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A characterization of the simple group $S_p(6, 2)$

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§ 1. Introduction.

This is a continuation of our previous paper [**11**]. The purpose of this paper is to give a characterization of the finite simple group $S_p(6, 2)$, the symplectic group of 6 variables over the field of 2 elements, by the structure of the centralizer of an element of order 2 contained in the center of its Sylow 2-subgroup. Let V be a 6-dimensional vector space over the finite field $GF(2)$ and let f be a skew-symmetric non-degenerate bilinear form on V. The set of all non-singular linear transformations which leave f invariant form a group, the symplectic group over $GF(2)$. As is well-known, the structure of the symplectic group does not depend on the form f. So we may assume

$$f=x_1y_6+x_2y_5+x_3y_4+x_4y_3+x_5y_2+x_6y_1 .$$

If J is the matrix of the form f, then the set of non-singular matrices A such that

$${}^tAJA=J$$

may be identified with the symplectic group. Since this group has the trivial center, this is a simple group and of order $2^9 \cdot 3^4 \cdot 5 \cdot 7$ (cf. Artin [**1**]). Put

$$\hat{\alpha}=\begin{pmatrix} 1 & & & & & 1 \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

and $\hat{H}=C(\hat{\alpha})\cap S_p(6, 2)$. Then $\hat{\alpha}$ is a central involution of a Sylow 2-subgroup of $S_p(6, 2)$. Let A_n be the alternating group of degree n and $O_{2'}(G)$ be the maximal normal subgroup of odd order of the group G. Our main theorem of this paper is the following.

THEOREM. *Let G be a finite group such that G contains an element α of order 2 which is contained in the center of a Sylow 2-subgroup of G such that the centralizer $C_G(\alpha)$ is isomorphic to $\hat{H}$.*

Then (i) $G\cong A_{12}$ *or* A_{13} *or*

(ii) $G\cong S_p(6, 2)$ *or*

(iii) *$G = O_{2'}(G)C_G(\alpha)$ where $O_{2'}(G)$ is abelian.*

In this paper we are devoted to prove the case (ii) and (iii). But the case (iii) is an easy consequence of our previous paper in which we determined the fusion of the conjugacy classes of involutions and treated the case (i). In the course of our proof we construct a subgroup G_0 of G with a (B, N)-pair in the sense of Tits [9] which is isomorphic to $S_p(6, 2)$, and determine the structure of the centralizers of involutions. The coincidence of G with G_0 is accomplished by the result of Suzuki [6] (cf. Thompson [7]). The author wishes to express his thanks to Dr. Kondo for valuable suggestions and discussions.

We shall use the following notations which are fairly standard:

G'	the commutator subgroup of a group G.
$O^2(G)$	the smallest normal subgroup N of G such that G/N is a 2-group.
$Z(G)$	the center of a group G.
$\langle x, y, \cdots \rangle$	the group generated by the elements $x, y, \cdots$.
$\{u, v, \cdots\}$	the set of the elements $u, v, \cdots$.
Z_n	a cyclic group of order n.
$H < G$	H is a proper subgroup of G.
$H \lhd G$	H is a normal subgroup of G.
$[x, y]$	$x^{-1}y^{-1}xy$.
x^y	yxy^{-1}.
$x \sim y$ in G	an element x is conjugate to y in G.
$ccl_G(x)$	a conjugate class in a group G containing x.
$\mathcal{N}_G(X; 2')$	the set of $2'$-subgroups of G which X normalizes and which intersect X in the identity only.
S_n	the symmetric group of degree n.
$GL(n, q)$	the general linear group of degree n over the field of q elements.

REMARK. Since we defined $x^y = yxy^{-1}$, x^{yz} means $(x^z)^y$.

§ 2. Some properties of $\hat{H}$.

The group $\hat{H} = C(\hat{a}) \cap S_p(6, 2)$ is the set of matrices over $GF(2)$ such that

$$\left(\begin{array}{c:c:c} 1 & A & \gamma \\ \hdashline & M & B \\ \hdashline & & 1 \end{array}\right) \qquad \begin{array}{l} {}^tMJ_0M = J_0 \\ A = {}^tBJ_0M \end{array}$$

where A is a row vector, $\gamma \in GF(2)$ and

$$J_0=\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}.$$

More precisely $\hat{H}$ is generated by the following elements:

$$\hat{\pi}_1=\left(\begin{array}{ccc|ccc} 1 & & & 1 & & 1 \\ & 1 & & & & \\ & & 1 & & & 1 \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right) \qquad \hat{\pi}_1'=\left(\begin{array}{ccc|ccc} 1 & & & & & 1 \\ & 1 & & & & \\ & & 1 & 1 & & \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right)$$

$$\hat{\mu}=\left(\begin{array}{ccc|ccc} 1 & & 1 & 1 & & 1 \\ & 1 & & & & \\ & & 1 & & & 1 \\ \hline & & & 1 & & 1 \\ & & & & 1 & \\ & & & & & 1 \end{array}\right) \qquad \hat{\sigma}=\left(\begin{array}{ccc|ccc} 1 & & & & & 1 \\ & 1 & & & & \\ & & 1 & & & \\ \hline & & 1 & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right)$$

$$\hat{\pi}_2=\left(\begin{array}{ccc|ccc} 1 & & & 1 & 1 & 1 \\ & 1 & & & & 1 \\ & & 1 & & & 1 \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right) \qquad \hat{\pi}_2'=\left(\begin{array}{ccc|ccc} 1 & & & & & 1 \\ & 1 & & 1 & 1 & \\ & & 1 & 1 & 1 & \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right)$$

$$\hat{\mu}'=\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & & & 1 \\ & 1 & & & & \\ & & 1 & & & \\ \hline & & & 1 & & 1 \\ & & & & 1 & 1 \\ & & & & & 1 \end{array}\right) \qquad \hat{\sigma}'=\begin{pmatrix} 1 & & & & & 1 \\ & & & & 1 & \\ & & 1 & & & \\ & & & 1 & & \\ & 1 & & & & \\ & & & & & 1 \end{pmatrix}$$

$$\hat{\pi}_3=\left(\begin{array}{ccc|ccc} 1 & & & & 1 & 1 \\ & 1 & & & & 1 \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right) \qquad \hat{\pi}_3'=\left(\begin{array}{ccc|ccc} 1 & & & & & 1 \\ & 1 & & & 1 & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right)$$

$$\hat{\alpha}=\hat{\pi}_1\hat{\pi}_2\hat{\pi}_3 \qquad \hat{\alpha}'=\hat{\pi}_1'\hat{\pi}_2'\hat{\pi}_3'$$

In the isomorphism from $H=C_G(\alpha)$ onto $\hat{H}$ let the inverse images of $\hat{\pi}_1$, $\hat{\mu}$, $\hat{\pi}_2$, $\hat{\mu}'$, $\hat{\pi}_3$, $\hat{\pi}_1'$, $\hat{\sigma}$, $\hat{\pi}_2'$, $\hat{\sigma}'$, $\hat{\pi}_3'$ be π_1, μ, π_2, μ', π_3, π_1', σ, π_2', σ', π_3' respectively. Then one has $\alpha=\pi_1\pi_2\pi_3$. We denote this isomorphism by θ. We put $S=\langle\pi_1, \pi_1', \pi_2, \pi_2', \pi_3, \pi_3'\rangle$ and $\hat{S}=\theta(S)=\langle\hat{\pi}_1, \hat{\pi}_1', \hat{\pi}_2, \hat{\pi}_2', \hat{\pi}_3, \hat{\pi}_3'\rangle$. Then $\hat{S}$ is the set of matrices over $GF(2)$ such that

$$\left(\begin{array}{ccc|ccc} 1 & & & \alpha & \beta & \gamma \\ & 1 & & \delta & \varepsilon & \beta \\ & & 1 & \omega & \delta & \alpha \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right) \qquad \alpha, \beta, \cdots, \omega \in GF(2)$$

and is an elementary abelian subgroup of order 2^6. Put $\alpha'=\pi_1'\pi_2'\pi_3'$, $\rho=\pi_1'\sigma$, $\xi=(\pi_1'\pi_2')^{\sigma}(\pi_2'\pi_3')^{\sigma'}$ and $\tau=(\pi_1'\pi_2')^{\sigma}$. Then $\rho^3=\xi^3=\tau^2=1$. H is the semidirect product of $\langle\pi_1, \mu, \pi_2, \mu', \pi_3\rangle$ and $\langle\pi_1', \sigma, \pi_2', \sigma', \pi_3'\rangle$, $\langle\pi_1, \mu, \pi_2, \mu', \pi_3\rangle$ is normal in H and is an elementary abelian group of order 2^5, and $\langle\pi_1', \sigma, \pi_2', \sigma', \pi_3'\rangle$ is isomorphic to S_6. The action of the elements π_1', σ, π_2', σ', π_3' on π_1, μ, π_2, μ', π_3 by conjugation is given by the following table.

