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Diffeomorphism groups and classification of manifolds

By Hajime SATO

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§ 0. Introduction.

The purpose of this paper is to investigate the groups of the pseudo-diffeotopy classes of diffeomorphisms of manifolds, which are total spaces of disk bundles over spheres or sphere bundles over spheres. The results are applied to the diffeomorphism classification of simply-connected manifolds, which are homological tori.

Let Diff M denote the group of orientation preserving diffeomorphisms of an oriented manifold M and let $\tilde{\pi}_0$ (Diff M) denote the group of pseudo-diffeotopy classes of Diff M. Let $\mathcal{E}_f$ and $\mathcal{F}_f$ be the D^{q+1} bundle over S^p and S^q bundle over S^p with characteristic map $f: S^{p-1} \to \mathrm{SO}_{q+1}$. In § 1, we study $\tilde{\pi}_0$ (Diff $\mathcal{E}_f$). In case where $\mathcal{E}_f = S^p \times D^{q+1}$, we prove the following theorem.

THEOREM 1.5. *Let $p < 2q-1$. The order of* $\tilde{\pi}_0$ (Diff $S^p \times D^{q+1}$) *is equal to the order of the direct sum group* π_p $(\mathrm{SO}_{q+1}) \oplus \boldsymbol{Z}_2$.

The concordance classes of (framed) embeddings of S^q in $\mathcal{F}_f$ are discussed in § 2. The set of framed embedding classes are related to the pairing

$$\mathrm{F}: \pi_{p-1}(\mathrm{SO}_q) \times \pi_q(S^p) \to \pi_{q-1}(\mathrm{SO}_p)$$

introduced by Wall [14]. In § 3, we define a map C from $\tilde{\pi}_0$ (Diff $S^p \times S^q$) to Θ^{p+q+1} and study its properties. Making use of the results of § 1∼3, the study of $\tilde{\pi}_0$ (Diff $\mathcal{F}_f$) is carried out in § 4. In case $\mathcal{F}_f = S^p \times S^q$, we obtain the following theorem.

THEOREM 4.17. *For $p < q < 2p-4$, the order of* $\tilde{\pi}_0$ (Diff $S^p \times S^q$) *is equal to the order of the direct sum group*

$$\boldsymbol{Z}_2 \oplus \pi_p(\mathrm{SO}_{q+1}) \oplus \pi_q(\mathrm{SO}_{p+1}) \oplus \Theta^{p+q+1}.$$

In § 5, as an application of our results in § 4, we deal with the classification of manifolds which satisfy the conditions,

$$\left\{ \begin{array}{l} M\text{: closed and simply connected} \\ \mathrm{H}_i(M) = \left\{ \begin{array}{ll} \boldsymbol{Z} & \text{for } 0,\ p,\ q+1,\ p+q+1 \\ 0 & \text{otherwise} \end{array} \right. \\ \pi_p(\mathrm{SO}_{q+1}) = 0 \\ p < q < 2p-4 \cdot \end{array} \right\} \qquad (*)$$

and in § 6, the conditions

$$\left.\begin{cases} M\text{: closed and simply connected} \\ \mathrm{H}_i(M)=\begin{cases} Z & \text{for } 0,\ p,\ q+1,\ p+q+1 \\ 0 & \text{otherwise} \end{cases} \\ \pi_q(M)=\pi_q(S^p) \\ p<q<2p-4\,. \end{cases}\right\} \qquad (**)$$

The classification of such manifolds in some cases were given by I. Tamura [12], [13]. We fix a basis of p-dimensional homology group, and we identify two manifolds if there exists a diffeomorphism preserving orientation and the preferred basis. We divide such manifolds into some classes by the characteristic class of the normal bundle of the generator of $\mathrm{H}_p(M)$. For each class we give a group structure by "connected sum along the cycle" operation, which extends the one defined by Novikov [10]. Then the calculation reduces to the structures of the groups $\tilde{\pi}_0$ (Diff $\mathcal{E}_f$) and $\tilde{\pi}_0$ (Diff $\mathcal{F}_f$). As corollaries of our classification, for example, we have the following results

PROPOSITION 5.5. *If $p\equiv 5, 6$ (mod 8), the number of differentiable manifolds satisfying (*), modulo diffeomorphisms preserving orientation and the preffered basis of p-dimensional homology group, is equal to*

$$\#(\pi_q(\mathrm{SO}_{p+1})\oplus\Theta^{p+q+1})\,.$$

PROPOSITION 6.10. *If $p\equiv 3, 5, 6, 7$ (mod 8), the number of differentiable manifolds satisfying (**), up to modulo one point diffeomorphism preserving orientation and basis, is equal to*

$$\#\{\mathrm{Im}\, s_*\colon \pi_p(\mathrm{SO}_p)\to\pi_q(\mathrm{SO}_{p+1})\}\,,$$

where s_* is induced by the inclusion $s\colon \mathrm{SO}_p\to\mathrm{SO}_{p+1}$.

A remark concerning the diffeotopy classes π_0 (Diff $\mathcal{E}_f$) and π_0 (Diff $\mathcal{F}_f$) is given in § 7, where we use the unpublished results due to Cerf [3]. In the appendix, we give a counter example to the linearity of the map C, using the index theorem due to Hirzebruch [7], which also shows the existence of non-trivial inertia group for some manifolds.

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Notations and Preliminaries.

Let M be an oriented differentiable manifold of class C^∞. On the set of orientation preserving diffeomorphisms of M, we define a topology by the C^∞-topology and a group structure by the composition of diffeomorphisms. Then we have a topological group. We write this topological group as Diff M. The arcwise connected components of Diff M, denoted by π_0 (Diff M), has also a group structure inherited from Diff M. Given two elements f and g of Diff M, we call f and g are diffeotopic if and only if there exists a diffeomorphism H of $M\times I$ onto itself such that

i) $H(x, t)=(H_t(x), t)$ (i.e. level preserving)

ii) $H_0=f$, $H_1=g$.

The map H_t is called a diffeotopy connecting f and g. Then it is known that the arcwise connected classes and the diffeotopy classes agree bijectively. On the other hand we call f and g are pseudo-diffeotopic if there exists a diffeomorphism H' of $M\times I$ such that $H'|M\times 0=f$, $H'|M\times 1=g$. We shall write $\tilde{\pi}_0$ (Diff M) for the set of pseudo-diffeotopy classes, which also has a group structure induced from that of Diff M. Analogous to the diffeotopy extension theorem, the following pseudo-diffeotopy extension theorem holds due to Smale's structure theorem [**11**].

THEOREM (*Pseudo-diffeotopy extension theorem*). *Suppose M^n is a closed manifold with the dimension $n\geqq 5$. Let V be a submanifold of M such that $\pi_1(M-V)=0$. If $e: V\times I\to M\times I$ is an embedding such that $e|V\times 0=$ identity, $e(V\times 1)\subset M\times 1$, then*

1) *there exists a diffeomorphism E of $M\times I$ onto itself which covers e, i.e. $E|V\times I=e$.*

2) *there exists a diffeomorphism h of M which is pseudo-diffeotopic to the identity and such that $h|V=e|V\times 1$.*

PROOF. 1) follows directly from Smale's theorem [**11**, Corollary 3.2]. Now prove 2). Let E_0 and E_1 be the restriction of E to $M\times 0$ and $M\times 1$. Then the diffeomorphism $E_1E_0^{-1}$ satisfies the condition for h of 2). Indeed, $E_1E_0^{-1}|V$ is equal to $e|V\times 1$ and the diffeomorphism $E(E_0^{-1}\times\text{identity})$ of $M\times I$ gives the pseudo-diffeotopy connecting the identity and $E_1E_0^{-1}$.

We call that two embeddings e_1 and e_2 of a differentiable manifold W in a differentiable manifold N are concordant if there exists an embedding H of $W\times I$ in $N\times I$ which agrees with e_1 and e_2 on $W\times 0$ and $W\times 1$ respectively, and they are isotopic if H is a level preserving embedding of $W\times I$ in $N\times I$.

Let $p\geqq 1$ and let $\mathcal{E}_f=\{E, S^p, D^{q+1}, \mathrm{SO}_{q+1}\}$ be a fibre bundle with total space E, base space S^p, fibre D^{q+1}, structure group SO_{q+1} and characteristic map $f: S^{p-1}\to \mathrm{SO}_{q+1}$. We define $D^n_+=\{x=(x_1, x_2, \cdots, x_{n+1})\,;\, |x|=1, |x_1|\geqq 0\}$ and D^n_-

$=\{x=(x_1, x_2, \cdots, x_{n+1}); |x|=1, |x_1|\leqq 0\}$. Then $\mathcal{E}_f$ is obtained from the disjoint union $D^p_+\times D^{q+1}\cup D^p_-\times D^{q+1}$ by identifying $(x, y)\in\partial D^p_+\times D^{q+1}$ with $\tilde{f}(x, y)\in\partial D^p_-\times D^{q+1}$, where the attaching map $\tilde{f}: S^{p-1}\times D^{q+1}\to S^{p-1}\times D^{q+1}$ is defined by $\tilde{f}(x, y)=(x, f(x)y)$. Let $\mathcal{F}_f=\{F, S^p, S^q, \mathrm{SO}_{q+1}\}$ be the sphere bundle associated with $\mathcal{E}_f$. If $p<q$, $\{f\}$, the homotopy class of f, is an image of $\{g\}\in\pi_{p-1}(\mathrm{SO}_q)$ by the homomorphism indeed by the inclusion $s: \mathrm{SO}_q\to\mathrm{SO}_{q+1}$. Then $\mathcal{F}_f$ admits a cross section and the exact sequence of the homotopy groups of $\mathcal{F}_f$ decomposes into split short exact sequences. Hence

$$\pi_i(F)\approx\pi_i(S^p)\oplus\pi_i(S^q) \qquad \text{for } i\geqq 2\,.$$

The homology group of F are as follows,

$$\mathrm{H}_i(F)=\begin{cases}\boldsymbol{Z} & \text{for } i=0,\ p,\ q,\ p+q\\ 0 & \text{otherwise}\,.\end{cases}$$

Obviously the total space E of $\mathcal{E}_f$ is homotopy equivalent to S^p. Both E and F can be naturally regarded as smooth manifolds with an orientation. We again denote these oriented smooth manifolds by $\mathcal{E}_f$ and $\mathcal{F}_f$. The manifold $\mathcal{F}_f$ can be regarded as the boundary of $\mathcal{E}_f$. The groups $\mathrm{Diff}\,\mathcal{E}_f$ and $\mathrm{Diff}\,\mathcal{F}_f$ are taken in this sense.

§ 1. Computation of $\tilde{\pi}_0(\mathrm{Diff}\,\mathcal{E}_f)$.

In this section we define homomorphisms

$$\mathrm{A}: \tilde{\pi}_0(\mathrm{Diff}\,\mathcal{E}_f)\to\boldsymbol{Z}_2$$

and

$$\mathrm{B}: \pi_p(\mathrm{SO}_{q+1})\to\mathrm{Ker}\,\mathrm{A}\,,$$

and study the images and the kernels of these homomorphisms.

