J. Math. Soc. Japan
Vol. 21, No. 2, 1969

On ruled surfaces of genus 1

By Tatsuo SUWA

(Received Aug. 24, 1968)

In this paper we study the complex structure of ruled surfaces of genus 1—complex analytic projective line bundles over non-singular elliptic curves. The classification of these bundles were given earlier by Atiyah [**1**]. In Section 1 of the present paper we make further classification of them as complex analytic surfaces. The underlying topological (or differentiable) manifold of a ruled surface of genus 1 is an S^2-bundle over $S^1 \times S^1$, where S^2 and S^1 denote, respectively, a 2-sphere and a circle. We prove, in Section 2, that such bundles have two types: a trivial bundle $E_0 = S^1 \times S^1 \times S^2$ and a non-trivial bundle E_1, and that if a surface S is topologically (or differentiably) homeomorphic to E_0 or E_1, then S is a ruled surface of genus 1 (Theorem 2). Combining this with the result of Section 1, we can determine all the complex structures on E_0 and E_1. We note that, while the set of all the complex structures on E_0 forms a continuum, E_1 admits only a countable number of complex structures. In Section 3 we give explicit construction of the complex analytic families of the above complex structures of which the existence is asserted by a theorem of Kodaira-Nirenberg-Spencer [**10**]. In those families we see the "jump" phenomenon of complex structures, which is characteristic to ruled surfaces.

F. Enriques ([**4**]) first discovered that, if an algebraic surface S has the numerical characters: $p_g = c_1^2 = 0$ and $q = 1$, then S is either a ruled surface (of genus 1) or an elliptic surface, where p_g, q and c_1 denote, respectively, the geometric genus, the irregularity and the first Chern class of S. In Section 4 we examine those surfaces which are both ruled and elliptic, in other words, we find the ruled surfaces which have another fibering of elliptic curves. A similar method used in proving Theorem 5 is applicable to the explicit determination of the structure of so called (irregular) hyperelliptic surfaces (Enriques-Severi [**5**]).

§1. Biholomorphic classification of ruled surfaces of genus 1.

By a surface we shall mean a connected compact complex manifold of complex dimension 2. We shall follow the notation and terminology of Kodaira

[9]. Thus we denote by S a surface and by p_g, q, b_ν, c_ν, $\cdots$ the geometric genus, the irregularity, the ν-th Betti number, the ν-th Chern class, $\cdots$ of S.

Let $P=\{z\mapsto(az+b)/(cz+d)\,;\ ad-bc\neq 0,\ a, b, c, d\in \boldsymbol{C}\}$ and $A=\{z\mapsto az+b\,;\ a\neq 0,\ a, b\in\boldsymbol{C}\}$ be, respectively, the 1-dimensional projective transformation group and the 1-dimensional affine transformation group. We may consider that $\boldsymbol{C}^*\subset A\subset P$, where $\boldsymbol{C}^*$ is the multiplicative group of complex numbers. By a *ruled surface* of genus g we mean (the bundle space of) a complex analytic fibre bundle over a non-singular algebraic curve X of genus g whose fibre is a projective line $\boldsymbol{P}^1$ and whose structure group is the group P. When we want to make explicit the base curve X, we call the surface a ruled surface over X. A surface S is said to be algebraic if there exists a biholomorphic embedding of S into a projective space $\boldsymbol{P}^N(\boldsymbol{C})$. Obviously every ruled surface is algebraic. For low values of the genus, Atiyah [1] classified ruled surfaces as P-bundles. In the case in which $g=0$, every P-bundles over $\boldsymbol{P}^1$ can be expressed uniquely as a $\boldsymbol{C}^*$-bundle of non-negative degree. Hence ruled surfaces of genus 0 are the Hirzebruch manifolds Σ_n, $n\geqq 0$ (Hirzebruch [6], see also [8] p. 86). Except Σ_1, they are relatively minimal models of rational surfaces. As for $g=1$, we have the following

THEOREM 1 (Atiyah [1], [2]). *Every P-bundle over an elliptic curve X can be expressed uniquely as one of the following:*

(i) *a $\boldsymbol{C}^*$-bundle of non-negative degree,*

(ii) A_0,

(iii) A_{-1},

where A_0 and A_{-1} are affine bundles.

Let X be a non-singular elliptic curve. We consider the exact sequence

$$0\longrightarrow \boldsymbol{Z}\longrightarrow \mathcal{O}\longrightarrow \mathcal{O}^*\longrightarrow 0\,,$$

where $\mathcal{O}$ and $\mathcal{O}^*$ are respectively the sheaves over X of germs of holomorphic functions and of non-vanishing holomorphic functions. We have the corresponding exact cohomology sequence

$$\cdots\longrightarrow H^1(X,\boldsymbol{Z})\xrightarrow{h} H^1(X,\mathcal{O})\xrightarrow{e} H^1(X,\mathcal{O}^*)\xrightarrow{c} H^2(X,\boldsymbol{Z})\longrightarrow 0\,. \tag{1}$$

Note that $H^1(X,\boldsymbol{Z})\cong\boldsymbol{Z}\oplus\boldsymbol{Z}$, $H^1(X,\mathcal{O})\cong\boldsymbol{C}$ and $H^2(X,\boldsymbol{Z})\cong\boldsymbol{Z}$. For any $\boldsymbol{C}^*$-bundle $\xi\in H^1(X,\mathcal{O}^*)$, $c(\xi)\in H^2(X,\boldsymbol{Z})=\boldsymbol{Z}$ is the *degree* of ξ. From (1), we infer that the collection of the $\boldsymbol{C}^*$-bundles of a fixed degree forms a complex analytic family parametrized by the *Picard variety* $\mathcal{P}(X)=\ker c\cong H^1(X,\mathcal{O})/hH^1(X,\boldsymbol{Z})$ of X. Note that $\mathcal{P}(X)$ is a complex torus which is isomorphic to X itself.

Now we make a biholomorphic classification of ruled surfaces associated with the bundles of Theorem 1. For any divisor $\mathfrak{d}$ on X, we denote by $[\mathfrak{d}]$

the C^*-bundle over X which is determined by $\mathfrak{d}$. We write the operation of the group of divisors on X multiplicatively. The following lemma is a direct consequence of the Abel-Jacobi theorem:

LEMMA 1. (i) *Fix a point p_0 on X, then the mapping $p \mapsto [p_0 p^{-1}]$ gives an isomorphism between complex torus X and $\mathscr{P}(X)$.*

(ii) *For any C^*-bundle ξ of degree $n \geqq 1$, there exists a point p on X such that $\xi = [p^n]$.*

From Lemma 1 (ii), it follows that all the ruled surfaces associated with C^*-bundles of a degree $n \geqq 1$ are biholomorphically equivalent to one and the same surface, which will be denoted by S_n. Moreover, we denote by S_0 the direct product $P^1 \times X$. For C^*-bundles of degree 0, we prove the following

LEMMA 2. *Let ξ_1 and ξ_2 be two non-trivial C^*-bundles of degree 0, and let p_1 and p_2 be, respectively, the corresponding points on X, i.e. $\xi_1 = [p_0 p_1^{-1}]$ and $\xi_2 = [p_0 p_2^{-1}]$. Then the ruled surfaces R_1 and R_2 associated with ξ_1 and ξ_2, respectively, are biholomorphically equivalent if and only if there exists an automorphism φ of the base curve X such that $\varphi(p_0) = p_0$ and $\varphi(p_1) = p_2$.*

PROOF. The "if" part is obvious. Assume that there exists a biholomorphic map Ψ of R_1 onto R_2. Let $\pi_\nu : R_\nu \to X$ be the canonical projections of the ruled surfaces R_ν $(\nu = 1, 2)$ onto X. As any fibre F of R_1 and its image $\Psi(F)$ are P^1, we see that Ψ is fibre preserving and induces an automorphism ϕ of X such that $\pi_2 \circ \Psi = \phi \circ \pi_1$. We represent X as a quotient group: $X = C/G$, where G is a discontinuous subgroup of the additive group C generated by ω and 1, $\operatorname{Im} \omega > 0$, and, for any $u \in C$, we denote by $[u]$ the corresponding element of $X = C/G$. Let u_ν be local coordinates with respective centers p_ν $(\nu = 0, 1, 2)$, and put $V_\nu = X - \{p_0, p_\nu\}$ $(\nu = 1, 2)$, $U_\nu = \{u_\nu \mid |u_\nu| < \varepsilon\}$ $(\nu = 0, 1, 2)$. We choose ε small enough so that $U_0 \cap U_\nu = \phi$, $(\nu = 1, 2)$. We indicate any point on P^1 by its inhomogeneous coordinate ζ. The surfaces R_ν $(\nu = 1, 2)$ can be described as follows:

$$R_\nu = (V_\nu \times P^1) \cup (U_0 \times P^1) \cup (U_\nu \times P^1),$$

where $([u], \zeta) \in V_\nu \times P^1$ and $(u_0, \zeta_0) \in U_0 \times P^1$ are identified if and only if $[u] = p_0 + u_0$, $\zeta = \dfrac{1}{u_0} \zeta_0$, and $([u], \zeta) \in V_\nu \times P^1$ and $(u_\nu, \zeta_\nu) \in U_\nu \times P^1$ are identified if and only if $[u] = p_\nu + u_\nu$, $\zeta = u_\nu \zeta_\nu$. Ruled surfaces R_ν have two mutually disjoint sections $\Gamma_0^{(\nu)}$ and $\Gamma_\infty^{(\nu)}$ defined, respectively, by the equations $\zeta = \zeta_0 = \zeta_\nu = 0$ and $\zeta = \zeta_0 = \zeta_\nu = \infty$. Besides, R_ν have sections $\Gamma^{(\nu)}$ defined by $\zeta = 1$, $\zeta_0 = u_0$ and $\zeta_\nu = \dfrac{1}{u_\nu}$. It is easy to see that for any section Γ of R_ν, there exists a global meromorphic function f (possibly $\equiv \infty$) on X, such that Γ is defined by $\zeta = f$, $\zeta_0 = u_0 f$ and $\zeta_\nu = \dfrac{1}{u_\nu} f$, respectively, on V_ν, U_0 and U_ν. Moreover, for any section Γ on R_1, $\Psi(\Gamma)$ is a section of R_2. Let f, g and h