	π_1'	σ	π_2'	σ'	π_3'
π_1	π_1	μ	π_1	π_1	π_1
μ	$\mu\pi_1$	π_1	$\mu\pi_2$	μ	μ
π_2	π_2	$\mu\pi_1\pi_2$	π_2	$\mu\mu'$	π_2
μ'	$\mu'\pi_1$	μ'	μ'	$\mu\pi_2$	$\mu'\pi_3$
π_3	π_3	π_3	π_3	$\mu\mu'\pi_2\pi_3$	π_3

Let D be a Sylow 2-subgroup of G contained in H. We may assume that $D=\langle\pi_1, \pi_1', \pi_2, \pi_2', \pi_3, \pi_3'\rangle\langle\tau, \mu, \mu'\rangle=S\langle\tau, \mu, \mu'\rangle$ and hence $Z(D)=\langle\pi_1\pi_2, \pi_3\rangle$, $D'=\langle\pi_1, \pi_2, \pi_3, \pi_1'\pi_2', \mu\rangle$. Then we have $N_H(S)=D\langle\xi\rangle$ and $C_G(Z(D))=D\langle\rho\rangle$. The group $\langle\tau, \mu, \mu'\rangle$ is a dihedral group of order 8 with center $\langle\mu\rangle$ and acts on S. There are ten conjugacy classes of involutions of H and they are as follows.

π_1	π_1'	$\pi_1\pi_2$	$\pi_1\pi_2'$	$\pi_1'\pi_2'$	α	$\alpha\pi_1'$	$\alpha\pi_1'\pi_2'$	$\alpha\alpha'$	α'
3	6	3	12	12	1	6	12	4	4

The first and second entries in the columns give respectively, representatives of the classes and the cardinalities of the intersections of the classes and S. This implies that every involution of D is conjugate to an element of S in H.

PROPOSITION 1. *Let H^* be the centralizer of the element* $(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)$ *in A_{12}. Then H is isomorphic to H^*.*

PROOF. We have an isomorphism by the following correspondence:

$$\begin{array}{ll} \pi_1 \longrightarrow (1, 2)(3, 4) & \pi_1' \longrightarrow (1, 3)(2, 4) \\ \mu \longrightarrow (1, 2)(5, 6) & \sigma \longrightarrow (3, 5)(4, 6) \\ \pi_2 \longrightarrow (5, 6)(7, 8) & \pi_2' \longrightarrow (5, 7)(6, 8) \\ \mu' \longrightarrow (1, 2)(9, 10) & \sigma' \longrightarrow (7, 9)(8, 10) \\ \pi_3 \longrightarrow (9, 10)(11, 12) & \pi_3' \longrightarrow (9, 11)(10, 12) \end{array}$$

§ 3. The generators and relations of the group $S_p(6, 2)$ and $GL(3, 2)$.

In this section we characterize the group $S_p(6, 2)$ and $GL(3, 2)$ by the defining relations.

Let D_0 be a finite group generated by the elements $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9$ subject to the following relations:

$$\begin{aligned} & u_i^2 = 1 \qquad \text{for } 1 \leqq i \leqq 9 \\ & [u_i, u_j] = 1 \qquad \text{for } 4 \leqq i, j \leqq 9 \\ & (u_1 u_3)^2 = u_2 \\ & [u_1, \langle u_4, u_5, u_6, u_9 \rangle] = 1 \\ & u_1 u_7 u_1 = u_4 u_7, \qquad u_1 u_8 u_1 = u_5 u_6 u_8 \\ & [u_3, \langle u_5, u_6, u_7, u_8 \rangle] = 1 \\ & u_3 u_4 u_3 = u_4 u_5, \qquad u_3 u_9 u_3 = u_7 u_8 u_9 . \end{aligned} \tag{3.1}$$

The group $\langle u_1, u_2, u_3 \rangle$ is a dihedral group of order 8 with the center $\langle u_2 \rangle$ and acts on $\langle u_4, u_5, u_6, u_7, u_8, u_9 \rangle$ which is an elementary abelian 2-group of order 2^6. The group D_0 is isomorphic to a Sylow 2-subgroup of $S_p(6, 2)$ by the correspondence

$$u_1 \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & & & & \\ & 1 & & & & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & 1 & 1 \\ & & & & & 1 \end{array}\right) \qquad u_2 \longrightarrow \left(\begin{array}{ccc|ccc} 1 & & 1 & & & \\ & 1 & & & & \\ & & 1 & & & \\ \hline & & & 1 & & 1 \\ & & & & 1 & \\ & & & & & 1 \end{array}\right)$$

$$u_3 \longrightarrow \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & 1 & & & \\ & & 1 & & & \\ \hline & & & 1 & 1 & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right) \qquad u_4 \longrightarrow \left(\begin{array}{ccc|ccc} 1 & & & 1 & & \\ & 1 & & & & \\ & & 1 & & & 1 \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right)$$

$$u_5 \longrightarrow \left(\begin{array}{ccc|ccc} 1 & & & & 1 & \\ & 1 & & & & 1 \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right) \qquad u_6 \longrightarrow \left(\begin{array}{ccc|ccc} 1 & & & & & 1 \\ & 1 & & & & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right) \tag{3.2}$$

$$u_7 \longrightarrow \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & & 1 & & \\ & & 1 & & 1 & \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right) \qquad u_8 \longrightarrow \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & & & 1 & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right)$$

$$u_9 \longrightarrow \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & & & & \\ & & 1 & 1 & & \\ \hline & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{array}\right) .$$

Let W_0 be a finite group generated by the elements w_1, w_2, w_3 subject to the following relations:

$$\begin{aligned} & w_i^2 = 1 \qquad \text{for } 1 \leqq i \leqq 3 \\ & (w_1 w_2)^3 = (w_2 w_3)^4 = (w_1 w_3)^2 = 1 . \end{aligned} \tag{3.3}$$

The group W_0 is isomorphic to the associated Weyl group of a (B, N)-pair of $S_p(6, 2)$ by the following correspondence:

$$w_1 \longrightarrow \begin{pmatrix} & 1 & & & & \\ 1 & & & & & \\ & & & 1 & & \\ & & 1 & & & \\ & & & & & 1 \\ & & & & 1 & \end{pmatrix} \qquad w_2 \longrightarrow \begin{pmatrix} 1 & & & & & \\ & & 1 & & & \\ & 1 & & & & \\ & & & & 1 & \\ & & & 1 & & \\ & & & & & 1 \end{pmatrix}$$

$$w_3 \longrightarrow \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & & 1 & & \\ & & 1 & & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}. \tag{3.4}$$

Put $V_1 = \langle u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9 \rangle$, $V_2 = \langle u_1, u_2, u_4, u_5, u_6, u_7, u_8, u_9 \rangle$ and $V_3 = \langle u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8 \rangle$. Assume that

$$\begin{aligned} &(w_1u_1)^3 = (w_2u_3)^3 = (w_3u_9)^3 = 1 \\ &[w_i, V_i] \subset V_i \quad \text{for } 1 \leqq i \leqq 3 \end{aligned} \tag{3.5}$$

and the actions of the elements w_1, w_2, w_3 on V_1, V_2, V_3 by conjugation are given by the following table.

	w_1	w_2	w_3
u_1		u_2	u_1
u_2	u_3	u_1	u_4
u_3	u_2		u_7
u_4	u_7	u_5	u_2
u_5	u_5	u_4	u_5
u_6	u_8	u_6	u_6
u_7	u_4	u_7	u_3
u_8	u_6	u_9	u_8
u_9	u_9	u_8	

(3.6)

Let G_0 be a finite group generated by D_0 and W_0 with the defining relations (3.1), (3.3), (3.5), (3.6). Then we have

PROPOSITION 2. *The group G_0 is isomorphic to $S_p(6, 2)$.*

PROOF. The proof is given in various steps. We show that the homomorphism from G_0 onto $S_p(6, 2)$ defined by (3.2) and (3.4) is isomorphism. We put $B=D_0$ and $N=W=W_0$.

(a) *The set of elements in* $G_i=B\cup Bw_iB$ $(i=1, 2, 3)$ *is a subgroup of* G_0.

PROOF. The result follows immediately from (3.5) and (3.6).

DEFINITION. For any $w\in W$, let $l(w)=l$ be the smallest non-negative integer such that $w=w_{i_1}w_{i_2}\cdots w_{i_l}$ where $w_{i_j}\in\{w_1, w_2, w_3\}$.

(b) *For any i and $w\in W$, if $l(w_iw)\geqq l(w)$, then $w_iBw\subset Bw_iwB$.*

PROOF. By (3.3) we may identify w_1, w_2, w_3 with the elements (1, 2)(5, 6), (2, 3)(4, 5), (3, 4) in S_6 respectively. Let $C_0=\{1\}$, $C_1=\{w_1, w_2, w_3\}$. We shall give a method of constructing C_n for $n\geqq 2$. Suppose that the set $C_0, \cdots, C_{n-1}$ have been constructed. Let $\tilde{C}_n$ be the set of all 'words' of length n. Define $C_n=\tilde{C}_n-\bigcup_{0\leqq i\leqq n-1} C_i$. Then clearly elements w in C_n have $l(w)=n$.