An element $x\in\mathrm{Diff}\,\mathcal{E}_f$ induces an automorphisms of $\mathrm{H}_*(\mathcal{E}_f)$. In particular an automorphism of $\mathrm{H}_p(\mathcal{E}_f)\approx\boldsymbol{Z}$. Obviously the automorphism group of $\mathrm{H}_p(\mathcal{E}_f)$ is isomorphic to $\boldsymbol{Z}_2$. Since it is a pseudo-diffeotopy invariant, we have a well-defined homomorphism

$$\mathrm{A}: \tilde{\pi}_0(\mathrm{Diff}\,\mathcal{E}_f)\to\boldsymbol{Z}_2\,.$$

In order to compute Ker A, we will define a homomorphism B from $\pi_p(\mathrm{SO}_{q+1})$ to Ker A. Given $a\in\pi_p(\mathrm{SO}_{q+1})$, we can take a C^∞-map $r: S^p\to\mathrm{SO}_{q+1}$, which represents a and is the identity on the upper hemi-sphere D^p_+ of S^p. Recall that

$$\mathcal{E}_f=D^p_+\times D^{q+1}\cup D^p_-\times D^{q+1}\,.$$

We define the diffeomorphism $\mathrm{b}(r)$ of $\mathcal{E}_f$ by

$$\mathrm{b}(r)(x, y) = \begin{cases} (x, y) & \text{on } D^p_+ \times D^{q+1} \\ (x, r(x)y) & \text{on } D^p_- \times D^{q+1}. \end{cases}$$

Since this diffeomorphism $\mathrm{b}(r)$ keeps the zero cross section fixed, its pseudo-diffeotopy class belongs to $\mathrm{Ker}\,\mathrm{A}$. We will show that the pseudo-diffeotopy class of $\mathrm{b}(r)$ does not depend on the (C^∞-) homotopy class of representative r. If we take another representative r', there exists a C^∞-map $F: S^p \times I \to \mathrm{SO}_{q+1}$ which equals r on $S^p \times 0$, equals r' on $S^p \times 1$, and is the trivial map on $D^p_+ \times I$. Construct the level preserving diffeomorphism H of $\mathcal{E}_f \times I$ by

$$H(x, y, t) = \begin{cases} (x, y, t) & \text{on } D^p_+ \times D^{q+1} \times I \\ (x, F(x, t)y, t) & \text{on } D^p_- \times D^{q+1} \times I. \end{cases}$$

This gives a diffeotopy connecting $\mathrm{b}(r)$ to $\mathrm{b}(r')$. Hence we define a map

$$\mathrm{B}: \pi_p(\mathrm{SO}_{q+1}) \to \mathrm{Ker}\ \mathrm{A}$$

by $\mathrm{B}(a) = \{\mathrm{b}(r)\}$. This is clearly a homomorphism.

The next proposition holds.

PROPOSITION 1.1. *In case $p < 2q-1$, the homomorphism* $\mathrm{B}: \pi_p(\mathrm{SO}_{q+1}) \to \mathrm{Ker}\ \mathrm{A}$ *is epimorphic.*

PROOF. Given $x \in \mathrm{Ker}\ \mathrm{A}$, let $h \in \mathrm{Diff}\ \mathcal{E}_f$ be its representative. Let $c: S^p \to \mathcal{E}_f$ be the zero cross section of $\mathcal{E}_f$ and let $[S^p] \in \mathrm{H}_p(S^p)$ be a fixed generator. Then $c_*[S^p]$ is a generator of $\mathrm{H}_p(\mathcal{E}_f) \approx \boldsymbol{Z}$. As x belongs to $\mathrm{Ker}\,\mathrm{A}$, $h_*(c_*[S^p])$ is equal to $c_*[S^p]$. Since $\pi_p(\mathcal{E}_f) \approx \mathrm{H}_p(\mathcal{E}_f)$ by Hurewicz theorem, the embeddings c and $hc: S^p \to \mathcal{E}_f$ are homotopic. By the results of Haefliger [4], since $p < 2q-1$, they are also diffeotopic. According to the diffeotopy extension theorem the diffeotopy connecting $hc(S^p)$ to $c(S^p)$ is covered by a diffeotopy of $\mathcal{E}_f$. Therefore we can take h' in the diffeotopy class of h such that h' keeps $c(S^p)$ invariant. Let T be the tubular neighborhood of $c(S^p)$, which is the disk bundle, radius of the fibre being a half of that of $\mathcal{E}_f$. By the uniqueness theorem of tubular neighborhood, we can take h'' in the diffeotopy class of h' so that h'' restricted on T is a bundle map with the base space $c(S^p)$ fixed. Moreover it is diffeotopic to a map k which is a bundle map on T and does not move $T \cap D^p_+ \times D^{q+1}$. Complement $\mathcal{E}_f - T$ is diffeomorphic to $\mathcal{F}_f \times I$, where $\mathcal{F}_f$ is the boundary of $\mathcal{E}_f$. So we identify ∂T with $\mathcal{F}_f \times 1$ and $\mathcal{E}_f - T$ with $\mathcal{F}_f \times I$ by the natural diffeomorphism. Let k' be a diffeomorphism of $\mathcal{E}_f$, which is defined by

$$k' = \begin{cases} k & \text{on } T \\ (k \mid \partial T) \times \text{identity} & \text{on } \mathcal{E}_f - T = \mathcal{F}_f \times I. \end{cases}$$

We will give a pseudo-diffeotopy Q connecting k with k' (cf. Wall [14, Lemma 8]). By the usual 'normalization process' we replace k in its diffeotopy class by a diffeomorphism 'constant' near $\mathcal{F}_f \times 0$ and near $\mathcal{F}_f \times 1$, i.e., having there the form $k(p, u) = (k(p), u)$. Write $k(p, r) = (p'(r), \lambda(p, r)r)$ on $\mathcal{F}_f \times I$. Then define the map $Q: \mathcal{F}_f \times I \times I \to \mathcal{F}_f \times I \times I$ by

$$\begin{aligned} Q(p, u, t) &= (p'(r), \lambda(p, r)u, \lambda(p, r)t) \qquad r^2 = t^2 + u^2 \leqq 1 \\ &= (p'(1), \lambda(p, 1)u, \lambda(p, 1)t) \qquad r^2 = t^2 + u^2 \geqq 1 . \end{aligned}$$

Then $Q(p, u, 0) = (k(p, u), 0)$ and $Q(p, u, 1) = (k | \partial T, u, 1)$. Moreover our condition on k ensures that Q is indeed a diffeomorphism. Hence combining Q with the identity map on $T \times I$, we obtain a desired pseudo-diffeotopy. Since k' is in $\operatorname{Im} \mathrm{B}$, the homomorphism B is surjective, which completes the proof.

Let $\{f\}$ be the homotopy class of characteristic map f and let $R = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \in \mathrm{O}_{q+1}$ be the reflection. Denote by $R^{\#}$ the operation of R on $\pi_{p-1}(\mathrm{SO}_{q+1})$ by the inner automorphism. The operation $R^{\#}$ may be non-trivial since R is not arcwise connected to the identity element e of O_{q+1}. Concerning the image of the homomorphism $\mathrm{A}: \tilde{\pi}_0(\mathrm{Diff}\, \mathcal{E}_f) \to \boldsymbol{Z}_2$, we have the next proposition.

PROPOSITION 1.2. *Suppose* $p < 2q - 1$. *The homomorphism* A *is surjective if and only if* $R^{\#}\{f\} = -\{f\}$.

PROOF. Suppose that there exists an orientation preserving diffeomorphism h of $\mathcal{E}_f$ which maps a generator of $\mathrm{H}_p(\mathcal{E}_f) \approx \boldsymbol{Z}$ to the other generator. By the same argument of the above proof of Proposition 1.1, we can take in the diffeotopy class of h a diffeomorphism k which is a bundle map of $\mathcal{E}_f$ onto itself mapping the base space by degree -1. Let D^{q+1} be a fibre over a point. Then by the condition that k is orientation preserving, $k | D^{q+1}$ must be contained in the component of O_{q+1}, which does not contain the identity. Let $S = \begin{pmatrix} 1 & & & & 0 \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \in \mathrm{O}_{p+1}$ be the reflection, which, consequently, maps D^p_+ (resp. D^p_-) onto itself. Recall $\mathcal{E}_f = D^p_+ \times D^{q+1} \underset{\tilde{f}}{\cup} D^p_- \times D^{q+1}$. By using the diffeotopy extension theorem, we can take k' in the diffeotopy class of k such that

$$\begin{cases} k'(x, y) = (Sx, Ry)\,; \ (x, y) \in D^p_+ \times D^{q+1} \\ k'(x, y) = (Sx, m(x)y)\,; \ (x, y) \in D^p_- \times D^{q+1} , \end{cases}$$

where m is a C^∞-map: $D^p_- \to \mathrm{O}_{q+1}$. Since $k'(x, y) = (Sx, Ry)$ for $(x, y) \in \partial D^p_+ \times D^{q+1}$, we have $k'(x', y') = (Sx', f(Sx')Rf^{-1}(x')y')$ for $(x', y') \in \partial D^p_- \times D^{q+1}$. Therefore the map m restricted on $\partial D^p_- = S^{p-1}$ is equal to $f(Sx')Rf^{-1}(x')$. Since the map m is extendable to D^p_+, the homotopy class $R^{-1}m | S^{p-1}: S^{p-1} \to \mathrm{SO}_{q+1}$ is equal to zero. Hence it follows that $R^{\#}\{f\}=-\{f\}$. Conversely, if $R^{\#}\{f\}=-\{f\}$, such $m|S^{p-1}$ is extendable to D^p and we have an orientation preserving diffeomorphism which is mapped by A non-trivially, which completes the proof.

If the associated sphere bundle $\mathcal{S}_f$ has a cross section, then $R^{\#}\{f\}=\{f\}$. Consequently $R^{\#}\{f\}=-\{f\}$ if and only if $2\{f\}=0$. In case $p<q$, if $p\not\equiv 3 \pmod 4$, then $\pi_p(\mathrm{SO}_{q+1})=0$ or $\boldsymbol{Z}_2$; so $R^{\#}\{f\}=-\{f\}$ and the homomorphism A is surjective. If $p<q$ and $p\equiv 3 \pmod 4$, then $\pi_p(\mathrm{SO}_{q+1})\approx \boldsymbol{Z}$. Consequently $R^{\#}\{f\}=-\{f\}$ if and only if $\{f\}=0$, that is if and only if the bundle $\mathcal{E}_f$ is trivial.

Define the group X by

$$X=0 \quad \text{if } R^{\#}\{f\}\neq-\{f\}$$

$$X=\boldsymbol{Z}_2 \quad \text{if } R^{\#}\{f\}=-\{f\},$$

then combining Propositions 1.1 and 1.2, we have the following theorem.

THEOREM 1.3. *Let $p<2q-1$, then the order of $\tilde{\pi}_0(\mathrm{Diff}\, \mathcal{E}_f)$ is equal to the order of the direct sum group*

$$\pi_p(\mathrm{SO}_{q+1})/\mathrm{Ker}\, \mathrm{B}\oplus X.$$

In case $\mathcal{E}_f=S^p\times D^{q+1}$, we obtain the results more precisely.