be, respectively, meromorphic functions on X corresponding to the sections $\Psi(\Gamma^{(1)})$, $\Psi(\Gamma_0^{(1)})$ and $\Psi(\Gamma_\infty^{(1)})$ in the above manner. Suppose that neither g nor h is identically infinite. If the function $g-h$ has a zero q, then q must coincide with the point p_2 and p_2 is a zero of $g-h$ of order 1, since $\Psi(\Gamma_0^{(1)})$ and $\Psi(\Gamma_\infty^{(1)})$ are mutually disjoint. As there is no elliptic function of order 1, we see that $g-h$ reduces to a constant, but, in this case, $\Psi(\Gamma_0^{(1)})$ and $\Psi(\Gamma_\infty^{(1)})$ meet at a point on the fibre: $u_0=0$. This is a contradiction. Hence $g\equiv\infty$ or $h\equiv\infty$ and consequently $\Psi(\Gamma_0^{(1)})=\Gamma_\infty^{(2)}$ or $\Psi(\Gamma_\infty^{(1)})=\Gamma_\infty^{(2)}$. Moreover the mutual disjointness of $\Psi(\Gamma_0^{(1)})$ and $\Psi(\Gamma_\infty^{(1)})$ implies that if $\Psi(\Gamma_0^{(1)})=\Gamma_\infty^{(2)}$ then $\Psi(\Gamma_\infty^{(1)})=\Gamma_0^{(2)}$ and that if $\Psi(\Gamma_\infty^{(1)})=\Gamma_\infty^{(2)}$ then $\Psi(\Gamma_0^{(1)})=\Gamma_0^{(2)}$. Since $\Psi(\Gamma^{(1)})$ intersect transversally each of $\Gamma_0^{(2)}$ and $\Gamma_\infty^{(2)}$ at one point, the order of the elliptic function f is 0 or 2. In what follows we denote by (f) the divisor on X defined by f. Moreover let η be a base of holomorphic 1-forms on X which is invariant under translations. We examine the following cases separately:

(i) *The case in which $\Psi(\Gamma_0^{(1)})=\Gamma_0^{(2)}$, $\Psi(\Gamma_\infty^{(1)})=\Gamma_\infty^{(2)}$ and the order of f is* 0. Since f reduces to a constant distinct from 0 or ∞, we have $\psi(p_0)=p_0$ and $\psi(p_1)=p_2$. Hence it suffices to put $\varphi=\psi$.

(ii) *The case in which $\Psi(\Gamma_0^{(1)})=\Gamma_0^{(2)}$, $\Psi(\Gamma_\infty^{(1)})=\Gamma_\infty^{(2)}$ and the order of f is* 2. It follows that $(f)=p_2p_0^{-1}\psi(p_0)\psi(p_1)^{-1}$. By the Abel theorem we have

$$\int_{p_0}^{p_2}\eta+\int_{\psi(p_1)}^{\psi(p_0)}\eta\in G\,. \tag{2}$$

Let φ be the automorphism of X defined by $\varphi(p)=\psi(p)-\psi(p_0)+p_0$, where $p\in X$, then $\varphi(p_0)=p_0$ and (2) implies that $\int_{\varphi(p_1)}^{p_2}\eta\in G$. Hence we have $\varphi(p_1)=p_2$.

(iii) *The case in which $\Psi(\Gamma_0^{(1)})=\Gamma_\infty^{(2)}$, $\Psi(\Gamma_\infty^{(1)})=\Gamma_0^{(2)}$ and the order of f is* 0. Since f reduces to a constant distinct from 0 or ∞, we have $\psi(p_0)=p_2$ and $\psi(p_1)=p_0$. Hence it suffices to put $\varphi(p)=-\psi(p)+p_0+p_2$.

(iv) *The case in which $\Psi(\Gamma_0^{(1)})=\Gamma_\infty^{(2)}$, $\Psi(\Gamma_\infty^{(1)})=\Gamma_0^{(2)}$ and the order of f is* 2. It follows that $(f)=p_2p_0^{-1}\psi(p_1)\psi(p_0)^{-1}$. By the Abel theorem we have

$$\int_{p_0}^{p_2}\eta+\int_{\psi(p_0)}^{\psi(p_1)}\eta\in G\,. \tag{3}$$

Let φ be the automorphism of X defined by $\varphi(p)=-\psi(p)+\psi(p_0)+p_0$, then $\varphi(p_0)=p_0$ and (3) implies $\varphi(p_1)=p_2$, q. e. d.

The above lemma enables us to make biholomorphic classification of ruled surfaces associated with $\boldsymbol{C}^*$-bundles of degree 0, and to find the space of moduli for these surfaces. Let $\mathcal{S}\to\mathcal{P}(X)\cong X$ be the complex analytic family of ruled surfaces over X associated with $\boldsymbol{C}^*$-bundles of degree 0, parametrized by the Picard variety $\mathcal{P}(X)$ of X, which is identified with X by the isomorphism of Lemma 1 (i). We may assume that the point p_0 is the neutral element [0] of the additive group $X = \boldsymbol{C}/G$. The group of automorphisms of X which leave p_0 fixed is generated by $g: p \mapsto \sigma p$, where $\sigma = e^{\frac{2\pi\sqrt{-1}}{n}}$, and $n = 2, 4$ or 6 according as X is a general, a harmonic or an equianharmonic elliptic curve. Thus we see that the space of moduli of ruled surfaces associated with $\boldsymbol{C}^*$-bundles of degree 0 over an elliptic curve X is the quotient space $X/\{g\} \cong \boldsymbol{P}^1$. Choosing a representative from each biholomorphic equivalence class of ruled surfaces associated with $\boldsymbol{C}^*$-bundles of degree 0, we form a set $\mathcal{S}_0$ of surfaces. The elements of $\mathcal{S}_0$ are in one-to-one correspondence with the points on $X/\{g\} = \boldsymbol{P}^1$.

Now we make some remarks on the property of sections of ruled surfaces under consideration. The ruled surfaces associated with A-bundles A_0 and A_{-1} of Theorem 1 are denoted also by A_0 and A_{-1}. For any curves C and Γ on a surface, we write by $C\Gamma$ the intersection multiplicity of C and Γ. Note that for any two sections Γ_1 and Γ_2 of a ruled surface, we have $\Gamma_1^2 + \Gamma_2^2 = 2\Gamma_1\Gamma_2$, since the divisor $\Gamma_1 - \Gamma_2$ is homologous to a multiple of a fibre. A ruled surface S associated with a $\boldsymbol{C}^*$-bundle of degree n has two mutually disjoint sections Γ_0 and Γ_∞ defined respectively by $\zeta = 0$ and $\zeta = \infty$, where ζ denotes a fibre coordinate. We have $\Gamma_0^2 = n$ and $\Gamma_\infty^2 = -n$. Moreover for any section $\Gamma \neq \Gamma_\infty$ of S, the inequality $\Gamma^2 \geqq n$ holds. If S is the direct product $S_0 = X \times \boldsymbol{P}^1$, then of course there is an infinite number of sections with $\Gamma^2 = 0$. If $S \in \mathcal{S}_0$, $S \neq S_0$, then any section Γ of S meets either Γ_0 or Γ_∞. Thus if $\Gamma \neq \Gamma_0$, $\neq \Gamma_\infty$, then $\Gamma^2 \geqq 2$. The explicit construction of A_0 and A_{-1} in Section 3 shows that A_0 has a section Γ_∞ with $\Gamma_\infty^2 = 0$, and A_{-1} has an infinite number of sections Γ with $\Gamma^2 = 1$. It is not difficult to see that arbitrary two sections of A_0 or of A_{-1} intersect at least at one point. Thus for any section Γ of A_0, we have $\Gamma^2 \geqq 0$ and $\Gamma^2 = 0$ if and only if $\Gamma = \Gamma_\infty$, while for any section Γ of A_{-1}, we have $\Gamma^2 \geqq 1$.

From the facts mentioned above we see that the ruled surfaces associated with the bundles of Theorem 1 can be classified biholomorphically as follows:

$$\mathcal{S}_0,\ S_n\ (n \geqq 1),\ A_0,\ A_{-1}\,.$$

§ 2. Complex structures on S^2-bundles over $S^1 \times S^1$.

By an S^2-bundle we mean (the bundle space of) a differentiable fibre bundle over some differentiable manifold whose fibre is a 2-sphere S^2 and whose structure group is the group Diff S^2 of orientation preserving diffeomorphisms of S^2.

The underlying differentiable manifold of any ruled surface of genus 1 is an S^2-bundle over a 2-torus $S^1 \times S^1$, where S^1 denotes a circle. Note that the differentiable and topological classifications of Diff S^2-bundles coincide. Moreover as the group Diff S^2 has the same homotopy type as the special orthogonal group $SO(3)$ (Smale, Earle-Eells [3]), we may reduce the structure group to $SO(3)$.