Put $u_1=X_1$, $u_3=X_2$, $u_9=X_3$. To check that for those $w\in W$ with $l(w_iw)\geqq l(w)$, we have $w_iBw\subset Bw_iwB$, we need only to see that $w_iX_iw\in Bw_iwB$ for $i=1, 2, 3$. It is easily verified that for those $w\in W$ such that $l(w_iw)\geqq l(w)$, we can always write $w_iX_iw=w_iwY_i$ with $Y_i\in B$. The computations are summarized in the following table which is self-explanatory.

w		$l(w)$	$l(w_1w)$	$l(w_2w)$	$l(w_3w)$	Y_1	Y_2	Y_3
w_1	(1, 2)(5, 6)	1	0	2	2		u_2	u_9
w_2	(2, 3)(4, 5)	1	2	0	2	u_2		u_8
w_3	(3, 4)	1	2	2	0	u_1	u_7	
w_1w_2	(1, 2, 3)(4, 6, 5)	2	1	3	3		u_1	u_8
w_1w_3	(1, 2)(3, 4)(5, 6)	2	1	3	1		u_4	
w_2w_1	(1, 3, 2)(4, 5, 6)	2	3	1	3	u_3		u_6
w_2w_3	(2, 3, 5, 4)	2	3	1	3	u_4		u_8
w_3w_2	(2, 4, 5, 3)	2	3	3	1	u_2	u_7	
$w_1w_2w_1$	(1, 3)(4, 6)	3	2	2	4			u_6
$w_1w_2w_3$	(1, 2, 3, 6, 5, 4)	3	2	4	4		u_1	u_8
$w_1w_3w_2$	(1, 2, 4, 6, 5, 3)	3	2	4	2		u_5	
$w_2w_1w_3$	(1, 3, 5, 6, 4, 2)	3	4	2	4	u_7		u_6
$w_2w_3w_2$	(2, 5)	3	4	2	4	u_5		u_9
$w_3w_2w_1$	(1, 4, 5, 6, 3, 2)	3	4	4	2	u_3	u_4	
$w_3w_2w_3$	(2, 4)(3, 5)	3	4	4	2	u_4	u_3	

$w_1w_2w_1w_3$	(1, 3, 6, 4)	4	3	3	5			u_6
$w_1w_2w_3w_2$	(1, 2, 6, 5)	4	3	5	5		u_2	u_9
$w_1w_3w_2w_1$	(1, 4, 6, 3)	4	3	5	3		u_5	
$w_1w_3w_2w_3$	(1, 2, 4)(3, 6, 5)	4	3	5	3		u_5	
$w_2w_1w_3w_2$	(1, 3)(2, 5)(4, 6)	4	5	3	5	u_7		u_6
$w_2w_3w_2w_1$	(1, 5, 6, 2)	4	5	3	5	u_5		u_9
$w_2w_3w_2w_3$	(2, 5)(3, 4)	4	5	3	3	u_5		
$w_3w_2w_1w_3$	(1, 4, 2)(3, 5, 6)	4	5	5	3	u_7	u_2	
$w_1w_2w_1w_3w_2$	(1, 3, 2, 6, 4, 5)	5	4	4	6			u_6
$w_1w_2w_3w_2w_1$	(1, 6)	5	4	6	6		u_3	u_9
$w_1w_2w_3w_2w_3$	(1, 2, 6, 5)(3, 4)	5	4	6	4		u_4	
$w_1w_3w_2w_1w_3$	(1, 4)(3, 6)	5	4	6	4		u_5	
$w_2w_1w_3w_2w_1$	(1, 5, 4, 6, 2, 3)	5	6	4	6	u_4		u_8
$w_2w_1w_3w_2w_3$	(1, 3, 6, 4)(2, 5)	5	6	4	6	u_3		u_6
$w_2w_3w_2w_1w_3$	(1, 5, 6, 2)(3, 4)	5	6	4	4	u_5		
$w_3w_2w_1w_3w_2$	(1, 4, 6, 3)(2, 5)	5	6	6	4	u_7	u_1	
$w_1w_2w_1w_3w_2w_1$	(1, 6)(2, 3)(4, 5)	6	5	5	7			u_8
$w_1w_2w_1w_3w_2w_3$	(1, 3, 5)(2, 6, 4)	6	5	5	7			u_6
$w_1w_2w_3w_2w_1w_3$	(1, 6)(3, 4)	6	5	7	5		u_7	
$w_1w_3w_2w_1w_3w_2$	(1, 4, 5)(2, 6, 3)	6	5	7	5		u_4	
$w_2w_1w_3w_2w_1w_3$	(1, 5, 4)(2, 3, 6)	6	7	5	7	u_2		$\boldsymbol{u_8}$
$w_2w_3w_2w_1w_3w_2$	(1, 5, 3)(2, 4, 6)	6	7	5	5	u_4		
$w_3w_2w_1w_3w_2w_3$	(1, 4)(2, 5)(3, 6)	6	7	7	5	u_3	u_1	
$w_1w_2w_1w_3w_2w_1w_3$	(1, 6)(2, 3, 5, 4)	7	6	6	8			u_8
$w_1w_2w_3w_2w_1w_3w_2$	(1, 6)(2, 4, 5, 3)	7	6	8	6		u_2	

$w_1w_3w_2w_1w_3w_2w_3$	(1, 4, 2, 6, 3, 5)	7	6	8	6		u_2	
$w_2w_1w_3w_2w_1w_3w_2$	(1, 5)(2, 6)	7	8	6	8	u_1		u_9
$w_2w_3w_2w_1w_3w_2w_3$	(1, 5, 3, 6, 2, 4)	7	8	6	6	u_2		
$w_1w_2w_1w_3w_2w_1w_3w_2$	(1, 6)(2, 5)	8	7	7	9			u_9
$w_1w_2w_3w_2w_1w_3w_2w_3$	(1, 6)(2, 4)(3, 5)	8	7	9	7		u_3	
$w_2w_1w_3w_2w_1w_3w_2w_3$	(1, 5)(2, 6)(3, 4)	8	9	7	7	u_1		
$w_1w_2w_1w_3w_2w_1w_3w_2w_3$	(1, 6)(2, 5)(3, 4)	9	8	8	8			

(c) *For all* w_i $(i=1, 2, 3)$ *we have* $w_iBw_i \neq B$.

PROOF. It follows from (3.5) and (3.6).

(d) *The group* G_0 *is a group with a* (B, N)*-pair in the sense of Tits* [8].

PROOF. It follows from (a), (b), (c) and (3.5).

(e) *The group* G_0 *is a simple group.*

PROOF. Let B_0 be the intersection of all the conjugates of B in G_0. Assume that $B_0 \neq (1)$. Since B_0 is a normal subgroup of G_0, B_0 contains some element of $Z(B) = \langle u_5, u_6 \rangle$. On the other hand since

$$u_5^{w_1w_2w_3w_2} = u_4^{w_1w_2w_3} = u_2^{w_1w_2} = u_1^{w_1} \notin B$$

$$u_6^{w_3w_2w_1} = u_8^{w_3w_2} = u_9^{w_3} \notin B$$

$$(u_5u_6)^{w_3w_2w_1} = (u_5u_8)^{w_3w_2} = (u_4u_9)^{w_3} = u_2(u_9^{w_3}) \notin B$$

by (3.6), we have a contradiction. This implies that

$$B_0 = \bigcap_{g \in G_0} B^g = (1) \tag{3.7}$$

Let G_1 be the subgroup of G_0 generated by all the conjugates of B in G_0. It follows from (3.3) and (3.5) that $\langle w_1, u_1 \rangle$, $\langle w_2, u_3 \rangle$ and $\langle w_3, u_9 \rangle$ are isomorphic to S_3. Thus we have $w_1 \sim u_1$, $w_2 \sim u_3$, $w_3 \sim u_9$ and hence G_1 contains B and W. This yields $G_0 = G_1 = \langle B^g ; g \in G_0 \rangle = \langle B, W \rangle$. Since G_1' contains u_2, u_4, u_5, u_6 by (3.1), it follows from (3.4) that $G_1' \supset B$. By the definition of G_1 we must have

$$G_0 = G_1 = G_1' \tag{3.8}$$

Since the set of distinguished generators $\{w_1, w_2, w_3\}$ of W_0 is not the union of two commuting proper subsets by (3.3), (d) implies that we can apply the result of Tits [**9**]. Therefore every normal subgroup of G_0 is contained in B_0 or contains G_1'. (3.7) and (3.8) yield the simplicity of the group G_0. This proves (e).

(f) *The group* G_0 *is isomorphic to* $S_p(6, 2)$.

PROOF. By the correspondences (3.2) and (3.4) we get the homomorphism from G_0 onto $S_p(6, 2)$. Thus it follows from (e) that the kernel of this homomorphism is trivial and so G_0 is isomorphic to $S_p(6, 2)$. The proof of our proposition is complete.

By the similar argument as above we can prove the following proposition.

