}), \quad c\in M(m, n, \boldsymbol{R}), \quad \lambda\in \boldsymbol{C} \quad (|\lambda|=1).$$

And we have the natural isomorphisms as follows: $\mathfrak{g}^{-2}\cong H(m+n, \boldsymbol{R})$ is the direct sum of two vector spaces $\mathfrak{r}^{-2}$ and $\mathfrak{s}^{-2}$, where $\mathfrak{r}^{-2}\cong H(m, \boldsymbol{R})+M(m, n, \boldsymbol{R})$ and $\mathfrak{s}^{-2}\cong H(n, \boldsymbol{R})$; $\mathfrak{g}^{-1}\cong \boldsymbol{C}^{m}$; $\mathfrak{g}^{0}$ is the direct sum of two subalgebras $\mathfrak{r}^{0}$ and $\mathfrak{s}^{0}$, where $\mathfrak{r}^{0}\cong M(m, n, \boldsymbol{R})+\boldsymbol{C}$ (as vector spaces) and $\mathfrak{s}^{0}\cong \mathfrak{gl}(n, \boldsymbol{R})+\mathfrak{sl}(m, \boldsymbol{R})$ (as Lie algebras). If $m=1$ and $n=0$, then the domain D is symmetric and we have:

$$\dim \mathfrak{g}^{-2}=\dim \hat{\mathfrak{g}}^{2}=1, \qquad \dim \mathfrak{g}^{-1}=\dim \hat{\mathfrak{g}}^{1}=2, \qquad \dim \mathfrak{g}^{0}=2$$

and $\hat{\mathfrak{g}}^{k}=\{0\}$ $(k>3)$; $\mathfrak{g}=\hat{\mathfrak{g}}$; $\mathfrak{g}\cong\mathfrak{su}(2, 1)$. Suppose now that $n>0$ or $m>1$ and $n=0$. Then we can prove the followings:

(1) $\hat{\mathfrak{g}}^{1}=\{0\}$, $\hat{\mathfrak{g}}^{2}\cong H(n, \boldsymbol{R})$ and $\hat{\mathfrak{g}}^{k}=\{0\}$ $(k>2)$.

(2) $\mathfrak{g}=\hat{\mathfrak{g}}$.

(3) $\mathfrak{r}=\mathfrak{r}^{-2}+\mathfrak{r}^{-1}+\mathfrak{r}^{0}$ is the radical of $\mathfrak{g}$.

(4) $\mathfrak{s}=\mathfrak{s}^{-2}+\mathfrak{s}^{0}+\hat{\mathfrak{g}}^{2}$ is a semi-simple part of $\mathfrak{g}$ and is isomorphic with the direct sum $\mathfrak{sp}(n, \boldsymbol{R})+\mathfrak{sl}(m, \boldsymbol{R})$.

Finally we note that $\mathfrak{g}^{-2}=[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$ if and only if $n=0$.

Appendix

Let V (resp. V') be a convex cone in a real vector space W^{-2} (resp. W'^{-2}) and let F (resp. F') be a V-(resp. V'-) hermitian form on a complex vector space W^{-1} (resp. W'^{-1}). Assume that the corresponding Siegel domain D (resp. D') in $\widetilde{W}=W_{c}^{-2}+W^{-1}$ (resp. in $\widetilde{W}'=W_{c}'^{-2}+W'^{-1}$) is affine homogeneous. Given an object A such as a space or a mapping or$\cdots$ with respect to the domain D, we write as A' the corresponding object with respect to the domain D'. For the domain D, we use the notations given in § 3.

Let us identify the space $\widetilde{W}$ with an open submanifold of the complex manifold $\widetilde{M}=G_{c}/B$ by the imbedding h. Then the open submanifold $\widetilde{W}$ of $\widetilde{M}$ is characterized as the orbit of the complex Lie group N $(=\exp\mathfrak{n})$ through the origin o of $\widetilde{M}$, and the domain D as the orbit of the Lie group G through the point δo, where $\delta=\exp\sqrt{-1}s(v)$. Furthermore the Silov boundary S of the domain D is an open submanifold of the orbit M of G through the origin o.