PROPOSITION. *S^2-bundles over $S^1\times S^1$ are classified into two equivalence classes.*

PROOF. Let $BSO(3)=\lim_{n\to\infty} SO(n)/SO(n-3)\times SO(3)$ be the classifying space for $SO(3)$-bundles, then the equivalence classes of the bundles under consideration are in one-to-one correspondence with the homotopy classes $[S^1\times S^1, BSO(3)]$ of the maps of $S^1\times S^1$ into $BSO(3)$. Writing the Puppe exact sequence for $S^1\times S^1/S^1\vee S^1=S^2$, where $S^1\vee S^1$ denotes the one point join of two meridian circles on $S^1\times S^1$, we have

$$\begin{aligned}(3)\qquad &\cdots \longrightarrow [S(S^1\vee S^1), BSO(3)] \longrightarrow [S^2, BSO(3)] \\ &\longrightarrow [S^1\times S^1, BSO(3)] \longrightarrow [S^1\vee S^1, BSO(3)]\,,\end{aligned}$$

where $S(S^1\vee S^1)$ denotes the suspension of $S^1\vee S^1$. From isomorphisms $\pi_i(BSO(3))\cong \pi_{i-1}(SO(3))$, we have

$$[S^2, BSO(3)]\cong \pi_1(SO(3))\cong \boldsymbol{Z}_2\,,$$

$$[S^1\vee S^1, BSO(3)]\cong \pi_0(SO(3))\oplus\pi_0(SO(3))=0\,.$$

Thus the exact sequence (3) reduces to

$$[S(S^1\vee S^1), BSO(3)] \xrightarrow{h} [S^2, BSO(3)] \longrightarrow [S^1\times S^1, BSO(3)] \longrightarrow 0\,.$$

We infer readily that $h=0$, and consequently we obtain

$$[S^1\times S^1, BSO(3)]\cong [S^2, BSO(3)]\cong \boldsymbol{Z}_2\,, \qquad \text{q. e. d.}$$

THEOREM 2. *If a surface S is differentiably (or topologically) homeomorphic to the bundle space of an S^2-bundle over $S^1\times S^1$, then S is a ruled surface of genus 1.*

PROOF. Let M be the S^2-bundle over the universal covering manifold $\boldsymbol{R}^2$ of $S^1\times S^1$ which is induced from the S^2-bundle S by the covering map $\boldsymbol{R}^2\to S^1\times S^1$, then M is trivial: $M=S^2\times \boldsymbol{R}^2$. Thus we see that the universal covering manifold of the surface S is topologically homeomorphic to $S^2\times\boldsymbol{R}^2$. From the exact sequence of homotopy groups of the bundle, we have $\pi_1(S)=\boldsymbol{Z}\oplus\boldsymbol{Z}$. Hence $b_1=2$ and $q=1$. On the other hand, denoting by $\chi(N)$ the Euler number of a manifold N, we have $c_2=\chi(S)=\chi(S^1\times S^1)\chi(S^2)=0$. Hence $b_2=2$ and $p_g=0$. The Noether formula shows that $c_1^2=0$. The above values of the numerical characters of S show that S is a relatively minimal algebraic surface. Considering the Albanese map of S, we have a holomorphic map Ψ of S onto a non-singular elliptic curve Δ such that the fibres $\Psi^{-1}(u)$, $u\in\Delta$, are all connected. Let the genus of a general fibre be π. If $\pi=0$, then S is a ruled surface (over Δ), since S is relatively minimal. If $\pi=1$, then S is an elliptic surface over Δ. Since the Euler number c_2 of S vanishes, S has no other singular fibres than that of the form $m\Theta$, where Θ is a non-singular elliptic curve, and hence the functional invariant of S reduces to a constant. Thus, taking suitably branched simply connected covering $\mathcal{U}$ of Δ, we obtain an elliptic fibre space $\mathcal{S}$ over $\mathcal{U}$ induced from S by the covering map $\mathcal{U}\to\Delta$ which is free from singular fibres and forms an unramified covering manifold of S. As $\mathcal{U}$ is complex analytically homeomorphic to either $\boldsymbol{C}$ or $\boldsymbol{D}$, where $\boldsymbol{D}$ denotes a unit disk, we have $\mathcal{S}=\boldsymbol{C}\times C$ or $\boldsymbol{D}\times C$, where C is an elliptic curve. Accordingly the universal covering manifold of S is $\boldsymbol{C}\times\boldsymbol{C}$ or $\boldsymbol{C}\times\boldsymbol{D}$ which are both not homeomorphic to $\boldsymbol{R}^2\times S^2$. This is a contradiction. If $\pi\geqq 2$, then, as is shown in the proof of Theorem 51 in [**9**] IV, S has as an unramified covering manifold, the direct product $C_0\times\boldsymbol{C}$, where C_0 is a non-singular algebraic curve of genus $\geqq 2$. Thus, in this case, the universal covering manifold of S is $\boldsymbol{D}\times\boldsymbol{C}$, this is also a contradiction, q. e. d.

REMARK. The table II of [**9**] IV shows that if a surface S is topologically homeomorphic to a S^2-bundle over a compact orientable 2-manifold R of genus $\geqq 2$, then S is a ruled surface of the same genus as R.

Let E_0 and E_1 denote, respectively, the trivial and the non trivial bundle spaces of the Proposition. Examining the intersection matrices or the Stiefel-Whitney classes of the ruled surfaces of Section 1, we see that the differentiable manifold E_0 admits the complex structures $\mathcal{S}_0$, S_{2n} $(n\geqq 1)$, A_0 and no others and that E_1 admits S_{2n+1} $(n\geqq 0)$, A_{-1} and no others.

§ 3. Local complex analytic families.

We denote by Θ the sheaf over S of germs of holomorphic vector fields. It is easy to show the following

LEMMA 3. *If S is a ruled surface over an algebraic curve X, then $H^2(S,\Theta)=0$.*

Let S be a ruled surface over an elliptic curve X. As the Chern numbers c_1^2 and c_2 of S both vanish, by the Riemann-Roch-Hirzebruch theorem we have

$$\dim H^1(S,\Theta)=\dim H^0(S,\Theta)+\dim H^2(S,\Theta)\,.$$

Combining this with the preceding lemma we obtain

$$\dim H^1(S,\Theta)=\dim H^0(S,\Theta)\,.$$

Now we calculate $\dim H^0(S,\Theta)$. We represent X as a quotient group: $X=\boldsymbol{C}/G$, where G is a discontinuous subgroup of the additive group $\boldsymbol{C}$ generated by ω and 1, $\operatorname{Im}\omega>0$, and, for any $u\in\boldsymbol{C}$, we denote by $[u]$ the corresponding element of $X=C/G$. We denote by ζ an inhomogeneous coordinate of $\boldsymbol{P}^1$.

(i) $S=S_0=X\times\boldsymbol{P}^1$. A holomorphic vector field $\theta\in H^0(S,\Theta)$ has the following form:

$$\theta=a_0\frac{\partial}{\partial\zeta}+a_1\zeta\frac{\partial}{\partial\zeta}+a_2\zeta^2\frac{\partial}{\partial\zeta}+b\frac{\partial}{\partial u},$$

where a_0, a_1, a_2 and b are arbitrary constants. Hence we have $\dim H^0(S,\Theta)=4$.

(ii) $S\in\mathcal{S}_0$, $S\neq S_0$. By Lemma 1 (i) of § 1, S is associated with $[p_1^{-1}p_2]$, where p_1 and p_2 are two distinct points on X. Let u_ν $(\nu=1,2)$ be a local coordinate with the center p_ν and put $U_\nu=\{u_\nu|u_\nu<\varepsilon\}$. Let $U=X-\{p_1,p_2\}$. Taking ε sufficiently small, we may assume that $U_1\cap U_2=\phi$. The surface S can be described as follows: $S=U\times\boldsymbol{P}^1\cup U_1\times\boldsymbol{P}^1\cup U_2\times\boldsymbol{P}^1$, where $(u,\zeta)\in U\times\boldsymbol{P}^1$ and $(u_1,\zeta_1)\in U_1\times\boldsymbol{P}^1$ are identified if and only if $\zeta=u_1\zeta_1$, $[u]=p_1+u_1$, and $(u,\zeta)\in U\times\boldsymbol{P}^1$ and $(u_2,\zeta_2)\in U_2\times\boldsymbol{P}^1$ are identified if and only if $\zeta=\dfrac{1}{u_2}\zeta_2$, $[u]=p_2+u_2$. A holomorphic vector field $\theta\in H^0(S,\Theta)$ can be expressed as follows:

$$\theta=a_0(u)\frac{\partial}{\partial\zeta}+a_1(u)\zeta\frac{\partial}{\partial\zeta}+a_2(u)\zeta^2\frac{\partial}{\partial\zeta}+b(u)\frac{\partial}{\partial u},\qquad\text{on } U\times\boldsymbol{P}^1,\tag{4}$$

where $a_\nu(u)$ $(\nu=0,1,2)$ and $b(u)$ are holomorphic functions of $[u]\in U$. If we write (4) in terms of (u_1,ζ_1), we have

$$\theta=a_0(u)\frac{1}{u_1}\frac{\partial}{\partial\zeta_1}+\left\{a_1(u)-\frac{b(u)}{u_1}\right\}\zeta_1\frac{\partial}{\partial\zeta_1}+a_2(u)u_1\zeta_1^2\frac{\partial}{\partial\zeta_1}+b(u)\frac{\partial}{\partial u_1}.\tag{5}$$

Similarly we have

$$\theta=a_0(u)u_2\frac{\partial}{\partial\zeta_2}+\left\{a_1(u)+\frac{b(u)}{u_2}\right\}\zeta_2\frac{\partial}{\partial\zeta_2}+\frac{a_2(u)}{u_2}\zeta_2^2\frac{\partial}{\partial\zeta_2}+b(u)\frac{\partial}{\partial u_2}.\tag{6}$$

Equations (5) and (6) show that $b(u)$ is holomorphic everywhere on X and therefore reduces to a constant: $b(u)=b$ and that $a_2(u)$ has p_1 as its pole of order at most 1 and p_2 as its zero of order at least 1. As there exists no elliptic function of order 1, $a_2(u)$ vanishes identically. In a neighbourhood of p_1 and p_2, the function $a_1(u)$ has the forms

$$a_1(u)=\frac{b}{u_1}+\alpha_0+\alpha_1u_1+\cdots$$

and

$$a_1(u)=-\frac{b}{u_2}+\beta_0+\beta_1u_1+\cdots,$$

respectively, where α_i and β_j are constants. Hence we have $a_1(u)=c+b\zeta(u-p_1)-b\zeta(u-p_2)$, where $\zeta(u)$ denotes the Weierstrass ζ-function with the periods $(1,\omega)$ and c is an arbitrary constant. $a_0(u)$ is identically equal to zero by the same reason as for $a_2(u)$. Consequently we have $\dim H^0(S,\Theta)=2$.