PROPOSITION 1.4. *If $\mathcal{E}_f$ is a trivial bundle and $p<2q-1$, then the homomorphism* B *is a monomorphism, hence isomorphism.*

PROOF. From the condition $p<2q-1$, it follows that $q\geqq 2$. Suppose that two diffeomorphisms $\tilde{r}_1$ and $\tilde{r}_2$ of $S^p\times D^{q+1}$ defined by $\tilde{r}_i(x, y)=(x, r_i(x)y)$ are pseudo-diffeotopic, where r_i are C^∞-maps: $S^p\to \mathrm{SO}_{q+1}$ $(i=1, 2)$. Let $S^{p+q+1}=S^p\times D^{q+1}\cup D^{p+1}\times S^q$ be the decomposition. Then the two embeddings of $S^p\times D^{q+1}$ in S^{p+q+1} obtained from the natural embeddings followed by $\tilde{r}_i$ are concordant. Let C_p^{q+1} be the concordance classes of embeddings of S^p in S^{p+q+1} and let FC_p^{q+1} be the concordance class of orientation preserving embeddings of $S^p\times D^{q+1}$ in S^{p+q+1}. Then according to Haefliger [4], since $q+1\geqq 3$, C_p^{q+1} and FC_p^{q+1} have abelian group structures and the next exact sequence holds

$$\longrightarrow \mathrm{C}_{p+1}^{q+1}\xrightarrow{\partial}\pi_p(\mathrm{SO}_{q+1})\xrightarrow{j}\mathrm{FC}_p^{q+1}\xrightarrow{\pi}\mathrm{C}_p^{q+1}\longrightarrow.$$

Here the map ∂ is given by taking the characteristic class of normal bundle, j is defined by changing trivializations of the bundle $S^p\times D^{q+1}$ and π is a natural projection. Since $\partial: \mathrm{C}_{p+1}^{q+1}\to\pi_p(\mathrm{SO}_{q+1})$ is the zero map for $p<2q$ [6, Cor. 6.10], $\pi_p(\mathrm{SO}_{q+1})$ is mapped injectively in FC_p^{q+1}. Hence the above embeddings are pseudo-diffeotopic if and only if r_i belongs to the same class of $\pi_p(\mathrm{SO}_{q+1})$, which completes the proof.

The following theorem is a direct consequence of Propositions 1.2 and 1.4.

THEOREM 1.5. *If $p<2q-1$, the order of $\tilde{\pi}_0(\mathrm{Diff}\, S^p\times D^{q+1})$ is equal to the* *order of the direct sum group*

$$\pi_p(\mathrm{SO}_{q+1}) \oplus \boldsymbol{Z}_2 .$$

Let us study the group structure of $\tilde{\pi}_0(\mathrm{Diff}\, S^p \times D^{q+1})$. There exists an orientation preserving diffeomorphism l of $S^p \times D^{q+1}$, defined by $(x, y) \to (R_{p+1}x, R_{q+1}y)$, where R_{p+1} and R_{q+1} are reflections contained in O_{p+1} and O_{q+1} respectively. This diffeomorphism l is order 2 and mapped non-trivially by A.

LEMMA 1.6. *Let $p < 2q-1$. If $R^{\#}\{g_i\} = -\{g_i\}$, where $\{g_i\}$ are generators of homotopy group $\pi_p(\mathrm{SO}_{q+1})$, then the diffeomorphism l commutes with any element of* Ker A.

PROOF. Let a be an element of Ker A. Then by Proposition 1.1, we can take a representative h of a which is a bundle map of $S^p \times D^{q+1}$ defined by $(x, y) \to (x, r(x)y)$, where r is a C^∞-map: $S^p \to \mathrm{SO}_{q+1}$. Then $l^{-1}hl$ maps (x, y) to $(x, R^{-1}r(Sx)Ry)$. The homotopy class of $R^{-1}r(Sx)R : S^p \to \mathrm{SO}_{q+1}$ is equal to $R^{\#}\{-r\} \in \pi_p(\mathrm{SO}_{q+1})$. Since $R^{\#}\{g_i\} = -\{g_i\}$, it follows that $R^{\#}\{-r\} = -R^{\#}\{r\} = \{r\}$. Hence the diffeomorphism $l^{-1}hl$ is diffeomorphic to h, which completes the proof.

Let L be the group generated by l, which is isomorphic to $\boldsymbol{Z}_2$. Since l is mapped by A non-trivially, we can write $\tilde{\pi}_0(\mathrm{Diff}\, S^p \times D^{q+1}) = \mathrm{Ker\,A} \cdot L$. Further $\mathrm{Ker\,A} \cap L = \{\text{identity}\}$. Therefore the next theorem follows directly from Lemma 1.6.

THEOREM 1.7. *Let $p < 2q-1$. If $R^{\#}\{g_i\} = -\{g_i\}$, where $\{g_i\}$ are generators of homotopy groups $\pi_p(\mathrm{SO}_{q+1})$, then $\tilde{\pi}_0(\mathrm{Diff}\, S^p \times D^{q+1})$ is isomorphic to*

$$\pi_p(\mathrm{SO}_{q+1}) \oplus \boldsymbol{Z}_2 .$$

§ 2. Embeddings of sphere in sphere bundle.

In this section we assume that p and q satisfy the equation $p < q < 2p-3$. Let $\mathrm{Q}_q^p(f)$ be the set of concordance classes of embeddings of S^q in $\mathcal{F}_f$ whose homotopy classes fall on the fixed generator 1 of $\pi_q(S^q)$ by the natural projection $p_* : \pi_q(\mathcal{F}_f) \approx \pi_q(S^q) \oplus \pi_q(S^p) \to \pi_q(S^q)$. Since $p < q < 2p-3$, by Haefliger's theorem [4], any homotopy class is realizable by a differentiable embedding and two homotopic embeddings are diffeotopic. Since two concordant embeddings are homotopic, $\mathrm{Q}_q^p(f)$ corresponds bijectively to the homotopy group $p_*^{-1}(1) \approx \pi_q(S^p)$. Therefore we can define an abelian group structure on the set $\mathrm{Q}_q^p(f)$ by that of $\pi_q(S^p)$. We will give a geometric interpretation of this structure. Take two copies of $\mathcal{E}_g$, D^q bundle over S^p, and denote them by $\mathcal{E}_g^+$ and $\mathcal{E}_g^-$, where $s_*\{g\} = \{f\}$, s being the inclusion $\mathrm{SO}_q \to \mathrm{SO}_{q+1}$. By identifying their boundaries by the identity map, we have a manifold which is diffeomorphic to $\mathcal{F}_f$. We fix this diffeomorphism. Further identify S^q with the fibre in $\mathcal{E}_f$

over a fixed point x_0. Then we have

LEMMA 2.1. *Any embedding d of S^q in $\mathfrak{T}_f$, whose concordance class is an element of $\mathrm{Q}_q^p(f)$, is isotopic to an embedding e such that*

i) *$e|D_-^q$ is the identity embedding*

ii) $e(\mathrm{Int}\, D_+^q)\subset \mathrm{Int}\, \mathcal{E}_g^+$.

PROOF. Let $c: S^p \to \mathcal{E}_g^-$ be the zero cross section. Then the homological intersection number of $c(S^p)$ and $d(S^q)$ is equal to one. By the method of Whitney [**15**], since $p<q<2p-3$ we can take an embedding b of S^p in $\mathfrak{T}_f$ in the homotopy class of $c: S^p \to \mathfrak{T}_f$ such that $b(S^p)$ intersects transversely at one point x with $d(S^q)$. By Haefliger [**4**], there exists a diffeotopy H_t of $\mathfrak{T}_f$ moving $b(S^p)$ to $c(S^p)$ and $x\in b(S^p)$ to $y=c(S^p)\cap p^{-1}(x_0)\in c(S^p)$. Then $H_1d(D_-^q)$ and D_-^q are fibres over y of tubular neighborhoods. Therefore by the tubular neighborhood theorem, there exists a diffeotopy K_t scuh that K_0 is the identity and

$$\begin{cases} K_1H_1d\,|\,D_-^q = \text{identity} \\ K_1H_1d(\mathrm{Int}\, D_+^q)\subset \mathrm{Int}\, \mathcal{E}_q^+ . \end{cases}$$

Hence the embedding K_1H_1d is the required embedding e.

Analogously we have;

LEMMA 2.1′. *Any embedding d of S^q in $\mathfrak{T}_f$ whose concordance class is an element of $\mathrm{Q}_q^p(f)$, is isotopic to an embedding e' such that*

i) *$e'|D_+^q$ is the identity embedding*

ii) $e'(\mathrm{Int}\, D_-^q)\subset \mathrm{Int}\, \mathcal{E}_g^-$.

Let a_1 and a_2 be two elements of $\mathrm{Q}_q^p(f)$. We can represent a_1 by e_1 which satisfies conditions (i) and (ii) of Lemma 2.1 and a_2 by e_2 which satisfies (i) and (ii) of Lemma 2.1′. The class a_1+a_2 is defined as the class represented by the embedding defined by

$$(e_1+e_2)(x)=\begin{cases} e_1(x) & \text{for } x\in D_+^q \\ e_2(x) & \text{for } x\in D_-^q . \end{cases}$$

Since the homotopy class of a_1+a_2 is given by the sum of those of a_1 and a_2, this definition of the sum actually agrees with the preceding one defined by $\pi_q(S^p)$, and so a_1+a_2 is well-defined.

Let

$$\alpha_f:\ \mathrm{Q}_q^p(f)\to \pi_{q-1}(\mathrm{SO}_p)$$

be a map which associates to the embedded sphere the characteristic class of its normal bundle. It does not depend on the concordance class and the map α_f is well-defined. By the above interpretation of the sum operation in $\mathrm{Q}_q^p(f)$, it is easy to see that α_f is a homomorphism.