PROPOSITION 3. *Let K be a finite group generated by the elements r_1, r_2, a_1, a_2 subject to the following relations:*

$$\begin{aligned} &r_i^2 = a_j^2 = (r_i a_i)^3 = (r_1 r_2)^3 = 1 \qquad \text{for } 1 \leqq i, j \leqq 2 \\ &r_1 a_2 r_1 = r_2 a_1 r_2 = (a_1 a_2)^2 . \end{aligned} \tag{3.9}$$

Then the group K is isomorphic to $GL(3, 2)$.

PROOF. We put $B = \langle a_1, a_2 \rangle$, $N = W = \langle r_1, r_2 \rangle$ and $a_3 = (a_1 a_2)^2$.

(a) *The set of elements in $K_i = B \cup B r_i B$ $(i = 1, 2)$ is a subgroup of K.*

PROOF. It follows from (3.9).

(b) *For any i and $r \in W$, if $l(r_i r) \geqq l(r)$, then $r_i B r \subset B r_i r B$.*

PROOF. By (3.9) we may identify r_1, r_2 with the elements (1, 2), (2, 3) in S_3 respectively. By (3.9) we need only to see that $r_i a_i r \in B r_i r B$ for $i = 1, 2$. It is easily verified that for those $r \in W$ such that $l(r_i r) \geqq l(r)$, we can always write $r_i a_i r = r_i r Y_i$ with $Y_i \in B$. The computations are summarized in the following table which is self-explanatory.

r		$l(r)$	$l(r_1 r)$	$l(r_2 r)$	Y_1	Y_2
r_1	(1, 2)	1	0	2		a_3
r_2	(2, 3)	1	2	0	a_3	
$r_1 r_2$	(1, 2, 3)	2	1	3		a_1
$r_2 r_1$	(1, 3, 2)	2	3	1	a_2	
$r_1 r_2 r_1$	(1, 3)	3	2	2		

(c) *For all r_i $(i = 1, 2)$ we have $r_i B r_i \neq B$.*

PROOF. It follows from (3.9).

(d) *The group K is a group with a (B, N)-pair in the sense of Tits* [8].

PROOF. The result follows from (a), (b), (c) and (3.9).

(e) *The group K is a simple group.*

PROOF. Let B_0 be the intersection of all conjugates of B in K. Assume that $B_0 \neq (1)$. Since B_0 is a normal subgroup of K, B_0 contains $Z(B) = \langle a_3 \rangle$. On the other hand since

$$a_3^{r_2 r_1} = a_2^{r_2} \notin B$$

by (3.9) we have a contradiction. This implies that

$$B_0 = \bigcap_{g \in K} B^g = (1)\,. \tag{3.10}$$

Let K_1 be the subgroup of K generated by all conjugates of B in K. It follows from (3.9) that $r_1 \sim a_1$ and $r_2 \sim a_2$. Hence K_1 contains B and W. This implies that $K = K_1 = \langle B^g\,;\, g \in K \rangle = \langle B, W \rangle$. Since K_1' contains a_3 by (3.9), we must have $K_1' \supset B$ and then by the definition of K_1 we get

$$K = K_1 = K_1'\,. \tag{3.11}$$

By (3.10), (3.11) and Tits [**9**], K is a simple group.

(f) *The group K is isomorphic to $GL(3, 2)$.*

PROOF. By the correspondence

$$r_1 \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad r_2 \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$a_1 \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad a_2 \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

we get the homomorphism from K onto $GL(3, 2)$. It follows from (e) that K is isomorphic to $GL(3, 2)$. The proof is complete.

§ 4. Conjugacy classes of involutions of G and the case $G > O^2(G)$.

In this section we summarize the result in our previous paper [**11**]. But our proof in [**11**] is a little insufficient and so we give some additional explanations in Remark 1 below.

LEMMA 1. *Assume that $N_G(S) > N_H(S)$. Then the group G posesses precisely three or four conjugacy classes of involutions. If notations are chosen suitably, the possibilities for the fusion of the conjugacy classes of involutions of G are*

$$\text{Case I} \begin{cases} \pi_1 \sim \pi_1' \\ \pi_1\pi_2 \sim \pi_1\pi_2' \sim \pi_1'\pi_2' \\ \alpha \sim \alpha\pi_1' \sim \alpha\pi_1'\pi_2' \sim \alpha\alpha' \sim \alpha' \end{cases}$$

$$\text{Case II} \begin{cases} \pi_1 \sim \pi_1' \sim \pi_1'\pi_2' \\ \pi_1\pi_2 \sim \alpha\alpha' \\ \alpha \sim \alpha\pi_1' \\ \alpha' \sim \pi_1\pi_2' \sim \alpha\pi_1'\pi_2'\,. \end{cases}$$

REMARK 1 (cf. [11] p. 681 Lemma 9). If we replace π'_3 with $\pi_3\pi'_3$ the same relations still hold in H. Moreover this replacement is independent of the replacement π'_1, σ, π'_2, σ', π'_3 with $\alpha\pi'_1$, $\alpha\sigma$, $\alpha\pi'_2$, $\alpha\sigma'$, $\alpha\pi'_3$ in this order (cf. [11] Lemma 7). Thus we may assume that $\alpha'\sim\pi_1\pi'_3\sim\alpha\pi'_1\pi'_2$ and so $\pi_1\pi_2\sim\alpha\alpha'$.

REMARK 2. In [11] our assumption is that $G=O^2(G)$. From this we can easily deduce the condition $N_G(S)>N_H(S)$ and for the proof of Lemma 1 we only used this condition. In both cases of Lemma 1 focal subgroup theorem [4] implies that $G=O^2(G)$. Hence if $G>O^2(G)$, then we must have $N_G(S)=N_H(S)$. Since S is weakly closed in D with respect to G and is an abelian subgroup, Burnside's argument implies that if two elements of S are conjugate in G, they are conjugate in $N_G(S)$. Also it follows from the structure of H that every involution is conjugate to an element of S. Thus $N_G(S)=N_H(S)$ implies that $ccl_G(\alpha)\cap D=\{\alpha\}$. By the theorem of Glauberman [3] we have $\alpha\in Z(G \bmod. O_{2'}(G))$ and then $G\triangleright\langle\alpha\rangle O_{2'}(G)$. Frattini argument yields

$$G=C_G(\alpha)O_{2'}(G).$$

Since $C_G(\alpha)\cap O_{2'}(G)=(1)$, α acts fixed-point-freely on $O_{2'}(G)$ and so $O_{2'}(G)$ is an abelian group by a theorem of Zassenhaus [12]. This proves the case (iii) of our theorem.

REMARK 3. In [11] we deduced $G\cong A_{12}$ or A_{13} in the Case I (cf. Kondo [5]). Thus in the following *we shall assume that the fusion of the conjugacy classes of involutions is as in the Case II.*

DEFINITION. We call the representatives π_1, $\pi_1\pi_2$, α, α', *canonical representatives* of the conjugacy classes of involutions.

LEMMA 2. (i) $N_G(D)=D$

(ii) *The group* $S\langle\mu\rangle$ *is conjugate to* $S\langle\tau\rangle$ *in* $N_G(S)$

(iii) *The extension of* $N_G(S)$ *over* S *splits and* $N_G(S)/S$ *is isomorphic to* $GL(3, 2)$.

PROOF. See Lemmas 3, 9 and p. 682 in [11].

LEMMA 3. *If* α^x *is in* D *for some element* x *in* G, *then* α^x *is in* S. *More precisely we have the following*:

$$ccl_G(\alpha)\cap D=\{\alpha, \alpha\pi'_1, \alpha\pi'_2, \alpha\pi'_3, \pi'_1\pi_2\pi_3, \pi_1\pi'_2\pi_3, \pi_1\pi_2\pi'_3\}.$$

PROOF. Since $ccl_H(\alpha\pi'_1)\cap D\subset S$ by the structure of H, the result follows immediately from Lemma 1.

§ 5. Construction of a group isomorphic to $S_p(6, 2)$.

The main purpose of this section is to prove Lemma 4 and to show the existence of the subgroup G_0 of G isomorphic to $S_p(6, 2)$.

LEMMA 4. *There exists an one to one correspondence between* $C_G(\alpha)\cup N_G(S)$

and $(C(\hat{\alpha}) \cup N(\hat{S})) \cap S_p(6, 2)$ inducing isomorphisms of $C_G(\alpha)$ with $C(\hat{\alpha}) \cap S_p(6, 2)$ and $N_G(S)$ with $N(\hat{S}) \cap S_p(6, 2)$.

To prove this lemma we need several lemmas.