LEMMA A. *Suppose that there is given a holomorphic homeomorphism* φ *of* D *onto* D'.

(1) φ *is extended to a (unique) holomorphic homeomorphism, denoted by the same letter* φ, *of* $\tilde{M}$ *onto* $\tilde{M}'$ *and there is a (unique) complex isomorphism* Φ *of* G_c *onto* G'_c *such that* $\Phi(G)=G'$ *and* $\varphi(ax)=\Phi(a)\varphi(x)$ $(a\in G_c,\ x\in\tilde{M})$.

(2) $\varphi(M)=M'$.

PROOF. (1) Since D' and M' are orbits of G', we may assume that $\varphi(\delta o)=\delta' o'$. The holomorphic homeomorphism φ gives rise to an isomorphism Φ of $H(D)$ onto $H(D')$ such that $\varphi(ax)=\Phi(a)\varphi(x)$ $(a\in G,\ x\in D)$. We clearly have $\Phi(G)=G'$. The isomorphism Φ of G onto G' is extended to a (complex) isomorphism, denoted by the same letter Φ, of G_c onto G'_c. Let $\hat{\Phi}$ be the (complex) isomorphism of G_c onto G'_c defined by $\hat{\Phi}(a)=\delta'^{-1}\Phi(\delta a\delta^{-1})\delta'$ $(a\in G_c)$. We show that $\hat{\Phi}(B)=B'$. Indeed, $\Phi(K)=K'$, because $\varphi(\delta o)=\delta' o'$. Since φ is a holomorphic homeomorphism of $D=G/K$ onto $D'=G'/K'$, we find $\Phi_*(jX)\equiv j'\Phi_*(X)$ (mod $\mathfrak{k}$) for all $X\in\mathfrak{g}$, where Φ_* denotes the (complex) isomorphism of $\mathfrak{g}_c$ onto $\mathfrak{g}'_c$ induced by Φ. It follows that $\Phi_*(\bar{P}(\mathfrak{g}_c)+\mathfrak{k}_c)=\bar{P}'(\mathfrak{g}'_c)+\mathfrak{k}'_c$, $\mathfrak{b}=Ad\,\delta^{-1}(\bar{P}(\mathfrak{g}_c)+\mathfrak{k}_c)$. Therefore $\hat{\Phi}_*(\mathfrak{b})=\mathfrak{b}'$ and hence $\hat{\Phi}(B)=B'$, proving our assertion. Since $\hat{\Phi}(B)=B'$, $\hat{\Phi}$ induces a holomorphic homeomorphism $\hat{\varphi}$ of $\tilde{M}$ onto $\tilde{M}'$ such that $\hat{\varphi}\circ\pi=\pi'\circ\hat{\Phi}$. Let $x=a\delta o\in D$ $(a\in G)$. Then $\varphi(x)=\Phi(a)\varphi(\delta o)=\Phi(a)\delta' o'=\pi'(\Phi(a)\delta')=\pi'(\delta'\hat{\Phi}(\delta^{-1}a\delta))=\delta'\hat{\Phi}(\delta^{-1})\hat{\varphi}(x)$. This implies that φ is extended to a holomorphic homeomorphism of $\tilde{M}$ onto $\tilde{M}'$. It is clear that $\varphi(ax)=\Phi(a)\varphi(x)$ $(a\in G_c,\ x\in\tilde{M})$.