Let $\mathrm{FQ}_q^p(f)$ be the set of concordance classes of orientation preserving embeddings of $S^q\times D^p$ in $\mathcal{F}_f$ whose homotopy classes fall on the fixed generator 1 of $\pi_q(S^q)$ by the natural projection $p_*: [S^q\times D^p, \mathcal{F}_f]\approx\pi_q(\mathcal{F}_f)\approx\pi_q(S^q)\oplus\pi_q(S^p)\to\pi_q(S^q)$ $(p<q<2p-3)$, Let i be the isomorphism $\mathrm{Q}_q^p\approx\pi_q(S^p)$. Let $t\in i(\mathrm{Ker}\,\alpha_f)\subset\pi_q(S^p)$, then there exist some elements in $\mathrm{FQ}_q^p(f)$ whose homotopy class is equal to $1+t\in\pi_q(S^q)\oplus\pi_q(S^p)$. Denote by ${}_t\mathrm{FQ}_q^p(f)$ such subset of $\mathrm{FQ}_q^p(f)$. Then the set $\mathrm{FQ}_q^p(f)$ is equal to the disjoint union $\bigcup_{t\in i(\mathrm{Ker}\,\alpha f)}{}_t\mathrm{FQ}_q^p(f)$. We now study ${}_t\mathrm{FQ}_q^p(f)$. Let a and b be two embeddings of $S^q\times D^p$ in $\mathcal{F}_f$ representing elements x and y respectively, where $x, y\in{}_t\mathrm{FQ}_q^p$. Let $S^q\times 0$ be the zero cross section of the trivial bundle $S^q\times D^p$ over S^q. Then, since $q<2p-3$, by Haefliger, there exists a diffeotopy connecting the identity and a diffeomorphism h of $\mathcal{F}_f$ such that $ha|S^q\times 0=b|S^q\times 0$. Further by the tubular neighborhood theorem, we can take h so that ha and b differ by a bundle map of $S^q\times D^p$, which corresponds to an element of $\pi_q(\mathrm{SO}^p)$. Hence the number of elements of ${}_t\mathrm{FQ}_q^p(f)$ does not exceed the order of $\pi_q(\mathrm{SO}_p)$. But they are not the same, because two embeddings of ${}_t\mathrm{FQ}_q^p(f)$ differing by a non trivial bundle map may be concordant. Let $S_f(t)$ be the subset of $\pi_q(\mathrm{SO}_p)$, consisting of those elements $r\in\pi_q(\mathrm{SO}_p)$ such that there exists two concordant embedding c and d of ${}_t\mathrm{FQ}_q^p$ which differ by a bundle map of $S^q\times D^p$ corresponding to the element r. This is a well-defined subgroup. Therefore the set ${}_t\mathrm{FQ}_q^p(f)$ corresponds bijectively to the group $\pi_q(\mathrm{SO}_p)/S_f(t)$ and we can write $\mathrm{FQ}_q^p(f)=\bigcup_{t\in i(\mathrm{Ker}\,\alpha f)}\pi_q(\mathrm{SO}_p)/S_f(t)$.

Let a and b embeddings of $S^q\times D^p$ in $\mathcal{F}_f$ whose concordance classes belong to $\mathrm{FQ}_q^p(f)$, and let the homotopy classes of a and b be $1+t_1$ and $1+t_2$ respectively, where $t_1, t_2\in\pi_q(S^p)$. Then the next lemma is easy to see.

LEMMA 2.2. *Suppose that there exists a diffeomorphism k of $\mathcal{F}_f$ such that $ka=b$, then $S_f(t_1)=S_f(t_2)$.*

REMARK. Let x be an element of $\mathrm{Ker}\,\alpha_{f_0}\in\mathrm{Q}_q^p(f_0)$, where f_0 is the constant characteristic map. Then later in § 4, it will be proved that $S_{f_0}(i(x))=S_{f_0}(0)$, where 0 is the trivial element of $\pi_q(S^p)$.

The next lemma is also easy to prove.

LEMMA 2.3. *A framed embedding $e: S^q\times D^p\to\mathcal{F}_f$ representing an element of $\mathrm{FQ}_q^p(f)$, is concordant to the identity embedding if and only if there exists an orientation preserving embedding of $D^{q+1}\times D^p$ in $\mathcal{E}_f$ which is an extension of e.*

PROPOSITION 2.4. *Suppose $p<q<2p-4$, then $S_f(0)$ is equal to the image of $\alpha_{sf}: \mathrm{Q}_{q+1}^p(sf)\to\pi_q(\mathrm{SO}_p)$, where $sf: S^{p-1}\to\mathrm{SO}_{q+2}$ is the composition of f with the inclusion $s: \mathrm{SO}_{q+1}\to\mathrm{SO}_{q+2}$.*

PROOF. Given $\lambda\in\mathrm{Q}_{q+1}^p(sf)$, by Lemma 2.1, we can take a representative $e: S^{q+1}\to\mathcal{F}_{sf}$ such that $e(\mathrm{Int}\,D_+^{q+1})\subset\mathrm{Int}\,\mathcal{E}_f^+$ and $e|D_-^{q+1}$ is the identity map. Then $\alpha_{sf}(\lambda)$ can be regarded as the difference between the natural trivialization of the D^p-bundle $S^q\times D^p$ in $\mathcal{F}_f=\partial\mathcal{E}_f^+$ and the trivialization induced from the frame over $e|D_+^{q+1}$. Let $h: S^q\times D^p\to\mathscr{F}_f$ denote the framed embedding induced from $e|D_+^{q+1}$. Since h is extendable to an embedding $h': D^{q+1}\times D^p\to\mathcal{E}_f^+$, it is concordant to the identity by Lemma 2.3. Hence we have $\alpha_{sf}(\lambda)\subset S_f(0)$. Conversely given $\mu\in S_f(0)$, define a diffeomorphism $\tilde{\mu}$ of $S^q\times D^p$ by $\tilde{\mu}(x, y)=(x, \mu(x)y)$ and define an embedding $e(\mu)$ of $S^q\times D^p$ by the identity embedding composed with $\tilde{\mu}$. Then by definition, $e(\mu)$ is concordant to the indentity. By Lemma 2.3, it is extendable to an embedding k of $D^{q+1}\times D^p$ in $\mathcal{E}_f$. Let $k'=k|D^{q+1}\times 0$. Define an embedding l of S^{q+1} in $\mathscr{F}_{sf}$ by the identity embedding on D_-^{q+1} (we identify D_-^{q+1} with the fibre of $\mathcal{E}_f^-$) and by k' on D_+^{q+1} ($k': D_+^{q+1}\to\mathcal{E}_f^+$). Then $\alpha_{sf}(l)=\mu$, which shows that $\mu\in\operatorname{Im}\alpha_{sf}$. The proof is complete.

If the bundle is trivial, we can also define $S_{f_0}(t)\in\pi_q(\mathrm{SO}_p)$ and $\mathrm{Q}_q^p(f_0)$ for $q<p$ as in the case $p<q<2p-3$. But for $q<p$, since $\pi_q(S^p)=0$, $\mathrm{Q}_q^p(f_0)$ is the trivial group. The proof of the next lemma is analogous to Proposition 2.4.

LEMMA 2.5. *Suppose $q+1<p$, then $S_{f_0}(0)$ is the trivial group.*

Let $p<q<2p-2$. Given $x\in\pi_q(S^p)$, choose an embedding h of S^q in $\mathscr{F}_f$ whose homotopy class is equal to $0+x\in\pi_q(S^q)\oplus\pi_q(S^p)\approx\pi_q(\mathscr{F}_f)$. Define a map $F_f: \pi_q(S^p)\to\pi_{q-1}(\mathrm{SO}_p)$ by taking the characteristic class of the normal bundle of $h(S^q)$. By the Haefliger's theorem [**4**] and the Wall's result [**14**, Lemma 1], this map F_f is well-defined. For $p<q<2p-3$, we have denoted by i the isomorphism $\mathrm{Q}_q^p(f)\approx\pi_q(S^p)$. Concerning the homomorphism $\alpha_f: \mathrm{Q}_q^p(f)\to\pi_{q-1}(\mathrm{SO}_p)$, we have the next theorem (cf. Wall [**14**, Theorem 1]).

THEOREM 2.6 (Wall). *If $p<q<2p-3$, we have*

$$\alpha_f(e)=F_f(i(e))+\partial(i(e)),$$

where $\partial: \pi_q(S^p)\to\pi_{q-1}(\mathrm{SO}_p)$ is the boundary operation of the homotopy exact sequence of the fibration $\mathrm{SO}_p\to\mathrm{SO}_{p+1}\to S^p$.

PROOF. The homotopy class of $e\in\mathrm{Q}_q^p(f)$ is equal to $1+i(e)\in\pi_q(\mathscr{F}_f)$. We represent 1 and $i(e)$ by spheres S_1^q and S_2^q transverse to each other. The normal bundle of S_1^q is trivial and the normal bundle of S_2^q is characterized by $F_f(i(e))$. These spheres can be joined by a small tube obtained by thickening an arc which joins S_1 to S_2, but is disjoint from them except at the ends. We obtain an immersed sphere $\tilde{e}(S^q)$ in $\mathscr{F}_f$ representing $1+i(e)$ with normal bundle $F_f(i(e))$. We must modify this to be an embedding and see how this changes the normal bundle. For this modification, we choose an immersion d of S^q in S^{p+q} as follows. Consider S^{p+q} as $S^p\times D^q\cup D^{p+1}\times S^{q-1}$. Let $t: D^q\to S^p$ be a map whose homotopy class relative to the boundary is equal to $-i(e)$ and let $\mu: D^q\to\mathrm{SO}_{p+1}$ be its lifting. We can take μ as a C^∞-map. Then we define an immersion $d: S^q=D_+^q\cup D_-^q\to S^p\times D^q\subset S^{p+q}$ by

$$\begin{cases} d(q)=(p, q) & \text{if } q\in D_+^q \\ d(q)=(\mu(q)p, q) & \text{if } q\in D_-^q\,, \end{cases}$$

where p is a fixed point of S^p. Then the normal bundle of $d(S^q)$ in S^{p+q} is equal to $\partial(-i(e))=-\partial(i(e))$. Take the relative connected sum of $(\mathcal{T}_f, \tilde{e}(S^q))$ with $(S^{p+q}, d(S^q))$. In the new pair $(\mathcal{T}_f, e(S^q))$, two hemispheres are embedded. By the main theorem of Haefliger [**5**] and by our construction removing the intersection invariant, this immersion c is regularly homotopic to an embedding. The normal bundle of $c(S^q)$ is given by $F_f(i(e))-(-\partial(i(e))=F_f i(e)+\partial(i(e))$, which completes the proof of this theorem.

The map F_f can be regarded as a part of a pairing

$$F:\ \pi_{p-1}(\mathrm{SO}_q)\times\pi_q(S^p)\to\pi_{q-1}(\mathrm{SO}_p) \qquad (p<q<2p-2)$$

by changing the characteristic class of the sphere bundle. Several properties are described in Wall [**14**, Lemma 5], but complete homotopy-theoretic interpretation is posed to be a problem. Obviously if the first variable is trivial, F is the zero map and we have $\alpha_{f_0}(e)=\partial(i(e))$, where f_0 is the constant map.

§ 3. Map from $\tilde{\pi}_0(\mathrm{Diff}\ S^p\times S^q)$ to Θ^{p+q+1}.