We take a point p on X, let u_1 be a local coordinate of the center p and put $U_1=\{u_1 \mid |u_1|<\varepsilon\}$, $U=X-p$. We construct the rest of ruled surfaces by patching $U\times \boldsymbol{P}^1$ and $U_1\times \boldsymbol{P}^1$ in the manner described below. A holomorphic vector field $\theta\in H^0(S,\Theta)$ can be expressed in the form

$$\theta=a_0(u)\frac{\partial}{\partial\zeta}+a_1(u)\zeta\frac{\partial}{\partial\zeta}+a_2(u)\zeta^2\frac{\partial}{\partial\zeta}+b(u)\frac{\partial}{\partial u}, \qquad \text{on } U\times \boldsymbol{P}^1, \tag{7}$$

where $a_0(u)$, $a_1(u)$, $a_2(u)$ and $b(u)$ are holomorphic functions of $[u]\in U$.

(iii) $S=S_n$ $(n\geqq 1)$. By Lemma 1 (ii) of § 1, S is associated with $[p^{-n}]$. And $(u,\zeta)\in U\times \boldsymbol{P}^1$ is identified with $(u_1,\zeta_1)\in U_1\times \boldsymbol{P}^1$ if and only if $\zeta=u_1^n\zeta_1$ and $[u]=p+u_1$. If we write (7) in terms of (u_1,ζ_1), we have

$$\theta=a_0(u)\frac{1}{u_1^n}\frac{\partial}{\partial\zeta_1}+\left\{a_1(u)-\frac{nb(u)}{u_1}\right\}\zeta_1\frac{\partial}{\partial\zeta_1}+a_2(u)u_1^n\zeta_1^2\frac{\partial}{\partial\zeta_1}+b(u)\frac{\partial}{\partial u}.$$

Hence we infer that $b(u)$ reduces to a constant and that $a_0(u)$ is identically equal to zero. In a neighbourhood of p, $a_1(u)$ has the following form:

$$a_1(u)=\frac{nb}{u_1}+\alpha_0+\alpha_1u_1+\cdots, \qquad \alpha_i\in \boldsymbol{C}.$$

As there exists no elliptic function of order 1, we have $b=0$ and $a_1(u)$ reduces to a constant: $a_1(u)=a_1$. Finally, p is a pole of $a_2(u)$ of order at most n. Hence we get

$$a_2(u)=c_0+c_1\wp(u-p)+c_2\wp'(u-p)+\cdots+c_{n-1}\wp^{(n-2)}(u-p),$$

where c_i $(i=0,\cdots,n-1)$ are arbitrary constants and $\wp(u)$ is the Weierstrass $\wp$-function with the periods $(1,\omega)$. Moreover $\wp^{(k)}(u)$ denotes the k-th derivative of $\wp(u)$. Thus we obtain $\dim H^0(S,\Theta)=n+1$.

(iv) $S=A_0$. We identify $(u,\zeta)\in U\times \boldsymbol{P}^1$ and $(u_1,\zeta_1)\in U_1\times \boldsymbol{P}^1$ if and only if $\zeta=\zeta_1+\dfrac{1}{u_1}$ and $[u]=p+u_1$. We write θ in terms of (u_1,ζ_1):

$$\begin{aligned}\theta=&\left\{a_0(u)+\frac{a_1(u)}{u_1}+\frac{a_2(u)}{u_1^2}+\frac{b(u)}{u_1^2}\right\}\frac{\partial}{\partial\zeta_1}\\&+\left\{a_1(u)+\frac{2a_2(u)}{u_1}\right\}\zeta_1\frac{\partial}{\partial\zeta_1}+a_2(u)\zeta_1^2\frac{\partial}{\partial\zeta_1}+b(u)\frac{\partial}{\partial u_1}.\end{aligned}$$

Hence we infer that $b(u)$ and $a_2(u)$ reduce to constants: $b(u)=b$, $a_2(u)=a_2$. In a neighbourhood of p, $a_1(u)$ has the following form:

$$a_1(u)=-\frac{2a_2}{u_1}+\alpha_0+\alpha_1u_1+\alpha_2u_1^2+\cdots, \qquad \alpha_i\in \boldsymbol{C}.$$

As there exists no elliptic function of order 1, we have $a_2=0$ and $a_1(u)$ reduces to a constant: $a_1(u)=a_1$. In a neighbourhood of p, $a_0(u)$ has the following form:

$$a_0(u) = -\frac{b}{u_1^2} - \frac{a_1}{u_1} + \beta_0 + \beta_1 u_1 + \beta_2 u_1^2 + \cdots, \qquad \beta_i \in \boldsymbol{C}.$$

Since an elliptic function has no residue, we have $a_1 = 0$ and $a_0(u) = c - b\wp(u-p)$. Thus we obtain $\dim H^0(S, \Theta) = 2$.

(v) $A = A_{-1}$. We identify $(u, \zeta) \in U \times \boldsymbol{P}^1$ and $(u_1, \zeta_1) \in U_1 \times \boldsymbol{P}^1$ if and only if $\zeta = u_1\zeta_1 + \frac{1}{u_1}$ and $[u] = p + u_1$. Writing in terms of (u_1, ζ_1), we have

$$\theta = \left\{ \frac{a_0(u)}{u_1} + \frac{a_1(u)}{u_1^2} + \frac{a_2(u)}{u_1^3} + \frac{b(u)}{u_1^3} \right\} \frac{\partial}{\partial \zeta_1}$$
$$+ \left\{ a_1(u) + 2\frac{a_2(u)}{u_1} - \frac{b(u)}{u_1} \right\} \zeta_1 \frac{\partial}{\partial \zeta_1} + a_2(u) u_1 \zeta_1^2 \frac{\partial}{\partial \zeta_1} + b(u) \frac{\partial}{\partial u_1}.$$

Hence we infer that $b(u)$ and $a_2(u)$ reduce to constants: $b(u) = b$, $a_2(u) = a_2$. In a neighbourhood of p, $a_1(u)$ has the following form

$$a_1(u) = \frac{b - 2a_2}{u_1} + \alpha_0 + \alpha_1 u_1 + \alpha_2 u_1^2 + \cdots, \qquad \alpha_i \in \boldsymbol{C}.$$

Hence we have $b = 2a_2$ and $a_1(u)$ reduces to a constant: $a_1(u) = a_1$. Finally, in a neighbourhood of p, $a_0(u)$ has the following form:

$$a_0(u) = -\frac{a_2 + b}{u_1^2} - \frac{a_1}{u_1} + \beta_0 u_1 + \beta_1 u_1^2 + \cdots, \qquad \beta_i \in \boldsymbol{C}.$$

Hence we have $a_1 = 0$ and $a_0(u) = c - (a_2 + b)\wp(u-p)$. Consequently we obtain $\dim H^0(S, \Theta) = 1$.

We summarize the above results as follows:

THEOREM 3. *Let S be a ruled surface over an elliptic curve X and let Θ be the sheaf over S of germs of holomorphic vector fields. Then we have*

$$\dim H^0(S, \Theta) = \dim H^1(S, \Theta) = \begin{cases} 4, & \text{for } S = S_0 = X \times \boldsymbol{P}^1, \\ 2, & \text{for } S \in \mathcal{S}_0,\ S \neq S_0, \\ n+1, & \text{for } S = S_n \ (n \geqq 1), \\ 2, & \text{for } S = A_0, \\ 1, & \text{for } S = A_{-1}, \end{cases}$$

$$\dim H^2(S, \Theta) = 0.$$

Let S be a ruled surface of genus 1. Since $\dim H^2(S, \Theta) = 0$, theorems of Kodaira-Nirenberg-Spencer [10] and Kodaira-Spencer [11] assert the existence of a complex analytic family $\mathcal{S} \xrightarrow{\varpi} M$ such that $\varpi^{-1}(o) = S$ for a certain point $o \in M$ and $\dim M = \dim H^1(S, \Theta)$ which is effectively parametrized and complete at o. This family can be constructed explicity as follows:

Take a point p on an elliptic curve X and let u_1 be a local coordinate at

p. Let $U_1=\{u_1 \mid |u_1|<\varepsilon\}$, $U_0=X-p$ and $\mathfrak{U}=\{U_0, U_1\}$. We consider the exact sequence (1) of § 1. A $\boldsymbol{C}^*$-bundle is of degree zero if and only if it is in the image of e. For the Stein covering $\mathfrak{U}$, we have $H^1(\mathfrak{U}, \mathcal{O}) \cong H^1(X, \mathcal{O})$ $(\cong \boldsymbol{C})$. We define a 1-cocycle $\eta=\{\eta_{ij}\}_{i,j=0,1}$ on $\mathfrak{U}$ by $\eta_{01}=1/2\pi i u_1$. Then as there exists no elliptic function of order 1, η is not cohomologous to 0 and defines a basis of the complex vector space $H^1(\mathfrak{U}, \mathcal{O})$. Any $\boldsymbol{C}^*$-bundle of degree zero can be represented by a 1-cocycle $\eta(t)=\{\eta_{ij}(t)\}_{i,j=0,1}$, $\eta_{01}(t)=e^{\frac{t}{u_1}}$ for some $t\in\boldsymbol{C}$. $\eta(t)$ represents the trivial bundle if and only if $\eta(t)$ is in the image of h. If this is the case, we say that t belongs to the lattice. Let $\mathcal{H}=\{z\in\boldsymbol{C} \mid \operatorname{Im} z>0\}$ be the upper half plane.