LEMMA 5. *There exists an element $\tilde{w}_1$ in $N_G(S)$ such that the action of $\tilde{w}_1$ on S is given as follows:*

$$\pi_1^{\tilde{w}_1} = \pi_1'\pi_2', \qquad \pi_2^{\tilde{w}_1} = \pi_1\pi_1'\pi_2\pi_2', \qquad \pi_3^{\tilde{w}_1} = \pi_3\pi_3',$$
$$\pi_3'^{\tilde{w}_1} = \pi_3', \qquad \pi_1'^{\tilde{w}_1} = \pi_1'\pi_3', \qquad \pi_2'^{\tilde{w}_1} = \pi_1\pi_1'\pi_3'.$$

PROOF. By Lemma 2 (ii), $(S\langle\mu\rangle)^x = S\langle\tau\rangle$ for some element x in $N_G(S)$. Thus $(S\langle\mu\rangle)'^x = (S\langle\tau\rangle)'$ and $Z(S\langle\mu\rangle)^x = Z(S\langle\tau\rangle)$. Since $(S\langle\mu\rangle)' = \langle\pi_1, \pi_2\rangle$ and $(S\langle\tau\rangle)' = \langle\pi_1\pi_2, \pi_1'\pi_2'\rangle$ we may assume that

$$\pi_1^x = \pi_1'\pi_2', \qquad \pi_2^x = \pi_1\pi_1'\pi_2\pi_2', \qquad (\pi_1\pi_2)^x = \pi_1\pi_2$$

by Lemma 1. It follows from $[x, \alpha] \neq 1$ that $[x, \pi_3] \neq 1$. Because

$$ccl_G(\pi_1) \cap Z(S\langle\mu\rangle) = \{\pi_1, \pi_2, \pi_3, \pi_3', \pi_3\pi_3'\}$$

and

$$ccl_G(\pi_1) \cap Z(S\langle\tau\rangle) = \{\pi_1'\pi_2', \pi_1\pi_1'\pi_2\pi_2', \pi_3, \pi_3', \pi_3\pi_3'\}$$

we must have $\pi_3^x = \pi_3'$ or $\pi_3\pi_3'$. If $\pi_3^x = \pi_3'$, then $(\pi_1\pi_3)^x = \alpha'$ which is impossible by Lemma 1. Hence we have

$$\pi_3^x = \pi_3\pi_3' \qquad \text{and} \qquad \pi_3'^x = \pi_3' \qquad \text{or} \qquad \pi_3'^x = \pi_3.$$

Now $(\pi_1'\pi_2\pi_3)^x = \pi_1'^x\alpha\alpha'$ and then by Lemma 3 and the following table which is self-explanatory

$(\pi_1'\pi_2\pi_3)^x$	$\pi_1'^x$	$(\pi_1'\pi_3)^x$
α	α'	
$\alpha\pi_1'$	$\pi_2'\pi_3'$	$\pi_2'\pi_3$
$\alpha\pi_2'$	$\pi_1'\pi_3'$	$\pi_1'\pi_3$
$\alpha\pi_3'$	$\pi_1'\pi_2'$	$\alpha'\pi_3$
$\pi_1'\pi_2\pi_3$	$\pi_1\pi_2'\pi_3'$	
$\pi_1\pi_2'\pi_3$	$\pi_1'\pi_2\pi_3'$	
$\pi_1\pi_2\pi_3'$	$\pi_1'\pi_2'\pi_3$	

we have

$$\pi_1'^x = \pi_1'\pi_3' \qquad \text{or} \qquad \pi_1'^x = \pi_2'\pi_3'.$$

Similarly by the following table

$(\pi_1\pi_2'\pi_3)^x$	$\pi_2'^x$	$(\pi_2'\pi_3)^x$
α	$\alpha'\pi_1\pi_2$	
$\alpha\pi_1'$	$\pi_1\pi_2\pi_2'\pi_3'$	
$\alpha\pi_2'$	$\pi_1\pi_1'\pi_2\pi_3'$	
$\alpha\pi_3'$	$\pi_1\pi_1'\pi_2\pi_2'$	$\alpha\alpha'$
$\pi_1'\pi_2\pi_3$	$\pi_2\pi_2'\pi_3'$	$\pi_2\pi_2'\pi_3$
$\pi_1\pi_2'\pi_3$	$\pi_1\pi_1'\pi_3'$	$\pi_1\pi_1'\pi_3$
$\pi_1\pi_2\pi_3'$	$\alpha\pi_1'\pi_2$	

we have

$$\pi_2'^x = \pi_2\pi_2'\pi_3' \quad \text{or} \quad \pi_2'^x = \pi_1\pi_1'\pi_3'.$$

Assume that $\pi_3'^x = \pi_3'$. If $\pi_1'^x = \pi_2'\pi_3'$ and $\pi_2'^x = \pi_1\pi_1'\pi_3'$, then $\alpha'^x = \alpha'\pi_1 \sim \alpha\alpha'$ which is impossible by Lemma 1. If $\pi_1'^x = \pi_1'\pi_3'$ and $\pi_2'^x = \pi_2\pi_2'\pi_3'$, then $\alpha'^x = \alpha'\pi_2 \sim \alpha\alpha'$ which is also impossible. Thus we have

$$\begin{cases} \pi_1'^x = \pi_2'\pi_3' \\ \pi_2'^x = \pi_2\pi_2'\pi_3' \end{cases} \quad \text{or} \quad \begin{cases} \pi_1'^x = \pi_1'\pi_3' \\ \pi_2'^x = \pi_1\pi_1'\pi_3'. \end{cases}$$

Assume that $\pi_3'^x = \pi_3$. If $\pi_1'^x = \pi_2'\pi_3'$ and $\pi_2'^x = \pi_2\pi_2'\pi_3'$, then $\alpha'^x = \pi_2\pi_3$ which is impossible by Lemma 1. If $\pi_1'^x = \pi_1'\pi_3'$ and $\pi_2'^x = \pi_1\pi_1'\pi_3'$, then $\alpha'^x = \pi_1\pi_3$ which is also impossible. Thus we have

$$\begin{cases} \pi_1'^x = \pi_1'\pi_3' \\ \pi_2'^x = \pi_2\pi_2'\pi_3' \end{cases} \quad \text{or} \quad \begin{cases} \pi_1'^x = \pi_2'\pi_3' \\ \pi_2'^x = \pi_1\pi_1'\pi_3'. \end{cases}$$

Hence we have the following four possibilities of the action of the element x on S.

	I	II	III	IV
π_1^x	$\pi_1'\pi_2'$			
π_2^x	$\pi_1\pi_1'\pi_2\pi_2'$			
π_3^x	$\pi_3\pi_3'$			
$\pi_1'^x$	$\pi_1'\pi_3'$	$\pi_2'\pi_3'$	$\pi_2'\pi_3'$	$\pi_1'\pi_3'$
$\pi_2'^x$	$\pi_1\pi_1'\pi_3'$	$\pi_2\pi_2'\pi_3'$	$\pi_1\pi_1'\pi_3'$	$\pi_2\pi_2'\pi_3'$
$\pi_3'^x$	π_3'	π_3'	π_3	π_3
$\tilde{w}_1$	x	τx	$x\mu'$	$x\mu\mu'$

For the cases I, II, III, IV, put $\tilde{w}_1 = x,\ \tau x,\ x\mu',\ x\mu\mu'$, respectively. This proves our lemma.

LEMMA 6. *Let K be a complement of S in $N_G(S)$. Then we may assume that*

$$K = \langle \mu\pi_1\pi_2,\ \mu'\pi_1,\ \xi\pi_2'\pi_3',\ \tau,\ w_1^*\rangle,\ \tau^{w_1^*} = \mu\pi_1\pi_2$$

or

$$K = \langle \mu\pi_2\pi_3,\ \mu'\alpha,\ \xi\pi_2'\pi_3',\ \tau\alpha\alpha',\ w_1^*\rangle,\ (\tau\alpha\alpha')^{w_1^*} = \mu\pi_2\pi_3\,,$$

and in both cases

$$w_1^{*2} = 1, \qquad \tilde{w}_1 \equiv w_1^* \ (\text{mod.}\, S)\,.$$

PROOF. It is easily verified that the complement of S in $N_H(S)$ is conjugate to one of the following groups:

$$\langle \mu\pi_1\pi_2,\ \mu'\pi_1,\ \xi\pi_2'\pi_3',\ \tau\rangle$$

$$\langle \mu\pi_1\pi_2,\ \mu'\pi_1,\ \xi\pi_2'\pi_3',\ \tau\alpha\rangle$$

$$\langle \mu\pi_2\pi_3,\ \mu'\alpha,\ \xi\pi_2'\pi_3',\ \tau\alpha'\rangle$$

$$\langle \mu\pi_2\pi_3,\ \mu'\alpha,\ \xi\pi_2'\pi_3',\ \tau\alpha\alpha'\rangle\,.$$

Since K is isomorphic to $GL(3, 2)$, every involution in K is conjugate and hence $\tau\alpha \not\sim \mu\pi_1\pi_2$, $\tau\alpha' \not\sim \mu\pi_2\pi_3$ imply that we have two possibilities of K as stated in our lemma. Since $K \cap S = (1)$ and $C_G(S) = S$, there exists an element w_1^* in K such that $w_1^{*2} = 1$ and $w_1^* \equiv \tilde{w}_1$ (mod. S). By the maximality of $N_H(S)$ in $N_G(S)$ we get $N_G(S) = \langle w_1^*, N_H(S)\rangle$. By Lemma 5 we have $(S\langle\mu\rangle)^{\tilde{w}_1} = S\langle\tau\rangle$ and hence $\tau^{\tilde{w}_1^{-1}} = \mu,\ \mu\pi_1,\ \mu\pi_2,\ \mu\pi_1\pi_2$ by Lemma 1. Now $\tau^{w_1^*} \in K$ or $(\tau\alpha\alpha')^{w_1^*} \in K$ yields our result.