At first we will define a map C from $\tilde{\pi}_0(\mathrm{Diff}\ S^p\times S^q)$ to Θ^{p+q+1}, the group of $p+q+1$-dimensional homotopy spheres, for $3\leqq p<2q-1$, $3\leqq q<2p-1$, $p\neq q$. Let f be a representative of $x\in\tilde{\pi}_0(\mathrm{Diff}\ S^p\times S^q)$. The manifold $\mathrm{c}(f)$ is constructed from the disjoint sum $S^p\times D^{q+1}\cup D^{p+1}\times S^q$ by identifying $(u, v)\in S^p\times\partial D^{q+1}$ with $f(u, v)\in\partial D^{p+1}\times S^q$. Differentiable structure is defined on $\mathrm{c}(f)$ by the canonical way. It is easy to see by the Van Kampen theorem and the Mayer-Vietoris exact sequence that $\mathrm{c}(f)$ is simply connected and is a homology sphere, hence a homotopy sphere. The orientation is chosen to be compatible with that of $S^p\times D^{q+1}$ and so with the inverse orientation of $D^{p+1}\times S^q$. We will prove that an orientation preserving diffeomorphism class of the homotopy sphere $\mathrm{c}(f)$ is independent of the pseudo-diffeotopy class of the representative f. Suppose f' be another representative of $x\in\tilde{\pi}_0(\mathrm{Diff}\ S^p\times S^q)$. Then there exists a diffeomorphism of $S^p\times S^q\times I$ onto itself which is equal to $f'f^{-1}$ and to the identity on $S^p\times S^q\times 0$ and $S^p\times S^q\times 1$ respectively. Consequently the diffeomorphism $f'f^{-1}$ is extendable to a diffeomorphism F of $D^{p+1}\times S^q$. Define a map d from $\mathrm{c}(f)$ to $\mathrm{c}(f')$ by

$$d=\begin{cases}\text{identity} & \text{on } S^p\times D^{q+1}\\ F & \text{on } D^{p+1}\times S^q.\end{cases}$$

The map d is well-defined and is an orientation preserving diffeomorphism. Hence we define a map

$$\mathrm{C}:\ \tilde{\pi}_0(\mathrm{Diff}\ S^p\times S^q)\to\Theta^{p+q+1}$$

by $\mathrm{C}(x)=\mathrm{c}(f)$. Let i be the identity map of $S^p\times S^q$, then obviously $\mathrm{c}(i)=S^{p+q+1}$, the natural $(p+q+1)$-sphere. But the map C is not necessarily a homomorphism. The counter example will be given in the appendix. We write by Ker C the subset of $\tilde{\pi}_0(\text{Diff } S^p \times S^q)$ consisting of those elements that are mapped by C to the natural sphere. Let η: Diff $S^p \times D^{q+1} \to$ Diff $S^p \times S^q$ and ω: Diff $D^{p+1} \times S^q \to$ Diff $S^p \times S^q$ be the restriction map to their boundary and let $\eta_*: \tilde{\pi}_0(\text{Diff } S^p \times D^{q+1}) \to \tilde{\pi}_0(\text{Diff } S^p \times S^q)$ and $\omega_*: \tilde{\pi}_0(\text{Diff } D^{p+1} \times S^q) \to \tilde{\pi}_0(\text{Diff } S^p \times S^q)$ be the induced homomorphisms.

LEMMA 3.1. Ker C *is equal to a subset consisting of elements expressed as*

$$\omega_*(\tilde{\pi}_0(\text{Diff } D^{p+1} \times S^q)) \cdot \eta_*(\tilde{\pi}_0(\text{Diff } S^p \times D^{q+1})),$$

where $\cdot$ *is the composition in* $\tilde{\pi}_0(\text{Diff } S^p \times S^q)$.

PROOF. We may suppose $p < q$. Assume that $\{f\} = x \in \tilde{\pi}_0(\text{Diff } S^p \times S^q)$ belongs to Ker C, where $f \in$ Diff $S^p \times S^q$ and $\{f\}$ means its pseudo-diffeotopy class. Then there exists a diffeomorphism d from $c(i)$ to $c(f)$. Recall $c(i) = S^p \times D^{q+1} \bigcup_i D^{p+1} \times S^q$, $c(f) = S^p \times D^{q+1} \bigcup_f D^{p+1} \times S^q$. By the tubular neighborhood theorem and by the Haefliger's theorem, we can replace d in its diffeotopy class such that d maps $S^p \times D^{q+1}$ onto $S^p \times D^{q+1}$. Let $e \in$ Diff $S^p \times D^{q+1}$ be the restriction of d on $S^p \times D^{q+1}$. Then on the boundary of $D^{p+1} \times S^q$, the diffeomorphism d is equal to $f \cdot \eta(e)$. Since $d \mid \partial D^{p+1} \times S^q$ is extendable to $D^{p+1} \times S^q$, $f \cdot \eta(e) \in \omega(\text{Diff } D^{p+1} \times S^q)$. Therefore $f \in \omega(\text{Diff } D^{p+1} \times S^q) \cdot \eta(\text{Diff } S^p \times D^{q+1})$ and $x \in \omega_*(\tilde{\pi}_0(\text{Diff } D^{p+1} \times S^q)) \cdot \eta_*(\tilde{\pi}_0(\text{Diff } S^p \times D^{q+1}))$. Conversely let $y = \omega_*\{g\} \cdot \eta_*\{h\} \in \tilde{\pi}_0(\text{Diff } S^p \times S^q)$, where $g \in$ Diff $(D^{p+1} \times S^q)$ and $h \in$ Diff $(S^p \times D^{q+1})$. Remark that $C(y) = c(\omega(g) \cdot \eta(h)) = S^p \times D^{q+1} \bigcup_{\omega(g) \cdot \eta(h)} D^{p+1} \times S^q$. Then we define a map k: $c(i) \to C(y)$ by

$$k = \begin{cases} h^{-1}: & S^p \times D^{q+1} \to S^p \times D^{q+1} \\ g: & D^{p+1} \times S^q \to D^{p+1} \times S^q. \end{cases}$$

This is a well-defined (orientation preserving) diffeomorphism, which shows that y is contained in Ker C.

Denote by m the diffeomorphism of $S^p \times S^q$ defined by $m(x, y) = (R_{p+1}x, R_{q+1}y)$, where R_{p+1} and R_{q+1} are reflections of S^p and S^q respectively.

LEMMA 3.2. *Let* $2 \leqq p < 2q-1$, $2 \leqq q < 2p-1$. *Then*

i) *the homomorphisms* η_* *and* ω_* *are monomorphisms,*

ii) $\eta_*(\tilde{\pi}_0(\text{Diff } S^p \times D^{q+1})) \cap \omega_*(\tilde{\pi}_0(\text{Diff } D^{p+1} \times S^q)) = \{i, m\}$.

PROOF. To prove the injectivity of η_*, it is sufficient to show the injectivity of η_* for Ker A. By Proposition 1.4, Ker A is isomorphic to $\pi_p(SO_{q+1})$. We identify them. We again use the Haefliger's exact sequence

$$\longrightarrow C_{p+1}^{q+1} \longrightarrow \pi_p(SO_{q+1}) \xrightarrow{j} FC_p^{q+1} \longrightarrow C_p^{q+1} \longrightarrow,$$

where j is injective for $p < 2q$. Suppose that $\eta_*(r) = 0$, where $\{r\} \in \pi_p(SO_{q+1}) = \text{Ker A}$. Then the diffeomorphism of $S^p \times S^q$ given by $(x, y) \to (x, r(x)y)$ is extendable to a diffeomorphism of $D^{p+1}\times S^q$. Let $S^{p+q+1}=D^{p+1}\times S^q \bigcup_i S^p\times D^{q+1}$ be the decomposition. Denote by $\tilde{r}$ the diffeomorphism of $S^p\times D^{q+1}$ defined by $\tilde{r}(x, y)=(x, r(x)y)$. Then $\tilde{r}$ is extendable to a diffeomorphism h, where $h\in \operatorname{Diff} S^{p+q+1}$. Given an element of $\pi_0(\operatorname{Diff} S^{p+q+1})$, we can choose a representative which fixes D^{p+q+1}. Consequently we can take h' which is an extension of r such that h' is diffeotopic to the identity map of S^{p+q+1}. Since the embedding $j(r)$ is defined to be the identity embedding composed with $\tilde{r}$, it follows that $j(r)=0$. Since j is injective, η_* is also injective for Ker A. The injectivity of ω_* is analogous and i) follows. We will prove ii). First we study $\eta_*(\operatorname{Ker} A)\cap\omega_*(\operatorname{Ker} A)=\eta_*(\pi_p(SO_{q+1}))\cap\omega_*(\pi_q(SO_{p+1}))$. If $\eta_*(s)$ lies in $\omega_*(\pi_q(SO_{p+1}))$, where $s\in\pi_p(SO_{q+1})$, then the diffeomorphism $\tilde{s}: S^p\times D^{q+1}\to S^p\times D^{q+1}$ defined by $\tilde{s}(x, y)=(x, s(x)y)$ is extendable to a diffeomorphism of S^{p+q+1}. Similar argument as above shows that $j(s)=0$ and consequently $\{s\}=0$. Hence $\eta_*(\operatorname{Ker} A)\cap\omega_*(\operatorname{Ker} A)$ contains the class of the identity element only. Let us define the diffeomorphism l of $S^p\times D^{q+1}$ by $l(x, y)=(R_{p+1}x, R_{q+1}y)$ and diffeomorphism l' of $D^{p+1}\times S^q$ by $l'(x, y)=(R_{p+1}x, R_{q+1}y)$. Then $\eta(l)=\omega(l')=m$. By § 1, any element of $\tilde{\pi}_0(\operatorname{Diff} S^p\times D^{q+1})$ (resp. $\tilde{\pi}_0(\operatorname{Diff} D^{p+1}\times S^q)$) which does not belong to Ker A is written as $l\cdot r$, where $r\in\pi_p(SO_{q+1})$. (resp. $l'\cdot r'$, where $r'\in\pi_q(SO_{p+1})$). Obviously $\eta_*(l\cdot\pi_p(SO_{q+1}))\cap\omega_*(\pi_q(SO_{p+1}))=\eta_*(\pi_p(SO_{q+1}))\cap\omega_*(l'\pi_q(SO_{p+1}))=\phi$. If $\eta_*(l\cdot s)=\omega_*(l'\cdot t)$, where $\{s\}\in\pi_p(SO_{q+1})$ and $\{t\}\in\pi_q(SO_{p+1})$, then it follows that $\eta_*(s)=\omega_*(t)$. Hence $\{s\}=\{t\}=0$ by the former part of this proof, which completes the proof of ii).

COROLLARY 3.3. *The order of* Ker C *is equal to the order of the direct sum group* $\pi_p(SO_{q+1})\oplus\pi_q(SO_{p+1})\oplus \boldsymbol{Z}_2$ *for* $3\leqq p<2q-1$, $3\leqq q\leqq 2p-1$, $p\neq q$.

PROOF. If $\omega_*(a)\cdot\eta_*(x)=\omega_*(a')\cdot\eta_*(x')$, where $a, a'\in\pi_q(SO_{p+1})$ and x, x' $\pi_p(SO_{q+1})$, then $\omega_*(a')^{-1}\omega_*(a)=\eta_*(x')\eta_*(x^{-1})$. From Lemma 3.2 ii), they are equal to zero, and the corollary follows by Lemma 3.2 i).

We identify $\operatorname{Ker} A\subset\tilde{\pi}_0(\operatorname{Diff} S^p\times D^{q+1})$ with $\pi_p(SO_{q+1})$ and $\operatorname{Ker} A\subset\pi_0(\operatorname{Diff} D^{p+1}\times S^q)$ with $\pi_q(SO_{p+1})$ for $2\leqq p<2q-1$, $2\leqq q\leqq 2p-1$ as before.