(i) $S=S_0=X\times\boldsymbol{P}^1$. We construct surfaces S_{ω,t_1,t_2,t_3} parametrized by $(\omega, t_1, t_2, t_3)\in\mathcal{H}\times\boldsymbol{C}^3$ as follows:

$S_{\omega,t_1,t_2,t_3}=U_0\times\boldsymbol{P}^1\cup U_1\times\boldsymbol{P}^1$, where $(u, \zeta)\in U_0\times\boldsymbol{P}^1$ and $(u_1, \zeta_1)\in U_1\times\boldsymbol{P}^1$ are identified if and only if

$$[u]=p+u_1,\quad \zeta=\frac{e^{\frac{t_1}{u_1}}\zeta_1+\dfrac{t_2}{u_1}}{\dfrac{t_3}{u_1}\zeta_1+1}. \tag{8}$$

Then $S_{\omega,0,0,0}=S_0$, and $\{S_{\omega,t_1,0,0}\}_{t_1\in\boldsymbol{C}}$ is a complex analytic family of ruled surfaces associated with $\boldsymbol{C}^*$-bundles of degree 0. It is easy to show that if $t_2\neq 0$, then $S_{\omega,0,t_2,0}=A_0$. If $t_3\neq 0$, then $S_{\omega,0,0,t_3}=A_0$.

(ii) $S\in\mathcal{S}_0$, $S\neq S_0$. We construct surfaces $S_{\omega,t}$ parametrized by $(\omega, t)\in\mathcal{H}\times\boldsymbol{C}$ as follows:

$S_{\omega,t}=U_0\times\boldsymbol{P}^1\cup U_1\times\boldsymbol{P}^1$, where $(u, \zeta)\in U_0\times\boldsymbol{P}^1$ and $(u_1, \zeta_1)\in U_1\times\boldsymbol{P}^1$ are identified if and only if

$$[u]=p+u_1,\quad \zeta=e^{\frac{t_0+t}{u_1}}\zeta_1, \tag{9}$$

where t_0 is a complex number not belonging to the lattice such that the ruled surface S is represented by the 1-cocycle $\{\eta_{ij}(t_0)\}$. The complex analytic family $\{S_{\omega,t}\}_{t\in\boldsymbol{C}}$ is associated with the family of $\boldsymbol{C}^*$-bundles of degree 0.

(iii) $S=S_n$ $(n\geqq 1)$. We construct surfaces $S_{\omega,t_1,t_2,\cdots,t_n}$ parametrized by $(\omega, t_1, t_2, \cdots, t_n)\in\mathcal{H}\times\boldsymbol{C}^n$ as follows:

$S_{\omega,t_1,t_2,\cdots,t_n}=U_0\times\boldsymbol{P}^1\cup U_1\times\boldsymbol{P}^1$, where $(u, \zeta)\in U_0\times\boldsymbol{P}^1$ and $(u_1, \zeta_1)\in U_1\times\boldsymbol{P}^1$ are identified if and only if

$$[u]=p+u_1,\quad \zeta=u_1^n\zeta_1+\frac{t_1}{u_1}+t_2u_1+t_3u_1^2+\cdots+t_nu_1^{n-1}. \tag{10}$$

Then $S_{\omega,0,0,\cdots,0}=S_n$. It is not difficult to show that for $t_k\neq 0$ $(k=1, 2, \cdots, n-1)$, we have $S_{\omega,t_1,0,\cdots,0}=S_{\omega,0,t_2,0,\cdots,0}=A_0$ or A_{-1} according as n is even or odd and $S_{\omega,0,\cdots,0,t_k,0,\cdots,0}=S_{n-2(k-1)}$, $3\leqq k\leqq n$, where we let $S_{-m}=S_m$.

(iv) $S=A_0$. We construct surfaces $S_{\omega,t}$ parametrized by $(\omega, t)\in\mathcal{H}\times\boldsymbol{C}$ as follows:

$S_{\omega,t}=U_0\times \boldsymbol{P}^1\cup U_1\times \boldsymbol{P}^1$, where $(u,\zeta)\in U_0\times \boldsymbol{P}^1$ and $(u_1,\zeta_1)\in U_1\times \boldsymbol{P}^1$ are identified if and only if

$$[u]=p+u_1,\quad \zeta=\frac{\zeta_1+\dfrac{1}{u_1}}{\dfrac{t}{u_1}\zeta_1+1}. \tag{11}$$

(v) $S=A_{-1}$. Since $\dim H^1(S,\Theta)=1$, ω is the only parameter, i.e., A_{-1} is rigid if the base curve X is fixed.

THEOREM 4. *The complex analytic families constructed above are effectively parametrized and complete, respectively, at the points* (i) $(\omega,0,0,0)$, (ii) $(\omega,0)$, (iii) $(\omega,0,0,\cdots,0)$, (iv) $(\omega,0)$ *and* (v) ω, *where* $\omega\in\mathfrak{H}$.

PROOF. Let $\partial S/\partial\omega$ and $\partial S/\partial t_\nu$ denote the infinitesimal deformations of S along the tangent vectors $\partial/\partial\omega$ and $\partial/\partial t_\nu$ respectively. Since it is obvious that $\partial S/\partial\omega$ can not be written as a linear combination of $\partial S/\partial t_1, \partial S/\partial t_2,\cdots$, it suffices to prove that $\partial S/\partial t_1, \partial S/\partial t_2,\cdots$ are linearly independent. Let $V_i=U_i\times\boldsymbol{P}^1$, $i=0,1$, then $\mathfrak{B}=\{V_0,V_1\}$ is an open covering of S.

(i) $S=S_0$. If we represent the infinitesimal deformations $\partial S/\partial t_\nu$ respectively by 1-cocycles $\theta^{(\nu)}=\{\theta_{01}^{(\nu)}\}$, we have from (8)

$$\theta_{01}^{(1)}=\frac{\zeta_1}{u_1}\frac{\partial}{\partial\zeta},\qquad \theta_{01}^{(2)}=\frac{1}{u_1}\frac{\partial}{\partial\zeta},\qquad \theta_{01}^{(3)}=-\frac{\zeta_1^2}{u_1}\frac{\partial}{\partial\zeta}.$$

Assume a cohomological relation $\sum_{\nu=1}^{3}c_\nu\theta^{(\nu)}\sim 0$. Then there exist holomorphic vector fields θ_j on V_j $(j=0,1)$ such that

$$\sum_{\nu=1}^{3}c_\nu\theta_{01}^{(\nu)}=\theta_1-\theta_0. \tag{12}$$

Writing

$$\begin{aligned}\theta_1&=a_{10}(u_1)\frac{\partial}{\partial\zeta_1}+a_{11}(u_1)\zeta_1\frac{\partial}{\partial\zeta_1}+a_{12}(u_1)\zeta_1^2\frac{\partial}{\partial\zeta_1}+b_1(u_1)\frac{\partial}{\partial u_1},\\ \theta_0&=a_0(u)\frac{\partial}{\partial\zeta}+a_1(u)\zeta\frac{\partial}{\partial\zeta}+a_2(u)\zeta^2\frac{\partial}{\partial\zeta}+b(u)\frac{\partial}{\partial u},\end{aligned} \tag{13}$$

where $a_{1i}(u_1)$ and $b_1(u_1)$ are holomorphic functions of $u_1\in U_1$ and where $a_i(u)$ and $b(u)$ are holomorphic functions of $[u]\in U_0$, we obtain from (12)

$$\begin{aligned}&\frac{c_1\zeta_1}{u_1}\frac{\partial}{\partial\zeta}+\frac{c_2}{u_1}\frac{\partial}{\partial\zeta}-\frac{c_3\zeta^2}{u_1}\frac{\partial}{\partial\zeta}\\ &\qquad=\{a_{10}(u_1)-a_0(u)\}\frac{\partial}{\partial\zeta}+\{a_{11}(u_1)-a_1(u)\}\zeta\frac{\partial}{\partial\zeta}\\ &\qquad\qquad+\{a_{12}(u_1)-a_2(u)\}\zeta^2\frac{\partial}{\partial\zeta}.\end{aligned}$$

Hence we infer that, in a neighbourhood of $u_1=0$,

$$a_0(u)=-\frac{c_2}{u_1}+\alpha_0+\alpha_1u_1+\alpha_2u_1^2+\cdots,$$

$$a_1(u)=-\frac{c_1}{u_1}+\beta_0+\beta_1u_1+\beta_2u_1^2+\cdots,$$

$$a_2(u)=\frac{c_3}{u_1}+\gamma_0+\gamma_1u_1+\gamma_2u_1^2+\cdots,$$

where α_i, β_i and γ_i are constants. It follows that $c_1=c_2=c_3=0$, as there is no elliptic function of order 1. Thus we see that $\partial S/\partial t_1$, $\partial S/\partial t_2$, and $\partial S/\partial t_3$ are linearly independent.

(ii) $S\in\mathcal{S}_0$, $S=S_0$. If we represent the infinitesimal deformation $\partial S/\partial t$ by a 1-cocycle $\theta=\{\theta_{01}\}$, we have from (9)

$$\theta_{01}=\frac{1}{u_1}e^{\frac{t_0}{u_1}}\zeta_1\frac{\partial}{\partial\zeta}.$$

Assume a cohomological relation $\theta\sim0$. Then there exists holomorphic vector fields θ_j on V_j such that $\theta_{01}=\theta_1-\theta_0$. Writing θ_j as (13), we obtain

$$\begin{aligned}\frac{\zeta_1}{u_1}\frac{\partial}{\partial\zeta_1}=a_{10}(u_1)\frac{\partial}{\partial\zeta_1}&+a_{11}(u_1)\zeta_1\frac{\partial}{\partial\zeta_1}+a_{12}(u_1)\zeta_1^2\frac{\partial}{\partial\zeta_1}+b_1(u_1)\frac{\partial}{\partial u_1}\\&-a_0(u)e^{-\frac{t_0}{u_1}}\frac{\partial}{\partial\zeta_1}-a_1(u)\zeta_1\frac{\partial}{\partial\zeta_1}-a_2(u)e^{\frac{t_0}{u_1}}\zeta_1^2\frac{\partial}{\partial\zeta_1}\\&-b(u)\left\{\frac{\partial}{\partial u_1}+\frac{t_0\zeta_1}{u_1^2}\frac{\partial}{\partial\zeta_1}\right\}.\end{aligned}$$

This implies that

$$\begin{cases}\dfrac{1}{u_1}=a_{11}(u_1)-a_1(u)-\dfrac{t_0b(u)}{u_1^2}\\ 0=b_1(u_1)-b(u)\end{cases}$$

The second equation shows that $u_1=0$ is a removable singularity of $b(u)$ and $b(u)$ reduces to a constant: $b(u)=b$. In a neighbourhood of $u_1=0$, we have

$$a_1(u)=\frac{-t_0b}{u_1^2}-\frac{1}{u_1}+\alpha_0+\alpha_1u_1+\alpha_2u_1^2+\cdots.$$

This contradicts the fact that any elliptic function has no residue.