LEMMA 7. *There exists an isomorphism from $N_G(S)$ onto $N(\hat{S}) \cap S_p(6, 2)$ such that the restriction to $N_H(S)$ is that of θ defined in section* 2.

PROOF. The actions of the elements w_1^*, $\tau\xi^2$, $\mu\mu'$, τ on S by conjugation are given by the following table:

	w_1^*	$\tau\xi^2$	$\mu\mu'$	τ
π_1	$\pi_1'\pi_2'$	π_3	π_1	π_2
π_1'	$\pi_1'\pi_3'$	π_3'	π_1'	π_2'
π_2	$\pi_1\pi_1'\pi_2\pi_2'$	π_2	π_2	π_1
π_2'	$\pi_1\pi_1'\pi_3'$	π_2'	$\pi_2\pi_2'$	π_1'
π_3	$\pi_3\pi_3'$	π_1	π_3	π_3
π_3'	π_3'	π_1'	$\pi_3\pi_3'$	π_3'

(5.1)

Assume that $K = \langle \mu\pi_1\pi_2,\ \mu'\pi_1,\ \xi\pi_2'\pi_3',\ \tau,\ w_1^*\rangle$. We put

$$r_1=\left(\begin{array}{ccc|ccc} & 1 & & 1 & 1 & 1\\ 1 & & & 1 & 1 & 1\\ & & 1 & & 1 & 1\\ \hline & & & 1 & & \\ & & & & & 1\\ & & & & 1 & \end{array}\right)\qquad r_2=\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & & 1 & & 1 & \\ & 1 & & 1 & & \\ \hline & & & & 1 & \\ & & & 1 & & \\ & & & & & 1\end{array}\right)$$

$$a_1=\left(\begin{array}{ccc|ccc} 1 & 1 & & & 1 & 1\\ & 1 & & & & 1\\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & 1 & 1\\ & & & & & 1\end{array}\right)\qquad a_2=\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & 1 & 1 & 1 & \\ & & 1 & & 1 & \\ \hline & & & 1 & 1 & \\ & & & & 1 & \\ & & & & & 1\end{array}\right)$$

Then r_1, r_2, a_1, a_2 satisfy the relations of Proposition 3 and so $\langle r_1, r_2, a_1, a_2\rangle$ is isomorphic to $GL(3, 2)$. By (5.1), the result follows from the correspondence

$$w_1^* \longrightarrow r_1\,,\qquad \tau(\xi\pi_2'\pi_3')^2 \longrightarrow r_2\,,$$
$$\mu\mu'\pi_2 \longrightarrow a_1\,,\qquad \tau \longrightarrow a_2\,.$$

Assume that $K=\langle \mu\pi_2\pi_3,\ \mu'\alpha,\ \xi\pi_2'\pi_3',\ \tau\alpha\alpha',\ w_1^*\rangle$. We put

$$r_1=\left(\begin{array}{ccc|ccc} & 1 & & & & \\ 1 & & & & & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & & 1\\ & & & & 1 & \end{array}\right)\qquad r_2=\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & & 1 & & & \\ & 1 & & & & \\ \hline & & & & 1 & \\ & & & 1 & & \\ & & & & & 1\end{array}\right)$$

$$a_1=\left(\begin{array}{ccc|ccc} 1 & 1 & & & & \\ & 1 & & & & \\ & & 1 & & & \\ \hline & & & 1 & & \\ & & & & 1 & 1\\ & & & & & 1\end{array}\right)\qquad a_2=\left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & 1 & & & \\ & & 1 & & & \\ \hline & & & 1 & 1 & \\ & & & & 1 & \\ & & & & & 1\end{array}\right) \tag{5.2}$$

Then Proposition 3 implies that $\langle r_1, r_2, a_1, a_2\rangle$ is isomorphic to $GL(3, 2)$. By (5.1) the result follows from the correspondence

$$w_1^* \longrightarrow r_1\,,\qquad (\tau\alpha\alpha')(\xi\pi_2'\pi_3')^2 \longrightarrow r_2\,,$$
$$\mu\mu'\pi_1 \longrightarrow a_1\,,\qquad \tau\alpha\alpha' \longrightarrow a_2\,.$$

The proof is complete. We note that Lemma 7 yields Lemma 4.

DEFINITION. Let w_1, w_2 be the inverse images of r_1, r_2, respectively given in (5.2). We put $w_3=(\alpha\pi_1')^{\sigma}$. Thus $w_2=\tau\xi^2\alpha\pi_2'$.

LEMMA 8.

$$w_1^2=w_2^2=w_3^2=(w_1w_2)^3=(w_2w_3)^4=1\,.$$

PROOF. Since $w_1,\ w_2\in N_G(S)$ and $w_2,\ w_3\in H=C_G(\alpha)$, the result follows immediately.

LEMMA 9.

$$(w_1w_3)^2\in\langle\pi_1\pi_2,\ \pi_3,\ \pi_3'\rangle\,.$$

PROOF. By definition, we have $w_1w_3\in C_G(\pi_1\pi_2,\ \pi_3')$. Since

$$\alpha^{w_1w_3w_1w_3}=\alpha^{w_1w_3w_1}=(\alpha\pi_3')^{w_1w_3}=(\alpha\pi_3')^{w_1}=\alpha$$

$$(\pi_2\pi_3)^{w_1w_3w_1w_3}=(\mu\pi_2\pi_3)^{w_1w_3w_1}=(\tau\alpha\alpha')^{w_1w_3}=(\alpha\alpha')^{w_1}=\pi_2\pi_3\,,$$

we have

$$(w_1w_3)^2\in C_G(\pi_1\pi_2,\ \pi_3',\ \alpha,\ \pi_2\pi_3)=C_G(\pi_1,\ \pi_2,\ \pi_3,\ \pi_3')=S\langle\mu\rangle\,.$$

Moreover

$$(\mu\pi_2\pi_3)^{w_1w_3w_1w_3}=(\pi_2\pi_3)^{w_1w_3w_1}=(\alpha\alpha')^{w_1w_3}=(\tau\alpha\alpha')^{w_1}=\mu\pi_2\pi_3\,,$$

$$(\tau\alpha\alpha')^{w_1w_3w_1w_3}=(\alpha\alpha')^{w_1w_3w_1}=(\pi_2\pi_3)^{w_1w_3}=(\mu\pi_2\pi_3)^{w_1}=\tau\alpha\alpha'\,,$$

$$(\alpha\alpha')^{w_1w_3w_1w_3}=(\tau\alpha\alpha')^{w_1w_3w_1}=(\mu\pi_2\pi_3)^{w_1w_3}=(\pi_2\pi_3)^{w_1}=\alpha\alpha'$$

and thus we have

$$(w_1w_3)^2\in C(\mu\pi_2\pi_3,\ \tau\alpha\alpha',\ \alpha\alpha')\cap S\langle\mu\rangle=\langle\pi_1\pi_2,\ \pi_3,\ \pi_3'\rangle\,.$$

This proves our lemma.

LEMMA 10. $(w_1w_3)^2=1$, *that is*, $[w_1,\ w_3]=1$.

PROOF. It follows from Lemma 9 that $(\alpha\pi_1')^{w_3w_1w_3w_1}=\alpha\pi_1'$ and so

$$(\alpha\pi_1')^{w_3w_1w_3}=\alpha\pi_1'\,.$$

This implies that

$$(\alpha\pi_1')^{w_1w_3}=(\alpha\pi_1')^{w_3}$$

and then

$$[(\alpha\pi_1')^{w_3},\ w_1]=[w_3\alpha\pi_1'w_3,\ w_1]=1\,.$$

Since $(w_3\alpha\pi_1')^3=1$ and $[w_1,\ \alpha\pi_1']=1$, we must have

$$[w_1,\ w_3]=1\,.$$

This proves our lemma.

LEMMA 11. *Let G_0 be a subgroup of G generated by the elements w_1, w_2, w_3 and the group D. Then G_0 is isomorphic to $S_p(6, 2)$. Moreover G_0 is generated by $N_G(S)$ and $H=C_G(\alpha)$.*

PROOF. We put

$$u_1=\mu\mu'\pi_1\,,\qquad u_2=\mu\pi_2\pi_3\,,\qquad u_3=\tau\alpha\alpha'\,,$$
$$u_4=\pi_2\pi_3\,,\qquad u_5=\pi_1\pi_2\,,\qquad u_6=\alpha\,,$$
$$u_7=\alpha\alpha'\,,\qquad u_8=\alpha\pi_3'\,,\qquad u_9=\alpha\pi_1'\,,$$

and

$$W_0=\langle w_1,\ w_2,\ w_3\rangle,\qquad D_0=D\,.$$

Then the group G_0 satisfies the conditions (3.1), (3.3), (3.5), (3.6) of Proposition 2 by the results of section 2 and Lemmas 5, 8, 10. Hence G_0 is isomorphic to $S_p(6, 2)$. The second assertion is trivial.

§ 6. Final characterization.