(2) Let us consider the holomorphic transformation group of the complex Lie group $N'=\exp\mathfrak{n}'$ on the complex manifold $\tilde{M}'$. Since $\dim N'=\dim\tilde{M}'$ and since $\tilde{W}'$ is an open orbit (=a regular orbit) of N', the union T' of all singular orbits of N' forms a proper analytic set of $\tilde{M}'$ which is locally defined by a single equation. Furthermore $T'=\tilde{M}'-\tilde{W}'$, because $\tilde{M}'-T'$ is connected. Suppose now that $\varphi(S)\subset T'$. Take any $p\in S$ and let $f'=0$ be a local equation of T' at $\varphi(p)$. Then the function $f'\circ\varphi$ is holomorphic and vanishes on a neighborhood of p in $S\subset M$. Therefore by (3.7), $f'\circ\varphi$ vanishes on a neighborhood of p in $\tilde{M}$. Hence T' contains an open set of $\tilde{M}$, contradicting to the fact that T' is a proper analytic set. We have thereby shown that there is a $p\in S$ such that $\varphi(p)\in\tilde{W}'$ (cf. [**9**], Proposition 10.1). Analogously to the proof of Lemma 3.5, then we can prove that $\varphi(p)\in S'$. Both $\varphi(M)$ and M' are orbits of G' and $\varphi(p)\in\varphi(S)\cap S'\subset\varphi(M)\cap M'$. Hence $\varphi(M)=M'$.

THEOREM A. *Let* D *(resp.* D'*) be the Siegel domain of the second kind associated with a convex cone* V *(resp.* V'*) in* W^{-2} *(resp. in* W'^{-2}*) and a* V*- (resp.* V'*-) hermitian form* F *(resp.* F'*) on* W^{-1} *(resp. on* W'^{-1}*). Assume that the domain* D *(resp.* D'*) is affine homogeneous. Let* $\mathfrak{g}=\sum_p\mathfrak{g}^p$ *(resp.* $\mathfrak{g}'=\sum_p\mathfrak{g}'^p$*)*

be the corresponding graded algebra given in Theorem 3.1. *If the domain D is isomorphic with the domain D', then there is an isomorphism θ of $\mathfrak{g}$ onto $\mathfrak{g}'$ as graded algebras such that $\theta(I)=I'$ and $\theta(V)=V'$, where I is the element in the centre of $\mathfrak{g}^0$ defined in* 2.5 *and V should be identified with a convex cone in $\mathfrak{g}^{-2}$ by the linear isomorphism $W^{-2}\ni x\rightarrow s(x)\in\mathfrak{g}^{-2}$.*

PROOF. Assume that D is isomorphic with D'. By Lemma A, we can find a holomorphic homeomorphism φ of $\tilde{M}$ onto $\tilde{M}'$ and a complex isomorphism Φ of G_c onto G_c' satisfying the followings: $\varphi(D)=D'$, $\varphi(M)=M'$, $\Phi(G)=G'$ and $\varphi(ax)=\Phi(a)\varphi(x)$ $(a\in G_c,\ x\in\tilde{M})$. Moreover we may assume that $\varphi(o)=o'$. The mapping $M=G/G\cap B\ni x\rightarrow\varphi(x)\in M'=G'/G'\cap B'$ maps the pseudo-complex structure (M, Δ, I) isomorphically onto (M', Δ', I'). Therefore the argument in 3.5 proves:

1) $\Phi_*(L^p)=L'^p$ $(p=-1, 0)$;

2) $\Phi_*([I, X])\equiv[I', \Phi_*(X)]$ $(\mathrm{mod}\ L'^0)$ $(X\in L^{-1})$.

As we have observed in the proof of Lemma 1.1, the subspaces L^{-1} and L^0 of $\mathfrak{g}$ yield a family (L^p) of subspaces of $\mathfrak{g}$ and then a graded algebra $\bar{\mathfrak{g}}=\sum_p\bar{\mathfrak{g}}^p$, $\bar{\mathfrak{g}}^p=L^p/L^{p+1}$. By 1), then we have $\Phi_*(L^p)=L'^p$ for all p. Hence Φ_* induces an isomorphism θ_0 of $\bar{\mathfrak{g}}$ onto $\bar{\mathfrak{g}}'$ as graded algebras. By 2), we find $\theta_0(\bar{I})=\bar{I}'$, where $\bar{I}$ denotes the image of $I\in L^0$ by the projection of L^0 onto $\bar{\mathfrak{g}}^0=L^0/L^1$. Since $L^p=\sum_{r\geqq p}\mathfrak{g}^r$, there is a natural isomorphism θ_1 (resp. θ_1') of $\mathfrak{g}$ (resp. of $\mathfrak{g}'$) onto $\bar{\mathfrak{g}}$ (resp. onto $\bar{\mathfrak{g}}'$) as graded algebras such that $\theta_1(I)=\bar{I}$ (resp. $\theta_1'(I')=\bar{I}'$). Therefore $\theta=\theta_1'^{-1}\circ\theta_0\circ\theta_1$ is an isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}'$ as graded algebras and satisfies $\theta(I)=I'$.