LEMMA 3.4. *Suppose* $3\leqq p<2q-4$, $3\leqq q<2p-4$, $p\neq q$. *If an element* $\tilde{\pi}_0(\operatorname{Diff} S^p\times S^q)$ *is written as* $\omega_*(a)\cdot\eta_*(x)=\eta_*(x')\cdot\omega_*(a')$, *where* $a, a'\in\pi_p(SO_{p+1})$ *and* $x, x'\in\pi_p(SO_{q+1})$, *and if* a, a' *are suspension elements, then* $a=a'$.

PROOF. We can take representatives $g, g'\in\operatorname{Diff} S^p\times S^q$ of $\eta_*(b), \eta_*(b')$ respectively, which keeps $D^p_+\times S^q$ invariant. Let s be the inclusion of SO_p in SO_{p+1}. Since a, a' are suspension elements, there exists mapping $r_i: S^q\to SO_p$ $(i=1, 2)$ such that $s_*(r_1)=a$ and $s_*(r_2)=a'$. Further we can regard that r_i are C^∞-maps. Hence we can take $f, f'\in\operatorname{Diff} S^p\times S^q$ as respective representatives of $\omega_*(a), \omega_*(a')$ which map $D^p_+\times S^q$ and $D^p_-\times S^q$ onto itself respectively such that

$$f(x, y)=(x, r_1(x)y)\qquad \text{for } (x, y)\in D^p_+\times S^q \text{ or } D^p_-\times S^q$$

$$f'(x, y) = (x, r_2(x)y) \qquad \text{for } (x, y) \in D^p_+ \times S^q \text{ or } D^p_- \times S^q .$$

Consequently we have representatives fg and $g'f'$ of $\omega_*(a) \cdot \eta_*(b)$ and $\eta_*(b') \cdot \omega_*(a')$ respectively, which map $D^p_- \times S^q$ onto itself by f and f'. These are pseudo-diffeotopic by our assumption $\omega_*(a) \cdot \eta_*(b) = \eta_*(b') \cdot \omega_*(a')$. Hence $f|D^p_- \times S^q$ and $f'|D^p_- \times S^q$ are concordant as embeddings of $D^p \times S^q$ in $S^p \times S^q$. Since these differ only by the bundle map $f'^{-1}f$, the element $\{r_2^{-1}r_1\} \in \pi_q(\mathrm{SO}_p)$ lies in $S_{f_0}(0)$ of § 2 (in case $q < p$, generalized $S_{f_0}(0)$ of Lemma 2.5.). By Proposition 2.4 or by Lemma 2.5, the element $\{r_2^{-1}r_1\}$ lies in the image of the boundary operator $\partial : \pi_{q+1}(S^p) \to \pi_q(\mathrm{SO}_p)$. This implies that $\{r_1\}$ and $\{r_2\}$ go into the same element by $s_* : \pi_q(\mathrm{SO}_p) \to \pi_q(\mathrm{SO}_{p+1})$, which shows $a = a'$.

LEMMA 3.5. *The map* C *is surjective.*

PROOF. Assume $p < q$. Given a homotopy sphere Σ, we can choose an orientation reversing embedding $i : S^q \times D^{p+1} \to \Sigma$. The complement $\Sigma - \mathrm{Int}\,(i(S^q \times D^{p+1}))$ is simply connected and has the homotopy type of S^p. Since $p < \frac{p+q+1}{2}$ and Σ is π-manifold, we can choose an orientation preserving embedding $j : S^p \times D^{q+1} \to \Sigma - \mathrm{Int}\,(i(S^q \times D^{p+1}))$ such that $j(S^p \times 0)$ represents the generator of $\mathrm{H}_p[\Sigma - \mathrm{Int}\,\{i(S^q \times D^{p+1})\}] \approx \boldsymbol{Z}$. By the h-cobordism theorem [**11**], $[\Sigma - \mathrm{Int}\,\{i(S^q \times D^{p+1})\}]$ is diffeomorphic to $j(S^p \times D^{q+1})$. Hence we can regard Σ as the sum of $S^q \times D^{p+1}$ and $S^p \times D^{q+1}$ glued by an orientation preserving diffeomorphism. In case $p > q$, the proof is similar.

PROPOSITION 3.6. *If* $p < q$ *and* $\pi_p(\mathrm{SO}_q) = 0$, *then*

i) $\tilde{\pi}_0(\mathrm{Diff}\, S^p \times S^q)$ *is abelian*

ii) *the map* C *is a homomorphism.*

PROOF. Let f be an element of Diff $S^p \times S^q$. Then, since $p < q$ and $\pi_p(\mathrm{SO}_q) = 0$, we can take f' in the diffeotopy class of f such that it keeps $S^p \times D^q_+$ invariant, by first moving $f|S^p \times 0$ to the identity and next moving $f|S^p \times D^q_+$ to the identity by the tubular neighborhood theorem. Similarly, given $g \in \mathrm{Diff}\, S^p \times S^q$, we can take g' in the diffeotopy class of g, such that it keeps $S^p \times D^q_-$ invariant. Then we have $f'g' = g'f'$, which proves ii). To show ii), it is sufficient to prove that $\mathrm{c}(fg)$ is orientation preserving diffeomorphic to $\mathrm{c}(f) \# \mathrm{c}(g)$. We can take f such that it fixes $S^p \times D^q$ as above. The manifold $\mathrm{c}(f)$ is made from $S^p \times D^{q+1} \cup D^{p+1} \times S^q$ attached their boundaries by f. Decompose $D^{p+1} \times S^q$ as $D^{p+1} \times D^q_- \cup D^{p+1} \times D^q_-$. Then manifold $\mathrm{c}(f)$ is made by firstly attaching $S^p \times D^{q+1}$ to $D^{p+1} \times D^q_+$ by the identity map from $S^p \times D^q_+ \subset S^p \times \partial D^{q+1} \subset \partial(S^p \times D^{q+1})$ to $S^p \times D^q_+ \subset (D^{p+1} \times D^q_+)$, the resulting manifold being $(p+q+1)$-dimensional disk, and secondly by attaching it to $D^{p+1} \times D^q_-$ by an attaching map $\tilde{f} : S^p \times D^q_- \cup D^{p+1} \times \partial D^q_+ \to S^p \times D^q_- \cup D^{p+1} \times S^{q-1} \subset (D^{p+1} \times D^q_-)$ which is equal to f on $S^p \times D^q_-$ and equal to the identity on $D^{p+1} \times \partial D^q_+ = D^{p+1} \times S^{q-1}$. We can regard $\tilde{f} \in \mathrm{Diff}\, S^{p+q}$. Let $p : \tilde{\pi}_0(\mathrm{Diff}\, S^{p+q}) \to \Theta^{p+q+1}$ be the natural isomorphism.

Then $p(\tilde{f})=\mathrm{c}(f)$ by the above construction. Similarly we can choose $\tilde{g}\in \mathrm{Diff}\, S^{p+q}$. The connected sum operation corresponds to the composition of $\mathrm{Diff}\, S^{p+q}$. Consequently $\mathrm{c}(fg)=p(\tilde{f}\tilde{g})=\mathrm{c}(f)\,\#\,\mathrm{c}(g)$, which completes the proof.

As a consequence of these lemmas we have the following theorem.

THEOREM 3.7. *If* $3\leqq p<q<2p-1$ *and* $\pi_p(\mathrm{SO}_q)=0$, *then the order of* $\tilde{\pi}_0(\mathrm{Diff}\, S^p\times S^q)$ *is equal to the order of the direct sum group*

$$\pi_q(\mathrm{SO}_{p+1})\oplus\Theta^{p+q+1}\oplus \boldsymbol{Z}_2\,.$$

This results will be extended to the case $\pi_p(\mathrm{SO}_q)\neq 0$ in the next section.

§ 4. Computation of $\tilde{\pi}_0(\mathrm{Diff}\, \mathcal{F}_f)$.

In this section we will investigate $\pi_0(\mathrm{Diff}\, \mathcal{F}_f)$ for $p<q<2p-3$. This restriction of dimension is assumed throughout the present section. The study is carried out as follows. First we define a homomorphism.

$$\mathrm{A}'\colon \tilde{\pi}_0(\mathrm{Diff}\, \mathcal{F}_f)\to \boldsymbol{Z}_2\,,$$

by the induced automorphism of $\mathrm{H}_*(\mathcal{F}_f)$ and investigate $\mathrm{Im}\,\mathrm{A}'$. The group $\mathrm{Ker}\,\mathrm{A}'$ is consisting of pseudo-diffeotopy classes of $\mathcal{F}_f$ which induce the identity map in $\mathrm{H}^*(\mathcal{F}_f)$. in order to study $\mathrm{Ker}\,\mathrm{A}'$, we define a homomorphism

$$\mathrm{K}\colon \mathrm{Ker}\,\mathrm{A}'\to\pi_q(S^p)\,,$$

by the induced automorphism of $\pi_q(\mathcal{F}_f)\approx\pi_q(S^q)\oplus\pi_q(S^p)$, and investigate $\mathrm{Im}\,\mathrm{K}$. In order to study $\mathrm{Ker}\,\mathrm{K}$, we define a homomorphism.

$$\mathrm{L}\colon \mathrm{Ker}\,\mathrm{K}\to\pi_q(\mathrm{SO}_p)/\mathrm{Im}\,\alpha_f$$

by the induced automorphism of the tubular neighborhood of a fibre S^q, and investigate $\mathrm{Im}\,\mathrm{L}$. For the study of $\mathrm{Ker}\,\mathrm{L}$, we define a homomorphism

$$\mathrm{M}\colon \mathrm{Ker}\,\mathrm{L}\to\Theta^{p+q}/G_f\,.$$

The investigation of the image and the kernel of M completes our calculation of $\tilde{\pi}_0(\mathrm{Diff}\, \mathcal{F}_f)$.

(1) We will define the homomorphism A' and study $\mathrm{Im}\,\mathrm{A}'$. A diffeomorphism of $\mathcal{F}_f$ induces an automorphism of $\mathrm{H}_*(\mathcal{F}_f)$. We have

$$\mathrm{H}_i(\mathcal{F}_f)=\begin{cases}\boldsymbol{Z} & \text{for } i=0,\ p,\ q,\ p+q\\ 0 & \text{otherwise}\,.\end{cases}$$

By an orientation preserving diffeomorphism, generators of $\mathrm{H}_0(\mathcal{F}_f)$ and $\mathrm{H}_{p+q}(\mathcal{F}_f)$ must be mapped to the same generators. Furthermore by the Poincaré duality, if a generator of $\mathrm{H}_p(\mathcal{F}_f)$ is mapped to the other generator, so is the generator of $\mathrm{H}_q(\mathcal{F}_f)$. Therefore the induced automorphism group of $\mathrm{H}_*(\mathcal{F}_f)$ is isomorphic to $\boldsymbol{Z}_2$. Hence we have a homomorphism

$$\mathrm{A}': \tilde{\pi}_0(\mathrm{Diff}\ \mathcal{F}_f) \to \boldsymbol{Z}_2$$

analogous to the homomorphism $\mathrm{A}: \tilde{\pi}_0(\mathrm{Diff}\ \mathcal{E}_f) \to \boldsymbol{Z}_2$ defined in § 1. We have the next proposition.