In this section we determine the structure of the centralizers of involutions and identify G with G_0 by the result of Suzuki [**6**].

LEMMA 12. *$И_H(S;2')$ is trivial.*

PROOF. Assume that $R\in И_H(S;2')$. Since $\langle\pi_1'\pi_2',\ \pi_2'\pi_3'\rangle$ normalizes R and is isomorphic to $Z_2\times Z_2$, Brauer-Wielandt's theorem [**10**] implies that

$$R=C_R(\pi_1'\pi_2')C_R(\pi_2'\pi_3')C_R(\pi_1'\pi_3')\,.$$

On the other hand it follows from the structure of H that $C_H(\pi_1'\pi_2')$ is a 2-group and so $C_R(\pi_1'\pi_2')=(1)$. Since $\pi_1'\pi_2'\sim\pi_2'\pi_3'\sim\pi_1'\pi_3'$ in H we have $C_R(\pi_2'\pi_3')=C_R(\pi_1'\pi_3')=(1)$ and hence $R=(1)$. This proves our lemma.

LEMMA 13. *$И_G(S;2')$ is trivial.*

PROOF. Assume that $R\in И_G(S;2')$. Since six four-groups $\langle\alpha,\ \alpha\pi_1'\rangle$, $\langle\alpha,\ \alpha\pi_2'\rangle$, $\langle\alpha,\ \alpha\pi_3'\rangle$, $\langle\alpha,\ \pi_1'\pi_2\pi_3\rangle$, $\langle\alpha,\ \pi_1\pi_2'\pi_3\rangle$ and $\langle\alpha,\ \pi_1\pi_2\pi_3'\rangle$ of S act on R, a theorem of Brauer-Wielandt [**10**] implies that

$$\begin{aligned}R&=C_R(\alpha)C_R(\alpha\pi_1')C_R(\pi_1')\\&=C_R(\alpha)C_R(\alpha\pi_2')C_R(\pi_2')\\&=C_R(\alpha)C_R(\alpha\pi_3')C_R(\pi_3')\\&=C_R(\alpha)C_R(\pi_1'\pi_2\pi_3)C_R(\pi_1\pi_1')\\&=C_R(\alpha)C_R(\pi_1\pi_2'\pi_3)C_R(\pi_2\pi_2')\\&=C_R(\alpha)C_R(\pi_1\pi_2\pi_3')C_R(\pi_3\pi_3')\,.\end{aligned}$$

Now $\alpha\sim\alpha\pi_1'\sim\alpha\pi_2'\sim\alpha\pi_3'\sim\pi_1'\pi_2\pi_3\sim\pi_1\pi_2'\pi_3\sim\pi_1\pi_2\pi_3'$ by Lemma 3 and therefore Lemma 12 implies that

$$\begin{aligned}C_R(\alpha)&=C_R(\alpha\pi_1')=C_R(\alpha\pi_2')=C_R(\pi_1'\pi_2\pi_3)\\&=C_R(\pi_1\pi_2'\pi_3)=C_R(\pi_1\pi_2\pi_3')=(1)\,.\end{aligned}$$

Thus we have

$$R = C_R(\pi_1') = C_R(\pi_2') = C_R(\pi_3')$$
$$= C_R(\pi_1\pi_1') = C_R(\pi_2\pi_2') = C_R(\pi_3\pi_3') .$$

Since $S = \langle \pi_1', \pi_2', \pi_3', \pi_1\pi_1', \pi_2\pi_2', \pi_3\pi_3' \rangle$ we get $R \subset C_G(S) = S$. It follows from $R \cap S = (1)$ that $R = (1)$. The proof is complete.

LEMMA 14. $C_G(\pi_3) = D\langle \rho \rangle$.

PROOF. Since π_3 is a central involution of D, D is a Sylow 2-subgroup of $C_G(\pi_3)$. Assume that $\alpha^x \in D$ for some element x in $C_G(\pi_3)$. Since $\alpha^x = (\pi_1\pi_2)^x\pi_3$, $(\pi_1\pi_2)^x = \alpha^x\pi_3$. It follows from Lemma 3 that we have the following table and thus $\alpha^x = \alpha$.

α^x	$(\pi_1\pi_2)^x$	
α	$\pi_1\pi_2$	$\pi_1\pi_2$
$\alpha\pi_1'$	$\pi_1\pi_1'\pi_2$	α'
$\alpha\pi_2'$	$\pi_1\pi_2\pi_2'$	α'
$\alpha\pi_3'$	$\pi_1\pi_2\pi_3'$	α
$\pi_1\pi_2\pi_3'$	$\alpha\pi_3'$	α
$\pi_1\pi_2'\pi_3$	$\pi_1\pi_2'$	α'
$\pi_1'\pi_2\pi_3$	$\pi_1'\pi_2$	α'

In the table above the first, second and third entries of the column give respectively, the posibilities of α^x, $(\pi_1\pi_2)^x$ and the canonical representative of $ccl_G((\pi_1\pi_2)^x)$. Hence applying a theorem of Glauberman [3] we have $\alpha \in Z(C_G(\pi_3) \bmod. O_{2'}(C_G(\pi_3)))$ and then $C_G(\pi_3) \triangleright \langle \alpha \rangle O_{2'}(C_G(\pi_3))$. Frattini argument yields $C_G(\pi_3) = C_G(\pi_3, \alpha)O_{2'}(C_G(\pi_3))$. Since $S \subset C_G(\pi_3)$, it follows from Lemma 13 that $O_{2'}(C_G(\pi_3)) = (1)$ and therefore

$$C_G(\pi_3) = C_G(\pi_3, \alpha) = D\langle \rho \rangle .$$

This proves our lemma.

LEMMA 15. $C_G(\alpha') = S\langle \xi, \tau \rangle$.

PROOF. Since $Z(D) = \langle \pi_1\pi_2, \pi_3 \rangle$, it follows from Lemma 1 that α' is a non-central involution in any Sylow 2-subgroup of G. On the other hand the order of the centralizer of any involution which is not conjugate to α, $\pi_1\pi_2$, π_3, is, at most, 2^7. Thus $C_G(\alpha') \supset S\langle \tau \rangle$ implies that $S\langle \tau \rangle$ is a Sylow 2-subgroup of $C_G(\alpha')$. Assume that $\alpha^x \in S\langle \tau \rangle$ for some element x in $C_G(\alpha')$. Then $(\alpha\alpha')^x = \alpha^x\alpha'$. By the following table and Lemma 3, we have $\alpha^x = \alpha$.

α^x	$\alpha^x\alpha'$	
α	$\alpha\alpha'$	$\pi_1\pi_2$
$\alpha\pi_1'$	$\alpha\pi_2'\pi_3'$	α'
$\alpha\pi_2'$	$\alpha\pi_1'\pi_3'$	α'
$\alpha\pi_3'$	$\alpha\pi_1'\pi_2'$	α'
$\pi_1\pi_2\pi_3'$	$\pi_1\pi_1'\pi_2\pi_2'$	π_1
$\pi_1\pi_2'\pi_3$	$\pi_1\pi_1'\pi_3\pi_3'$	π_1
$\pi_1'\pi_2\pi_3$	$\pi_2\pi_2'\pi_3\pi_3'$	π_1

In the table above the first, second and third entries of the column give respectively, the possibilities of α^x, $(\alpha\alpha')^x$ and the canonical representative of $ccl_G((\alpha\alpha')^x)$. Again by a theorem of Glauberman [3] and Lemma 13 we have

$$C_G(\alpha') = C_G(\alpha, \alpha') = S\langle \xi, \tau \rangle .$$

This proves our lemma.

LEMMA 16. (i) $C_G(\pi_1\pi_2, \pi_2\pi_3) = S\langle \mu, \mu' \rangle$.

(ii) $C_G(\pi_1\pi_2, \pi_1\pi_3') = C_G(\pi_1\pi_2, \pi_1\pi_3\pi_3') = S\langle \mu \rangle$.

PROOF. (i) Since $Z(D) = \langle \pi_1\pi_2, \pi_3 \rangle \not\sim \langle \pi_1\pi_2, \pi_2\pi_3 \rangle$ by Lemma 1 $S\langle \mu, \mu' \rangle$ is a Sylow 2-subgroup of $C_G(\pi_1\pi_2, \pi_2\pi_3)$. Assume that $\alpha^x \in S\langle \mu, \mu' \rangle$ for some element x in $C_G(\pi_1\pi_2, \pi_2\pi_3)$. Since $\alpha^x = \pi_1\pi_2(\pi_3^x) \in S$ by Lemma 3, we must have $\pi_3^x = \pi_3, \pi_3'$, or $\pi_3\pi_3'$. On the other hand $(\pi_1\pi_3)^x = \pi_1\pi_3$ and thus $\pi_1^x = \pi_1\pi_3(\pi_3^x)$. This yields $\pi_3^x = \pi_3$ and hence $\alpha^x = \alpha$. By Glauberman's theorem [3] and Lemma 13 we have

$$C_G(\pi_1\pi_2, \pi_2\pi_3) = C_G(\alpha, \pi_1\pi_2, \pi_2\pi_3) = S\langle \mu, \mu' \rangle .$$

(ii) Since $\alpha' \sim \pi_1\pi_3' \sim \pi_1\pi_3\pi_3'$ and $S\langle \mu \rangle \subset C_G(\pi_1\pi_2, \pi_1\pi_3') \cap C_G(\pi_1\pi_2, \pi_1\pi_3\pi_3')$, it follows from Lemma 15 that $S \lhd C_G(\pi_1\pi_2, \pi_1\pi_3') \cap C_G(\pi_1\pi_2, \pi_1\pi_3\pi_3')$. Since $C_S(\xi) = \langle \alpha, \alpha' \rangle \not\sim \langle \pi_1\pi_2, \pi_1\pi_3' \rangle$ and $\langle \alpha, \alpha' \rangle \not\sim \langle \pi_1\pi_2, \pi_1\pi_3\pi_3' \rangle$, $|C_G(\pi_1\pi_2, \pi_1\pi_3')|$ and $|C_G(\pi_1\pi_2, \pi_1\pi_3\pi_3')|$ are not divisible by 3. This implies that $C_G(\pi_1\pi_2, \pi_1\pi_3') = C_G(\pi_1\pi_2, \pi_1\pi_3\pi_3') = S\langle \mu \rangle$. The proof is complete.