Let us now prove that there are $X^p\in\mathfrak{g}'^p$ $(1\leqq p\leqq k)$ such that $Ad\,c^{-1}\Phi_*(E)=E'$, where $c=\exp X^1\cdots\exp X^k$. Indeed, $\Phi_*(E)\equiv\theta(E)$ $(\mathrm{mod}\ L'^1)$ and $\theta(E)=E'$. Therefore $\Phi_*(E)\equiv E'$ $(\mathrm{mod}\ L'^1)$. Suppose that there are $X^p\in\mathfrak{g}'^p$ $(1\leqq p<r)$ such that $Ad\,c_{r-1}^{-1}\Phi_*(E)\equiv E'$ $(\mathrm{mod}\ L'^r)$, where $c_{r-1}=\exp X^1\cdots\exp X^{r-1}$. Then there is an $X^r\in\mathfrak{g}'^r$ such that $Ad\,c_{r-1}^{-1}\Phi_*(E)\equiv E'-rX^r$ $(\mathrm{mod}\ L'^{r+1})$. $Ad(\exp X^r)E'=E'-rX^r$ and $Ad(\exp(-X^r))L'^{r+1}=L'^{r+1}$. It follows that $Ad\,c_r^{-1}\Phi_*(E)\equiv E'$ $(\mathrm{mod}\ L'^{r+1})$, where $c_r=c_{r-1}\exp X^r=\exp X^1\cdots\exp X^r$. Since $\mathfrak{g}'^p=\{0\}$ for sufficiently large p, this argument proves our assertion. c is in the group $G'\cap B'$. By considering $c^{-1}\varphi$ instead of φ, we may thereby assume that $\Phi_*(E)=E'$. Then we have $\Phi_*(\mathfrak{g}^p)=\mathfrak{g}'^p$ and hence $\Phi_*=\theta$.

The points $\varphi(\delta o)=\Phi(\delta)o'$ and $\delta'o'$ are in the domain D'. Therefore there is a $g=\exp(X^{-2}+X^{-1})\cdot a\in G_a'$ $(X^{-2}\in\mathfrak{g}'^{-2},\ X^{-1}\in\mathfrak{g}'^{-1},\ a\in G'^0)$ such that $\Phi(\delta)o'=g\delta'o'$. Since $a\in B'$, we have $t=\Phi(\delta)^{-1}g\delta'a^{-1}\in B'$. $\delta=\exp\sqrt{-1}v$ and hence

$$t=\exp(-\sqrt{-1}\Phi_*(v))\cdot\exp(X^{-2}+X^{-1})\cdot\exp(\sqrt{-1}Ad\,a\,v')$$
$$=\exp(-\sqrt{-1}\Phi_*(v)+\sqrt{-1}Ad\,a\,v'+X^{-2}+X^{-1}).$$

Since $Ad\,t\,E'\in\mathfrak{b}'$, it follows that

$$-2\sqrt{-1}\Phi_*(v)+2\sqrt{-1}\,Ad\,a\,v'+2X^{-2}+X^{-1}+\frac{1}{2}[X^{-1}, X^{-1}]\equiv 0 \pmod{\mathfrak{b}'}.$$

Therefore we get $\theta(v)=\Phi_*(v)=Ad\,a\,v'$. Since V is the orbit of G^0 through the point v, this shows $\theta(V)=V'$.

REMARK. Assume that $\mathfrak{g}^{-2}=[\mathfrak{g}^{-1}, \mathfrak{g}^{-1}]$. Then it can be proved that every isomorphism φ of (M, Δ, I) onto (M', Δ', I') is extended to a holomorphic homeomorphism, denoted by the same letter φ, of $\tilde{M}$ onto $\tilde{M}'$ such that $\varphi(D)=D'$.

Kyoto University

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