(iii) $S=S_n$ $(n\geqq1)$. If we represent the infinitesimal deformation $\partial S/\partial t_\nu$ by a 1-cocycle $\theta^{(\nu)}=\{\theta_{01}^{(\nu)}\}$ on $\mathfrak{V}$, we have from (10)

$$\theta_{01}^{(1)}=\frac{1}{u_1}\frac{\partial}{\partial\zeta},\qquad\theta_{01}^{(\nu)}=u_1^{\nu-1}\frac{\partial}{\partial\zeta}\qquad(\nu=2,\cdots,n).$$

Assume a cohomological relation $\sum_{\nu=1}^{n} c_\nu \theta^{(\nu)} \sim 0$.

Then there exist holomorphic vector fields θ_j on V_j such that $\sum c_\nu \theta_{01}^{(\nu)} = \theta_1 - \theta_0$.

Writing θ_j as (13), we obtain

$$\Big(\frac{c_1}{u^{n+1}} + \frac{c_2}{u_1^{n-1}} + \frac{c_3}{u_1^{n-2}} + \cdots + \frac{c_n}{u_1}\Big)\frac{\partial}{\partial \zeta_1}$$

$$= a_{10}(a_1)\frac{\partial}{\partial \zeta_1} + a_{11}(u_1)\zeta_1\frac{\partial}{\partial \zeta_1} + a_{12}(u_1)\zeta_1^2\frac{\partial}{\partial \zeta_1} + b_1(u_1)\frac{\partial}{\partial u_1}$$

$$-a_0(u)\frac{1}{u_1^n}\frac{\partial}{\partial \zeta_1} - a_1(u)\zeta_1\frac{\partial}{\partial \zeta_1} - a_2(u)u_1^n\zeta_1^2\frac{\partial}{\partial \zeta_1}$$

$$-b(u)\Big\{\frac{\partial}{\partial u_1} - n\frac{\zeta_1}{u_1}\frac{\partial}{\partial \zeta_1}\Big\}.$$

This implies that

$$\frac{a_0(u)}{u_1^n} = -\Big(\frac{c_1}{u_1^{n+1}} + \frac{c_2}{u_1^{n-1}} + \cdots + \frac{c_n}{u_1}\Big) + a_{10}(u_1)$$

and hence in a neighbourhood of $u_1 = 0$, we have

$$a_0(u) = -\frac{c_1}{u_1} - c_2 u_1 - c_3 u_1^2 - \cdots - c_n u_1^{n-1} + \alpha_0 u_1^n + \alpha_1 u_1^{n+1} + \cdots.$$

It follows that $c_1 = 0$ and consequently $a_0(u)$ vanishes identically. This implies that $c_2 = \cdots = c_n = 0$. Thus we infer that $\partial S/\partial t_1, \partial S/\partial t_2, \cdots, \partial S/\partial t_n$ are linearly independent.

(iv) $S = A_0$. If we represent the infinitesimal deformation $\partial S/\partial t$ by a 1-cocycle $\theta = \{\theta_{01}\}$, we have from (11)

$$\theta_{01} = -\Big(\frac{\zeta_1^2}{u_1} + \frac{\zeta_1}{u_1^2}\Big)\frac{\partial}{\partial \zeta}.$$

Assume a cohomological relation $\theta \sim 0$, then there exist holomorphic vector fields θ_j on V_j $(j = 0, 1)$ such that $\theta_{01} = \theta_1 - \theta_0$. Writing θ_j as (13), we obtain

$$-\Big(\frac{\zeta_1^2}{u_1} + \frac{\zeta_1}{u_1^2}\Big)\frac{\partial}{\partial \zeta_1} = a_{10}(u_1)\frac{\partial}{\partial \zeta_1} + a_{11}(u_1)\zeta_1\frac{\partial}{\partial \zeta_1} + a_{12}(u_1)\zeta_1^2\frac{\partial}{\partial \zeta_1} + b_1(u_1)\frac{\partial}{\partial u_1}$$

$$-a_0(u)\frac{\partial}{\partial \zeta_1} - a_1(u)\Big(\zeta_1 + \frac{1}{u_1}\Big)\frac{\partial}{\partial \zeta_1}$$

$$-a_2(u)\Big(\zeta_1 + \frac{1}{u_1}\Big)^2\frac{\partial}{\partial \zeta_1} - b(u)\Big(\frac{\partial}{\partial u_1} + \frac{1}{u_1^2}\frac{\partial}{\partial \zeta_1}\Big).$$

This implies that $-\dfrac{1}{u_1} = a_{12}(u_1) - a_2(u)$. Hence $a_2(u)$ is an elliptic function of order 1. This is a contradiction, q. e. d.

§ 4. Elliptic ruled surfaces.

A surface S is said to be an *elliptic surface* if there exists a holomorphic map Ψ of S onto a non-singular curve Δ such that the inverse image $\Psi^{-1}(u)$ of any general point $u \in \Delta$ is an elliptic curve.

For a ruled surface S of genus 1, we have $p_g = c_1^2 = 0$ and $q = 1$ ($b_1 = 2$). Conversely let S be a surface with the above numerical characters, then by general results of Kodaira, S is a relatively minimal algebraic surface. Moreover S is either a ruled surface (of genus 1) or an elliptic surface (Enriques [4], Kodaira [9] IV). In this section we examine the surfaces which are both ruled and elliptic. In other words, we find the ruled surfaces which have another fibering of elliptic curves. Note that if an elliptic surface has a structure of ruled surface, it is of genus 1. We shall freely use the results of [7] on the theory of elliptic surfaces.

LEMMA 4. *Let S be an elliptic surface with the base curve Δ and the canonical projection $\Psi: S \to \Delta$. Then the following four conditions are necessary and sufficient for S to be ruled:* 1) $\Delta = \boldsymbol{P}^1$, 2) $b_1 = 2$, 3) *S has no singular fibres over Δ other than that of the form $m\Theta$, where Θ is a non-singular elliptic curve,* 4) *the multiplicities m_i of the singular fibres $m_i\Theta_i$, $i = 1, 2, \cdots, r$, of S satisfies the inequality:* $\sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right) < 2$.

In what follows we call Θ_i an *m_i-ply* degenerate fibre of S over Δ.

REMARK 1. The condition 3) implies that the functional invariant $\mathcal{J}(u)$ of the elliptic surface S is holomorphic everywhere on Δ and is reduced to a constant. Hence any general fibre is complex analytically homeomorphic to one and the same elliptic curve C, and Θ_i can be represented as a quotient of C by a cyclic group of order m_i.

REMARK 2. Let S be an elliptic surface satisfying the conditions 1), 3) and 4) but not 2). Then the first Betti number b_1 of S is equal to 1 and S is a Hopf surface, i.e., the universal covering manifold of S is complex analytically homeomorphic to $\boldsymbol{W} = \boldsymbol{C}^2 - (0, 0)$ ([9] II Theorem 28).

PROOF OF LEMMA 4. Assume S to be a ruled surface over an elliptic curve X. Obviously we have $b_1 = 2$. Considering the analytic fibre space over the universal covering manifold $\boldsymbol{C}$ of X induced from S by the covering map $\boldsymbol{C} \to X$, we see that the universal covering manifold of S is complex analytically homeomorphic to $\boldsymbol{P}^1 \times \boldsymbol{C}$. The vanishing of the Euler nnmber c_2 of S implies 3). If the genus of the curve Δ is zero and if S has at most two multiple fibres, there is nothing to be done. So we suppose that either the genus of Δ is greater than 0 or S has at least three multiple fibres. There exists a simply connected covering Riemann surface $\mathcal{U}$ of Δ which is unramified over $\Delta-\{p_i\}$ and has branch points of order m_i-1 over each point p_i. Let S_1 be the analytic fibre space of elliptic curves over $\mathcal{U}$ which is induced from S by the covering map: $\mathcal{U}\rightarrow\Delta$. Obviously S_1 is free from singular fibres and forms an unramified covering manifold of S, thus the universal covering manifold of S coincides with that of S_1. The Riemann surface $\mathcal{U}$ is conformally equivalent to one of $\boldsymbol{P}^1$, $\boldsymbol{C}$ and $\boldsymbol{D}$, where $\boldsymbol{D}$ is the unit disk in $\boldsymbol{C}$. If $\mathcal{U}$ is $\boldsymbol{C}$ or $\boldsymbol{D}$, then S_1 is complex analytically homeomorphic to $\boldsymbol{C}\times C$ or $\boldsymbol{D}\times C$ and, accordingly, the universal covering manifold of S_1 is complex analytically homeomorphic to $\boldsymbol{C}\times\boldsymbol{C}$ or $\boldsymbol{D}\times\boldsymbol{C}$. This is a contradiction. Thus we have $\mathcal{U}=\boldsymbol{P}^1$. Consequently $\Delta=\boldsymbol{P}^1$ and $\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)<2$.

The sufficiency of the conditions can be proved in a similar manner as in the proof of Theorem 52 of Kodaira [**9**] IV (Enriques' criterion of ruled surfaces). But we enumerate subsequently all the elliptic surfaces satisfying the conditions of the lemma. The results show that they are all ruled, q. e. d.