PROPOSITION 4.1. $\mathrm{Im}\,\mathrm{A} = \mathrm{Im}\,\mathrm{A}'$.

The proof is postponed until the end of this section. We will investigate $\mathrm{Ker}\,\mathrm{A}'$ next.

(2) We now define the homomorphism $\mathrm{K}: \mathrm{Ker}\,\mathrm{A}' \to \pi_q(S^p)$ and investigate $\mathrm{Im}\,\mathrm{K}$. Given an element $x \in \mathrm{Ker}\,\mathrm{A}'$, let $g \in \mathrm{Diff}\ \mathcal{F}_f$ be its representative. The diffeomorphism g induces an automorphism of $\pi_q(\mathcal{F}_f) \approx \pi_q(S^q) \oplus \pi_q(S^p)$. By the Hurewicz homomorphism the direct summand $\pi_q(S^q) \approx \boldsymbol{Z}$ is mapped isomorphically onto $\mathrm{H}_q(\mathcal{F}_f) \approx \boldsymbol{Z}$, and the direct summand $\pi_q(S^p)$ is mapped to the zero element. Since $\{g\}$ belongs to $\mathrm{Ker}\,\mathrm{A}'$, g maps the generators of $\mathrm{H}_i(\mathcal{F}_f)$ to the same elements. Hence the induced automorphism g_* of $\pi_q(\mathcal{F}_f)$ maps the element $1+0$ to an element $1+\mathrm{k}(g)$, where 0 denotes the trivial element of $\pi_q(S^p)$, 1 denotes a generator of $\pi_q(S^q)$ and $\mathrm{k}(g) \in \pi_q(S^p)$. We define the map

$$\mathrm{K}: \mathrm{Ker}\,\mathrm{A}' \to \pi_q(S^p)$$

by $\mathrm{K}(x) = \mathrm{k}(g)$. Obviously it is well-defined.

LEMMA 4.2. *The map* K *is a homomorphism.*

PROOF. Let g_1 and g_2 be elements of $\mathrm{Diff}\ \mathcal{F}_f$, whose pseudo-diffeotopy classes belong to $\mathrm{Ker}\,\mathrm{A}'$. Since $p<q$ we can assume that g_i $(i=1, 2)$ keeps the cross section $c(S^p)$ invariant. Hence $g_{i*}(0+x) = 0+x$, where $0+x \in \pi_q(S^q) \oplus \pi_q(S^p) \approx \pi_q(\mathcal{F}_f)$. Consequently we have $(g_1g_2)_*(1+0) = g_{1*}(1+k(g_2)) = g_{1*}((1+0) + (0+k(g_2))) = 1+k(g_2)+k(g_1)$, which completes the proof.

In our range of dimension, $p<q<2p-3$, the isotopy classes of embeddings of S^q in $\mathcal{F}_f$ correspond bijectively to the homotopy classes $\pi_q(\mathcal{F}_f)$. We define a map

$$\beta_f: \pi_q(S^p) \to \pi_{q-1}(\mathrm{SO}_p)$$

by defining $\beta_f(x)$ to be the characteristic class of the normal bundle to the embedded S^q in $\mathcal{F}_f$ whose homotopy class is equal to $1+x \in \pi_q(S^q) \oplus \pi_q(S^p) \approx \pi_q(\mathcal{F}_f)$. We have defined the homomorphism $\alpha_f: \mathrm{Q}_q^p(f) \to \pi_{q-1}(\mathrm{SO}_p)$ and the isomorphism $i: \mathrm{Q}_q^p(f) \to \pi_q(S^p)$ in § 2. Then obviously $\beta_f = \alpha_f i^{-1}$. By theorem 2.6, β_f is a homomorphism and $\beta_f(x) = F_f(x) + \partial(x)$. Obviously we have the following lemma.

LEMMA 4.3. *The group* $\mathrm{Im}\,\mathrm{K}$ *is a subgroup of* $\mathrm{Ker}\,\beta_f$.

We will now investigate this subgroup $\mathrm{Im}\,\mathrm{K}$. Consider $\mathcal{F}_f$ to be $\mathcal{E}_g^+ \cup \mathcal{E}^- = D_+^p \times D_1^q \cup D_-^p \times D_1^q \cup \mathcal{E}_g^-$. Let D^{p+q} be a fixed disc in the interior of $D_+^p \times D_1^q$, and let N denote the closure of $\mathcal{F}_f - D^{p+q}$. Given an element x of $\mathrm{Ker}\,\beta_f$, we can choose an embedding $e: S^q \to N \subset \mathfrak{T}_f$ whose homotopy class is equal to $1+x \in \pi_q(\mathfrak{T}_f)$. Since x belongs to Ker β_f, the tubular neighborhood of $e(S^q)$ is trivial. Hence we can extend e to an embedding $d: S^q \times D^p \to N$, where $d|S^q \times 0 = e$. Let $S^q \times D^p = D^p_+ \times D^q \cup D^p_- \times D^q$ and $\mathcal{E}_g^- = D^p_+ \times D^q_2 \cup D^p_- \times D^q_2$ be the decompositions. By Lemma 2.1, we can replace d in its diffeotopy class by an embedding satisfying;

i) $d|D^p_- \times D^q =$ canonical embedding of $D^p_- \times D^q$ in $\mathcal{E}_g^-$

ii) $d(\mathrm{Int}\,(D^p_+ \times D^q)) \subset \mathrm{Int}\,(D^p_+ \times D^q_1)$.

Let $W = d(S^q \times D^p) \cup \mathcal{E}_g^-$ be the union space in $\mathfrak{T}_f$. Since the zero cross section of $\mathcal{E}_g^-$ and $d(S^q \times 0)$ generate $\mathrm{H}_p(\mathfrak{T}_f)$ and $\mathrm{H}_q(\mathfrak{T}_f)$ respectively, the inclusion $\partial W \to N - \mathrm{Int}\,W$ is a homotopy equivalence. Consequently ∂N and ∂W are h-cobordant. Therefore by Smale's theorem [11], N and W are diffeomorphic. Let j denote the diffeomorphism $N \to W$. Let $T_0 = D^p_- \times D^q_1 \cup D^p_- \times D^q_2 = D^p \times S^q$, which is the tubular neighborhood of the canonical fibre and let $V = T_0 \cup \mathcal{E}_g^-$. Then similarly as above N and V are diffeomorphic and we have a diffeomorphism $k: N \to V$. Write $T(x)$ for $d(S_q \times D_p)$. There exists a diffeomorphism $l: V = T_0 \cup \mathcal{E}_g^- \to W = T(x) \cup \mathcal{E}_g^-$ defined by the identity on $\mathcal{E}_g^-$ and by a bundle map from T_0 to $T(x)$ which is the identity on $T_0 \cap \mathcal{E}_g^- = T(x) \cap \mathcal{E}_g^-$. The diffeomorphism $j^{-1}lk: N \to N$ maps the homotopy class $1+0 \in \pi_q(S^q) \oplus \pi_q(S^p) \approx \pi_q(N)$ $\pi_q(\mathfrak{T}_f)$ to $1+x \in \pi_q(\mathfrak{T}_f)$. If this diffeomorphism $j^{-1}lk$ of N is extendable to a diffeomorphism of $\mathfrak{T}_f = N \cup D^{p+q}$, then x lies in Im K. To know this extendability we now define the set map

$$\Psi: \mathrm{Ker}\,\beta_f \to \{\text{Subsets of } \Theta^{p+q}\},$$

where Θ^{p+q} is the group of $p+q$-dimensional homotopy spheres. Take two copies of the disk bundles $\mathcal{E}_f$. Attach them on the portions $N \subset \partial\mathcal{E}_f = \mathfrak{T}_f$ by the map $j^{-1}lk: N \to N$. Then we obtain a $(p+q+1)$-dimensional manifold $X(l)$. Its boundary $\varphi(l)$ is a $(p+q)$-dimensional homotopy sphere, which naturally corresponds to the obstruction of extension of $j^{-1}lk$ to a diffeomorphism of whole $\mathfrak{T}_f$. Changing l by taking other bundle maps $T_0 \to T(x)$, we define a subset $\Psi(x)$ of Θ^{p+q} by $\{\varphi(l)\}$. If we define the map $l': V = T_0 \cup \mathcal{E}_g^- \to W = T(x) \cup \mathcal{E}_g^-$ by a bundle map from $\mathcal{E}_g^- \subset V$ to $\mathcal{E}_g^- \subset W$ and by a bundle map from T_0 to $T(x)$, then the set $\{\varphi(l')\}$ is equal to $\{\varphi(l)\}$, because any bundle map of $\mathcal{E}_g^- \subset \mathfrak{T}_f$ is extendable to a diffeomorphism of $\mathfrak{T}_f$ by the reflection to $\mathcal{E}_g^+$. Suppose we choose other embeddings $e': S^q \to N$ and $d': S^q \times D^p \to N$. Then by the Haefliger's theorem [4], and by the tubular neighborhood theorem, there exists a diffeomorphism h of $\mathfrak{T}_f$ mapping $W = T(x) \cup \mathcal{E}_g^-$ to $W' = T'(x) \cup \mathcal{E}_g^-$, where $T'(x) = d'(S^q \times D^p)$ such that $i'|T(x)$ and $i'|\mathcal{E}_g^-$ are bundle maps mapping $T(x)$ to $T'(x)$ and $\mathcal{E}_g^- \subset W$ to $\mathcal{E}_g^- \subset W'$. Consequently the definition $\Psi(x)$ does not depend on the choice of $T(x)$. Hence we obtain the map

$$\Psi: \mathrm{Ker}\,\beta \to \{\text{Subsets of } \Theta^{p+q}\}.$$

PROPOSITION 4.4. *An element* $x \in \mathrm{Ker}\,\beta_f$ *lies in* Im K *if and only if* $\Psi(x)$ *contains the natural sphere.*

PROOF. Suppose $\Psi(x)$ contains the natural sphere. Then there exists a diffeomorphism $l: V \to W$ mapping T_0 to $T(x)$ and $\mathcal{E}_g^- \subset V$ to $\mathcal{E}_g \subset W$ such that $j^{-1}lk \mid \partial N = S^{p+q+1}$ is extendable to a diffeomorphism of D^{p+q}. Hence $j^{-1}lk$ is extendable to the diffeomorphism of $\mathcal{F}_f$, which shows that x lies in Im K. Conversely suppose that $y \in \mathrm{Ker}\,\beta_f$ belongs to Im K. Then there exists a diffeomorphism m of $\mathcal{F}_f$ such that jmk^{-1} is a diffeomorphism: $V = T_0 \cup \mathcal{E}_g^- \to W = T(y) \cup \mathcal{E}_g^-$. We can suppose that both $m \mid T_0$ and $m \mid \mathcal{E}_g^-$ are bundle maps and the proposition holds.