LEMMA 17. *Let z be an element of order 2 in D.*

(i) *If $\pi_1 z \sim \pi_1$ and $\pi_2 z \sim \alpha$, then $z = \pi_1\pi_3, \pi_1\pi_3'$ or $\pi_1\pi_3\pi_3'$.*

(ii) *If $\pi_3 \sim z$ in $C_G(\pi_1\pi_2)$, then $z = \pi_3, \pi_3'$ or $\pi_3\pi_3'$.*

PROOF. (i) It follows from Lemma 3 that

$$\pi_2 z \in \{\alpha, \alpha\pi_1', \alpha\pi_2', \alpha\pi_3', \pi_1'\pi_2\pi_3, \pi_1\pi_2'\pi_3, \pi_1\pi_2\pi_3'\}$$

and then we have

$$z \in \{\pi_1\pi_3, \pi_1\pi_1'\pi_3, \pi_1\pi_2'\pi_3, \pi_1\pi_3\pi_3', \pi_1'\pi_3, \alpha\pi_2', \pi_1\pi_3'\} .$$

On the other hand since $\pi_1 z \sim \pi_1$, we have $z = \pi_1\pi_3$, $\pi_1\pi'_3$ or $\pi_1\pi_3\pi'_3$ by Lemma 1.

(ii) By our assumption, $\alpha = \pi_1\pi_2\pi_3 \sim \pi_1\pi_2 z \in S\langle \mu, \mu' \rangle$. Lemma 1 yields our result.

LEMMA 18. $C_G(\pi_1\pi_2) \triangleright \langle \pi_3, \pi'_3 \rangle$.

PROOF. Since $\pi_3 \nsim \pi_1$ in $C_G(\pi_1\pi_2)$ by Lemma 1, $\langle \pi_3^x, \pi_1 \rangle$ is a dihedral group with non-trivial center for any x in $C_G(\pi_1\pi_2)$ (cf. Brauer-Fowler [2]). We put $Z(\langle \pi_3^x, \pi_1 \rangle) = \langle a(x) \rangle$. Then $\pi_3^x \sim \pi_1 a(x)$ or $\pi_1 \sim \pi_1 a(x)$. Since $[a(x), \pi_1\pi_2] = [a(x), \pi_1] = 1$ we have $[a(x), \pi_2] = 1$. Lemma 14 implies that

$$a(x) \in C_G(\pi_1) \cap C_G(\pi_2) = S\langle \mu, \mu' \rangle .$$

Assume that $\pi_3^x \sim \pi_1 a(x)$. Since $\alpha^x = \pi_1\pi_2(\pi_3^x) \sim \pi_1\pi_2(\pi_1 a(x)) = \pi_2 a(x)$, it follows from Lemma 17 that $a(x) = \pi_1\pi_3$, $\pi_1\pi'_3$ or $\pi_1\pi_3\pi'_3$. Now $[\pi_3^x, a(x)] = 1$ and then $\pi_3^x \in C_G(\pi_1\pi_2, \pi_1\pi_3)$, $C_G(\pi_1\pi_2, \pi_1\pi'_3)$ or $C_G(\pi_1\pi_2, \pi_1\pi_3\pi'_3)$. By Lemma 16, we have

$$\pi_3^x \in S\langle \mu, \mu' \rangle$$

for all x in $C_G(\pi_1\pi_2)$ with $\pi_3^x \sim \pi_1 a(x)$. By Lemma 17, we must have

$$\pi_3^x \in \langle \pi_3, \pi'_3 \rangle .$$

Assume that $\pi_1 \sim \pi_1 a(x)$. In this case we have $\pi_3^x \sim \pi_3^x a(x)$ and then there exists an element $y(x) \in \langle \pi_3^x, \pi_1 \rangle$ with $(\pi_3^x)^{y(x)} = \pi_3^x a(x)$. This implies that $a(x) = \pi_3^x (\pi_3^x)^{y(x)}$ and $[\pi_3^x, (\pi_3^x)^{y(x)}] = 1$. On the other hand Lemma 14 yields $C_G(\pi_3^x) = D^x \langle \rho^x \rangle$ and hence there exists an element $w(x)$ in $C_G(\pi_3^x) \cap C_G(\pi_1\pi_2)$ with $[(\pi_3^x)^{y(x)}]^{w(x)} \in D^x$. Since $y(x)$, $w(x) \in C_G(\pi_1\pi_2)$ Lemma 17 yields

$$[(\pi_3^x)^{y(x)}]^{w(x)} \in \langle \pi_3^x, \pi_3'^x \rangle .$$

Now $C_G(\pi_3^x) \triangleright \langle \pi_3^x, \pi_3'^x \rangle$ and hence $(\pi_3^x)^{y(x)} \in \langle \pi_3^x, \pi_3'^x \rangle$. It follows from $(\pi_3^x)^{y(x)} = \pi_3^x a(x)$ that $a(x) = \pi_3'^x$ or $a(x) = (\pi_3\pi'_3)^x$. Since $a(x) \in S\langle \mu, \mu' \rangle$ Lemma 17 implies that

$$a(x) \in \langle \pi_3, \pi'_3 \rangle .$$

This is impossible by Lemma 1 because $\pi_1 \sim \pi_1 a(x)$. Thus we have proved $\pi_3^x \in \langle \pi_3, \pi'_3 \rangle$ for all x in $C_G(\pi_1\pi_2)$. On the other hand w_1^* is in $N_G(S) \cap C_G(\pi_1\pi_2)$ and $\pi_3^{w_1^*} = \pi_3\pi'_3$ by Lemma 6 and so $\pi_3'^{\mu'} = \pi_3\pi'_3$ implies that $\pi_3 \sim \pi'_3 \sim \pi_3\pi'_3$ in $C_G(\pi_1\pi_2)$. Hence we get $\langle \pi_3, \pi'_3 \rangle \triangleleft C_G(\pi_1\pi_2)$. The proof is complete.

LEMMA 19. $C_G(\pi_1\pi_2) = D\langle \rho, w_1^* \rangle$.

PROOF. Since $(N_G(\langle \pi_3, \pi'_3 \rangle) : C_G(\langle \pi_3, \pi'_3 \rangle))$ is divisible by 6 we have

$$N_G(\langle \pi_3, \pi'_3 \rangle) = \langle w_1^*, \mu' \rangle C_G(\langle \pi_3, \pi'_3 \rangle) .$$

By Lemma 14, $C_G(\langle \pi_3, \pi'_3 \rangle) = S\langle \tau, \mu, \rho \rangle$ and then the result follows.

REMARK. By Lemmas 14, 15 and 19, every centralizer of involution of G_0 is contained in the group G_0 generated by $H = C_G(\alpha)$ and $N_G(S)$.

LEMMA 20. $G = G_0$.

PROOF. (cf. Suzuki [6] or Thompson [7]). Assume by way of contradiction that $G > G_0$. If there is no element of order 2 in $G - G_0$, then every involution is contained in G_0. Since G_0 is generated by the elements of order 2, G_0 is a normal subgroup of G and contains D. Since $N_G(D) = D$ by Lemma 2 Frattini argument implies that

$$G = N_G(D)G_0 = DG_0 = G_0 .$$

This contradicts our assumption. Now let y be an element of order 2 in $G - G_0$ and x be any element of order 2 in G_0. If $x \not\sim y$ in G, then it is well known that $\langle x, y \rangle$ is a dihedral group with non-trivial center. We put $Z(\langle x, y \rangle) = \langle z \rangle$. Then $z \in C_G(x) \cap C_G(y)$ and so $y \in C_G(z)$. On the other hand by Lemmas 14, 15 and 19, we have $C_G(x) \subset G_0$ and so $z \in G_0$. This is impossible because $y \in C_G(z) \subset G_0$. Therefore all involutions in G are conjugate to each other. This is also impossible because of Lemma 1. Thus we have proved $G = G_0$. The proof of our theorem is complete.

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