An elliptic surface S satisfying the conditions 1) and 3) can be obtained from $\boldsymbol{P}^1\times C$ by means of a finite number of logarithmic transformations:

$$S=L_{p_r}(m_r,\beta_r),\cdots,L_{p_2}(m_2,\beta_2)L_{p_1}(m_1,\beta_1)(\boldsymbol{P}^1\times C)\,, \tag{14}$$

where $[\beta_i]$ is an element of C of order m_i ([**9**] I p. 771 see also [**9**] II p. 685). Generally, the surface of type (14) has its first Betti number equal to 1 or 2, and $b_1=2$ if and only if $\sum_{i=1}^{r}\beta_i=0$. The last condition is proved analytically in [**9**] II p. 686. We can also show it by purely topological considerations. Let us call the elliptic surface defined by (14) as of type $(m_1, m_2,\cdots, m_r)$. Then the surfaces satisfying the condition 4) are of the following types:

$$\begin{cases} 0) & \text{free from singular fibres over } \Delta\,, \\ 1) & (1), (2), \cdots, (m), \cdots \\ 2) & (2, 2), (2, 3), \cdots, (m_1, m_2), \cdots \\ 3) & (2, 2, 2), (2, 2, 3), \cdots, (2, 2, m), \cdots, (2, 3, 3), (2, 3, 4), (2, 3, 5)\,, \end{cases} \tag{15}$$

where m_1, m_2 and m are rational integers greater than 1. We shall pick up the surfaces with $b_1=2$ from the surfaces (14) satisfying (15).

0). An elliptic surface S over $\boldsymbol{P}^1$ free from singular fibres is written as follows:

$$S=L_p(1,\gamma)(\boldsymbol{P}^1\times C)\,,\qquad \gamma=h+k\omega,\ h,\ k\in\boldsymbol{Z}\,. \tag{16}$$

For this surface (16), $b_1=2$ if and only if $\gamma=0$, i.e., $S=\boldsymbol{P}^1\times C$.

REMARK 1. The fundamental group $\pi_1(S)$ of the surface (16) can be calculated by van Kampen's theorem. The result is that $\pi_1(S)=\boldsymbol{Z}\oplus\boldsymbol{Z}_d$, where d

denotes the greatest common divisor of h and k.

REMARK 2. By a result of Earle-Eells [3], any bundle space E of a fibre bundle over S^2 whose fibre is $S^1\times S^1$ and whose structure group is the group of orientation preserving diffeomorphisms of $S^1\times S^1$ is differentiably homeomorphic to a surface of type (16). Hence E always admits a complex structure. Moreover if a surface S is differentiably homeomorphic to E, then S is either a ruled surface of genus 1 or a Hopf surface according as E is differentiably trivial or not (see Theorem 2 in §2; cf also [9] III Theorem 41).

1). We write $S=L_p(m, \beta)(\boldsymbol{P}^1\times C)$, where $[\beta]$ is an element of order m of C. Since β never vanishes, any surface of this type is not ruled.

2). We write $S=L_{p_2}(m_2, \beta_2)L_{p_1}(m_1, \beta_1)(\boldsymbol{P}^1\times C)$, where $[\beta_i]$ is an element of order m_i of C. We set $\beta_i=\frac{n_i}{m_i}+\frac{l_i}{m_i}\omega+h_i+k_i\omega$, where n_i, l_i, h_i, k_i, are integers and $0\leqq n_i\leqq m_i-1$, $0\leqq l_i\leqq m_i-1$ $(i=1, 2)$. From $\beta_1+\beta_2=0$, we have $m_1[\beta_2]=m_2[\beta_1]=0$ and, consequently, $m_1=m_2$. Putting $m=m_1=m_2$, $h=h_1+h_2$ and $k=k_1+k_2$, the equality $\beta_1+\beta_2=0$ reduces to $n_1+n_2+mh=l_1+l_2+mk=0$. This implies that $h=-1$ or $k=-1$. It is not difficult to show that, by a suitable transformation of coordinates, we may assume that $\beta_1=\frac{q}{m}$ and $\beta_2=-\frac{q}{m}$, where $0<q<m$ and $(q, m)=1$. Thus, in this case, surfaces of type

$$S=L_{p_2}\left(m, -\frac{q}{m}\right)L_{p_1}\left(m, \frac{q}{m}\right)(\boldsymbol{P}^1\times C) \tag{17}$$

are the only surfaces with $b_1=2$. The surface (17) can be represented as follows: As any pair of points on $\boldsymbol{P}^1$ can be transformed by a projective transformation into any other pair of points, we see that the complex structure of the surface (17) is independent of p_1 and p_2. Hence we may fix an inhomogeneous coordinate z on $\boldsymbol{P}^1$ such that p_1 and p_2 are, respectively, the points $z=0$ and $z=\infty$. Let S_1 be the fibre space of elliptic curves over a projective line $\boldsymbol{P}^1$ with an inhomogeneous coordinate ζ which is induced from S by the mapping $\zeta\mapsto z=\zeta^m$ of the ζ-sphere $\boldsymbol{P}^1$ onto the z-sphere $\boldsymbol{P}^1$, then S_1 is free from singular fibres and is an unramified covering manifold of S whose first Betti number is equal to 2. Hence $S_1=\boldsymbol{P}^1\times C$. The surface S can be represented as a quotient space of $S_1=\boldsymbol{P}^1\times C$: $S=\boldsymbol{P}^1\times C/\mathcal{G}$, where $\mathcal{G}$ is a cyclic group of order m generated by an automorphism g' of $\boldsymbol{P}^1\times C$ defined by

$$g': (\zeta, [u])\mapsto\left(e^{\frac{2\pi i}{m}}\zeta, \left[u-\frac{q}{m}\right]\right).$$

In stead of g', we may take an automorphism

$$g: (\zeta, [u])\mapsto\left(e^{\frac{2\pi ip}{m}}\zeta, \left[u-\frac{1}{m}\right]\right) \tag{18}$$

as a generator of $\mathcal{G}$, where p is the smallest positive integer such that $pq\equiv 1$ (mod m). We write $\mathcal{G}=\mathcal{G}_m^p$. It is obvious that S is a ruled surface associated with a $\boldsymbol{C}^*$-bundle of degree 0.

3). We write $S=L_{p_3}(m_3, \beta_3)L_{p_2}(m_2, \beta_2)L_{p_1}(m_1, \beta_1)(\boldsymbol{P}^1\times C)$, where $[\beta_i]$ is an element of C of order m_i. We set

$$\beta_i=\frac{n_i}{m_i}+\frac{l_i}{m_i}\omega+h_i+k_i\omega,\quad 0\leqq n_i\leqq m_i-1,\quad 0\leqq l_i\leqq m_i-1\,.$$

From $\beta_1+\beta_2+\beta_3=0$, we obtain

(19) $$m_i m_j\equiv 0 \pmod{m_k}\,.$$

The surfaces which we have to examine are of types (2, 2, m), (2, 3, 3), (2, 3, 4) and (2, 3, 5). If $m_1=m_2=2$, then (19) implies that $m_3=2$ or 4, and if $m_1=2$, $m_2=3$, then $m_3=6$. Moreover if $m_1=m_2=2$, then we have $2[\beta_3]=0$ and hence $m_3=2$. Thus we see that $m_1=m_2=m_3=2$. Put $h=h_1+h_2+h_3$ and $k=k_1+k_2+k_3$. The equation $\beta_1+\beta_2+\beta_3=0$ reduces to

(20) $$n_1+n_2+n_3+2h=0\,,\qquad l_1+l_2+l_3+2k=0\,.$$

It follows that $h=0$ or -1 and that $k=0$ or -1. If $h=0$, then we have $n_1=n_2=n_3=0$ and consequently $l_1=l_2=l_3=1$. But this contradicts the second equation of (20). Hence $h=-1$ and similarly $k=-1$. We may assume that $n_1=l_2=n_3=l_3=1$, $l_1=n_2=0$, $h_1=k_2=-1$, $k_1=h_2=h_3=k_3=0$. Thus, in this case, the surface of the form

(21) $$S=L_{p_3}\Big(2,\, -\frac{1}{2}\Big)L_{p_2}\Big(2,\, -\frac{\omega}{2}\Big)L_{p_1}\Big(2,\, \frac{1}{2}+\frac{\omega}{2}\Big)(\boldsymbol{P}^1\times C)$$

is the only one with $b_1=2$. As any three points on a projective line $\boldsymbol{P}^1$ can be transformed into any other three points on $\boldsymbol{P}^1$ by a projective transformation, the complex structure of the surface S defined by (21) is independent of p_i, and is uniquely determined. We fix an inhomogeneous coordinate z of $\boldsymbol{P}^1$ such that p_1, p_2 and p_3 are, respectively, the points $z=1$, $z=\infty$ and $z=0$. Let S_1 be an analytic fibre space of elliptic curves over a projective line $\boldsymbol{P}^1$ with an inhomogeneous coordinate ζ induced from S by the mapping $\zeta\mapsto z=\Big(\dfrac{\zeta^2+1}{\zeta^2-1}\Big)^2$ of the ζ-sphere $\boldsymbol{P}^1$ onto the z-sphere $\boldsymbol{P}^1$. Similarly, as in the case 2), we have $S_1=\boldsymbol{P}^1\times C$. Hence S can be represented as a quotient space of $S_1=\boldsymbol{P}^1\times C$: $S=\boldsymbol{P}^1\times C/\mathcal{G}$, where $\mathcal{G}$ is a group isomorphic to $\boldsymbol{Z}_2\oplus\boldsymbol{Z}_2$ generated by two analytic automorphisms g and h of $\boldsymbol{P}^1\times C$ defined by

(22) $$\begin{cases} g\colon (\zeta,\, [u])\mapsto\Big(-\zeta,\, \Big[u+\dfrac{1}{2}\Big]\Big)\,, \\ h\colon (\zeta,\, [u])\mapsto\Big(\dfrac{1}{\zeta},\, \Big[u+\dfrac{\omega}{2}\Big]\Big)\,. \end{cases}$$

This representation of the surface S obviously implies that S is a ruled surface of genus 1.

Summarizing the above results, we obtain the following

THEOREM 5. *Any elliptic surface which has also a structure of ruled surface can be represented as one of the following:*

(i) $\boldsymbol{P}^1\times C$,

(ii) $\boldsymbol{P}^1\times C/\mathcal{G}_m^p$ *(of type* (m, m)*), where* m *and* p *are integers such that* $m\geqq 2$, $0<p<m$ *and* $(p, m)=1$, *and where* $\mathcal{G}_m^p$ *is a cyclic group of order* m *generated by the automorphism defined by* (18),

(iii) $\boldsymbol{P}^1\times C/\mathcal{G}$ *(of type* $(2, 2, 2)$*), where* $\mathcal{G}$ *is a group generated by two automorphisms defined by* (22). $\mathcal{G}$ *is isomorphic to* $\boldsymbol{Z}_2\oplus\boldsymbol{Z}_2$.

Now we identify the above surfaces with ruled surfaces of Section 1.

(i) $\boldsymbol{P}^1\times C=S_0$.

(ii) We may consider the surface $\boldsymbol{P}^1\times C/\mathcal{G}_m^p$ as a ruled surface over an elliptic curve C' with the periods $\left(\frac{1}{m}, \omega\right)$, which is associated with a $\boldsymbol{C}^*$-bundle of degree 0. The images in $S=\boldsymbol{P}^1\times C/\mathcal{G}_m^p$ of two curves on $\boldsymbol{P}^1\times C$ defined respectively by the equations $\zeta=0$ and $\zeta=\infty$ are the m-ply degenerate fibres Θ_1 and Θ_2 of the elliptic surface S over the z-sphere $\boldsymbol{P}^1$. If we see S as a ruled surface over C', then the curves Θ_1 and Θ_2 appear as two mutually disjoint sections. Thus we have $\Theta_1=\Gamma_0$ and $\Theta_2=\Gamma_\infty$ (see Section 1).

From now on, we consider all the surface of type $\boldsymbol{P}^1\times C/\mathcal{G}_m^p$ as ruled surfaces over one and the same elliptic curve X with the periods $(1, \omega)$, $\operatorname{Im}\omega>0$. Let γ be a meridian circle on the Riemann surface X, U_1 a thin open neighbourhood of γ in X and let $U_0=X-\gamma$. We consider the exact sequence (1) of § 1. For the Stein covering $\mathfrak{U}=\{U_0, U_1\}$ of X, we have $H^1(\mathfrak{U}, \mathcal{O})\cong H^1(X, \mathcal{O})(\cong\boldsymbol{C})$. The intersection of U_0 and U_1 is composed of two mutually disjoint subsets A and B of X. We define a 1-cocycle $\eta=\{\eta_{ij}(u)\}_{i,j=0,1}$ on $\mathfrak{U}$ by

$$\eta_{01}(u)=\begin{cases}0, & \text{for } u\in A\,,\\ 1, & \text{for } u\in B\,.\end{cases}$$

This 1-cocycle η obviously defines a base $\bar{\eta}$ of the 1-dimensional vector space $H^1(\mathfrak{U}, \mathcal{O})$ over the field $\boldsymbol{C}$ and the ruled surface $\boldsymbol{P}^1\times C/\mathcal{G}_m^p$ is associated with the $\boldsymbol{C}^*$-bundle $e\left(\frac{p}{m}\bar{\eta}\right)$. Parametrizing the ruled surfaces associated with the $\boldsymbol{C}^*$-bundles of degree 0 by the Picard variety $\mathcal{P}(X)$ of X which is identified with X, we see that, corresponding to each rational point $\left[\frac{q}{r}\right](q, r>0$, $0<\frac{q}{r}<1)$ of $\mathcal{P}(X)=X$, there is an elliptic surface of type $\left(\frac{r}{d}, \frac{r}{d}\right)$, where $d=(q, r)$. Thus only a countable number of ruled surfaces associated with $\boldsymbol{C}^*$-bundles of degree 0 are elliptic surfaces. The above complex analytic family presents an interesting example for the stability of elliptic curves on a surface (see Kodaira [8]). Let Γ be a non-singular curve on a surface S, let $N=[\Gamma]_{\Gamma}$ be the normal bundle of Γ in S and let $\Psi=\Omega(N)$ be the sheaf over Γ of germs of holomorphic section of N. The degree of the line bundle N over Γ is equal to the intersection multiplicity Γ^2 of Γ with itself. Let S be a ruled surface associated with a non-trivial $\boldsymbol{C}^*$-bundle ξ of degree 0 and suppose moreover that S is an elliptic surface, i.e., $S=\boldsymbol{P}^1\times C/\mathcal{G}_m^p$ for some m and p. The normal bundle of a degenerate fibre Θ_i ($i=1$, or 2) of the elliptic surface S is equivalent to ξ if we identify the curve Θ_i, which is a section, with the base curve of the ruled surface S. As ξ is non-trivial, we have $H^1(\Theta_i, \Psi)=0$. Hence Θ_i is stable ([8] Theorem 1). In fact the curve Θ_i deforms into the sections Γ_0 or the sections Γ_∞ of ruled surfaces in the family surrounding S. On the other hand the normal bundle of a general fibre C of the elliptic surface S is trivial and $H^1(C, \Psi)\neq 0$. In fact C is unstable, since, otherwise the surrounding ruled surfaces would be elliptic surfaces. This is a contradiction.

(iii) The elliptic surface $S=\boldsymbol{P}^1\times C/\mathcal{G}$ can be regarded as a ruled surface over an elliptic curve C' with the periods $\left(\frac{1}{2}, \frac{\omega}{2}\right)$. The images in $S=\boldsymbol{P}^1\times C/\mathcal{G}$ of three disconnected curves on $\boldsymbol{P}^1\times C$ defined respectively by the equations $\zeta=0$ or ∞, $\zeta=\pm 1$ and $\zeta=\pm i$ are the doubly degenerate fibres Θ_i ($i=1, 2, 3$) of the elliptic surface S over the z-sphere $\boldsymbol{P}^1$. If we regard S as a ruled surface over C', then the curves Θ_i ($i=1, 2, 3$) appear as double covering Riemann surfaces of C'. Note that there is a one-to-one correspondence between the sections of the ruled surface S and the elliptic functions $f(u)$ with the periods $(1, \omega)$ satisfying

$$f\left(u+\frac{1}{2}\right)=-f(u), \qquad f\left(u+\frac{\omega}{2}\right)=\frac{1}{f(u)}. \tag{23}$$

It is easy to see that if f_1 and f_2 are elliptic functions with the periods $(1, \omega)$ satisfying (23), then $f_1(u_0)=f_2(u_0)$ for a point u_0. Hence the ruled surface S cannot be associated with any $\boldsymbol{C}^*$-bundle. To see whether $S=A_0$ or A_{-1}, we construct a section. Let $\wp(u)$ be the Weierstrass $\wp$-function with the periods $(1, \omega)$ and put $\alpha_1=\wp\left(\frac{1}{2}\right)$, $\alpha_2=\wp\left(\frac{\omega}{2}\right)$, $\alpha_3=\wp\left(\frac{1}{2}+\frac{\omega}{2}\right)$. Define an elliptic function $f(u)$ by

$$f(u)=\frac{\wp'(u)}{2\sqrt{\alpha_3-\alpha_2}\,\{\wp(u)-\alpha_1\}}.$$

Then it is a simple calculation to verify that $f(u)$ satisfies (23). Hence $f(u)$ defines a section Γ of the ruled surface S. The elliptic function $f(u)$ has zeros of order 1 at $\left[\frac{\omega}{2}\right]$ and at $\left[\frac{1}{2}+\frac{\omega}{2}\right]$ and poles of order 1 at $[0]$ and at $\left[\frac{1}{2}\right]$. We have therefore $\Theta_1\Gamma=1$. Let F $(=\boldsymbol{P}^1)$ be a fibre of the ruled surface S. Then F and Γ form a Betti base of the 2-dimensional integral homology group $H_2(S, \boldsymbol{Z})$ of S, which is isomorphic to $\boldsymbol{Z}\oplus\boldsymbol{Z}$. Let $\Theta_1 \backsim a\Gamma+bF$, where $\backsim$ denotes homology and $a, b\in\boldsymbol{Z}$. Taking into account the fact that $\Theta_1^2=0$ and $\Theta_1F=2$, we obtain $a=2$, $b=-1$ and $\Gamma^2=1$. Thus we infer that $S=A_{-1}$.

University of Tokyo

References

[1] M. F. Atiyah, Complex fibre bundles and ruled surfaces, Proc. London Math. Soc., (3) **5** (1955), 407–434.
[2] M. F. Atiyah, Vector bundles over an elliptic curve, Proc. London Math. Soc., (3) **7** (1957), 414–452.
[3] C. J. Earle and J. Eells, The diffeomorphism group of a compact Riemann surface, Bull. Amer. Math. Soc., **73** (1967), 557–559.
[4] F. Enriques, Sulle superficie algebriche di genere geometrico zero, Rend. Circ. Mat. Palermo, **20** (1905), 1–33.
[5] F. Enriques et F. Severi, Mémoire sur les surfaces hyperelliptiques, Acta Math., **32** (1909), 283–392, **33** (1910), 321–403.
[6] F. Hirzebruch, Über eine Klasse von einfach zusammenhängenden komplexen Mannigfaltigkeiten, Math. Ann., **124** (1951), 77–86.
[7] K. Kodaira, On compact analytic surfaces, II, III, Ann. of Math., **77** (1963), 563–626, **78** (1963), 1–40.
[8] K. Kodaira, On stability of compact submanifolds of complex manifolds, Amer. J. Math., **85** (1963), 79–94.
[9] K. Kodaira, On the structure of compact complex analytic surfaces, I, II, III, IV, Amer. J. Math., **86** (1964), 751–798, **88** (1966), 682–721, **90** (1968), 55–83, to appear.
[10] K. Kodaira, L. Nirenberg and D. C. Spencer, On the existence of deformations of complex analytic structures, Ann. of Math., **68** (1958), 450–459.
[11] K. Kodaira and D. C. Spencer, A theorem of completeness for complex analytic fibre spaces, Acta Math., **100** (1958), 281–294.