The map Ψ will be interpreted as a homomorphism.

$$\Psi': \mathrm{Ker}\,\beta_f \to \text{a factor group of } \Theta^{p+q}$$

in Lemma 4.11.

We have shown in Lemma 4.3 that Im K is a subgroup of $\mathrm{Ker}\,\beta_f$. But we have the next proposition.

PROPOSITION 4.5. *If the bundle* $\mathcal{F}_f$ *is trivial, then* Im K *is equal to* $\mathrm{Ker}\,\beta_f$.

PROOF. Recall that $X(l) = \mathcal{E}_f \underset{j^{-1}lk}{\cup} \mathcal{E}_f$. The homology group of $X(l)$ is as follows.

$$\mathrm{H}_i(X(l)) = \begin{cases} \boldsymbol{Z} & \text{for } i = 0,\ p,\ q+1 \\ 0 & \text{otherwise}. \end{cases}$$

The boundary $\partial X(l)$ is a homotopy sphere. A generator of p-dimensional homology group of $X(l)$ can be realized by an embedded sphere S^p. Its tubular neighborhood U is trivial by our assumption that $\mathcal{F}_f$ is trivial. Therefore we can do the usual spherical modification [8]. We make a new manifold $Y = (X(l) - U) \cup D^{p+1} \times S^q$. Since Y is contractible, the boundary of Y, which is equal to $\varphi(l)$ is diffeomorphic to the natural sphere. The proposition follows directly from Proposition 4.4.

Since F_{f_0} is a trivial map for a trivial bundle $\mathcal{F}_{f_0}$, we obtain the following corollary by combining Proposition 4.5 and Theorem 2.6.

COROLLARY 4.6. *If the bundle is trivial, then* Im K *is isomorphic to the image of the projection map* $p_*: \pi_q(\mathrm{SO}_{p+1}) \to \pi_q(S^p)$ *for* $q < 2p-3$.

(3) At this stage we study Ker K. For this purpose, we define a homomorphism $\mathrm{L}: \mathrm{Ker\,K} \to \pi_q(\mathrm{SO}_p)/\mathrm{Im}\,\alpha_f$ and investigate Im L. Let S^q be the canonical fibre of $\mathcal{F}_f$ and let T_0 be the tubular neighborhood of S^q which is a trivial bundle. Let $(\kappa 1)$ be the subgroup of Diff $\mathcal{F}_f$ consisting of diffeomorphisms which keep S^q invariant and map T_0 by a bundle map, whose pseudo-diffeotopy classes belongs to Ker K. Let g be an element of $(\kappa 1)$. We define

$l(g) \in \pi_q(\mathrm{SO}_p)$ by the corresponding homotopy class of the bundle map. Then $l: (\kappa 1) \to \pi_q(\mathrm{SO}_p)$ is a well-defined homomorphism. But if we regard l as a homomorphism: $\mathrm{Ker\,K} \to \pi_q(\mathrm{SO}_p)$, it is not well-defined. Let $(\kappa 2)$ be the subgroup of $(\kappa 1)$ consisting of elements of $(\kappa 1)$ which are pseudo-diffeotopic to the identity.

LEMMA 4.7. *The image of* $(\kappa 2)$ *by* l *is contained in* $S_f(0)$.

PROOF. Let $g' \in (\kappa 2)$. Then the two embeddings of $S^q \times D^p$, one by the identity and the other by changing the trivialization of $S^q \times D^p$ by $l(g')$ are concordant. Hence $l(g')$ is mapped to the same element of $\mathrm{FQ}_q^p(f)$ and the lemma follows.

By Proposition 2.4, for $p<q<2p-4$, $S_f(0)$ is equal to the image of $\alpha_f: \mathrm{Q}_{q+1}^p(sf) \to \pi_q(\mathrm{SO}_p)$. Given an element $x \in \mathrm{Ker\,K}$, we can take $h \in \mathrm{Diff}\,\mathcal{F}_f$ in the class x such that h belongs to $(\kappa 1)$. If we choose another element $h' \in (\kappa 1)$, then $h^{-1}h'$ lies in $(\kappa 2)$ and so $l(h)^{-1}l(h') = l(h^{-1}h')$ lies in $S_f(0)$. Hence for $p<q<2p-4$, we can well-define a homomorphism

$$\mathrm{L}: \mathrm{Ker\,K} \to \pi_q(\mathrm{SO}_p)/\mathrm{Im}\,\alpha_f .$$

by $\mathrm{L}(x) = l(g)$.

To study the image of L, we should take notice of the following situation. Let $T_0 = D^p \times S^q$ be the tubular neighborhood of the fibre as before. Given a C^∞-map $R: S^q \to \mathrm{SO}_p$, define the diffoemorphism $\tilde{r}: T_0 \to T_0$ by $\tilde{r}(x, y) = (r(y)x, y)$. The question is whether this diffeomorphism is extendable to a diffeomorphism of whole $\mathcal{F}_f$. Since $\mathcal{F}_f$ can be regarded as the glueing of two copies of $D^p \times S^q$ by the attaching map $\tilde{f} \in \mathrm{Diff}\,S^{p-1} \times S^q$, where $\tilde{f}(x, y) = (x, f(x)y)$, the question reduces to the following; whether the diffeomorphism of $S^{p-1} \times S^q$ given by $\tilde{f}\tilde{r}\tilde{f}^{-1}$ is the restriction of a diffeomorphism of $D^p \times S^q$ or not. We have defined the map $\mathrm{C}: \tilde{\pi}_0(\mathrm{Diff}\,S^{p-1} \times S^q) \to \Theta^{p+q}$ in § 2, by mapping an element of $\mathrm{Diff}\,S^{p-1} \times S^q$ to the homotopy sphere obtained from $D^p \times S^q \cup S^{p-1} \times D^{q+1}$ by attaching their boundaries by the element.

LEMMA 4.8. *The diffeomorphism* $\tilde{f}\tilde{r}\tilde{f}^{-1}$ *of* $S^{p-1} \times S^q$ *is extendable to* $D^p \times S^q$ *if and only if* $\mathrm{c}(\tilde{f}\tilde{r}\tilde{f}^{-1}) = 0$.

PROOF. Suppose first that the diffeomorphism $\tilde{f}\tilde{r}\tilde{f}^{-1}$ of $S^{p-1} \times S^q$ is extendable to a diffeomorphism g of $D^p \times S^q$. Define a map

$$h: \mathrm{c}(\tilde{f}\tilde{r}\tilde{f}^{-1}) = D^p \times S^q \bigcup_{\tilde{f}\tilde{r}\tilde{f}^{-1}} S^{p-1} \times D^{q+1} \to \mathrm{c}(i) = D^p \times S^q \bigcup_i S^{p-1} \times D^{q+1}$$

by

$$h(x, y) = \begin{cases} g(x, y) & (x, y) \in D^p \times S^q \\ (x, y) & (x, y) \in S^{p-1} \times D^{q+1} . \end{cases}$$

Obviously h is a diffeomorphism, which shows that $\mathrm{c}(\tilde{f}\tilde{r}\tilde{f}^{-1}) = 0$. Conversely, suppose $\mathrm{c}(\tilde{f}\tilde{r}\tilde{f}^{-1}) = 0$. Then by Lemmas 3.1 and 3.2, $\mathrm{Ker\,C}$ is written as

$\pi_q(\mathrm{SO}_p)\cdot\pi_{p-1}(\mathrm{SO}_{q+1})$. Hence we have $\tilde{f}\tilde{r}\tilde{f}^{-1}=\tilde{s}\tilde{t}$, where $s\in\pi_q(\mathrm{SO}_p)$ and $t\in\pi_{p-1}(\mathrm{SO}_{q+1})$. Then $\tilde{f}\tilde{r}=\tilde{s}\tilde{t}\tilde{f}$. But since t and f are of suspension type, by Lemma 3.4, we have $f=tf$. Consequently t is trivial and we have $\tilde{f}\tilde{r}\tilde{f}^{-1}=\tilde{s}$. Therefore $\tilde{f}\tilde{r}\tilde{f}^{-1}$ is extendable to $D^p\times S^q$, which completes the proof.

By this lemma, it follows that $r\in\pi_q(\mathrm{SO}_p)$ is in $\mathrm{Im}\,\mathrm{L}$ if and only if $\mathrm{c}(\tilde{f}\tilde{r}\tilde{f}^{-1})=0$. But we will rewrite this fact using the following definition.

We define the pairing

$$\Gamma:\ \pi_{p-1}(\mathrm{SO}_{q+1})\times\pi_q(\mathrm{SO}_p)\to\Theta^{p+q}$$

for $p<q$, by $\Gamma(f,r)=\mathrm{c}(\tilde{f}\tilde{r}\tilde{f}^{-1})$. By Lemma 3.1, the left operation of $\pi_q(\mathrm{SO}_p)$ and the right operation of $\pi_{p-1}(\mathrm{SO}_{q+1})$ does not change the image of C. Hence we have

$$\Gamma(f,r)=\mathrm{c}(\tilde{f}\tilde{r})=\mathrm{c}(\tilde{f}^{-1}\tilde{f}\tilde{r})\,.$$

LEMMA 4.9. *The pairing $\Gamma:\ \pi_{p-1}(\mathrm{SO}_{q+1})\times\pi_q(\mathrm{SO}_p)\to\Theta^{p+q}$ is linear in the second variable. If the second is the suspension, Γ is linear in the first variable.*

PROOF. Since $p<q$, $\pi_{p-1}(\mathrm{SO}_{q+1})$ is of suspension type. Hence we can choose $\tilde{f}$ in its diffeotopy class such that $\tilde{f}$ maps $S^{p-1}\times D_+^q$ onto itself. Further we can take $\tilde{r}$ so that it keeps $S^{p-1}\times D_+^q$ invariant. Consequently $\mathrm{c}(\tilde{f}\tilde{r}\tilde{f}^{-1})$ corresponds to an element $\tilde{\tilde{r}}\in\mathrm{Diff}\,S^{p+q}$ which is defined by the identity on $D^p\times S^{q-1}$ and by $\tilde{f}\tilde{r}\tilde{f}^{-1}|S^{p-1}\times D_-^q$ on $S^{p-1}\times D^q$ (cf. Proof of (ii) of Proposion 3.6). Since the connected sum corresponds to the composition of $\mathrm{Diff}\,S^{p+q}$ we have

$$\mathrm{c}(\tilde{f}\tilde{r}_1\tilde{f}^{-1})\,\#\,\mathrm{c}(\tilde{f}\tilde{r}_2\tilde{f}^{-1})=\mathrm{c}(\tilde{f}\tilde{r}_1\tilde{f}^{-1}\tilde{f}\tilde{r}_2\tilde{f}^{-1})=\mathrm{c}(\tilde{f}\tilde{r}_1\tilde{r}_2\tilde{f}^{-1})\,,$$

which shows the linearity in $\pi_q(\mathrm{SO}_p)$. The second assertion of this lemma follows similarly from the equation $\Gamma(f,r)=\mathrm{c}(\tilde{r}^{-1}\tilde{f}\tilde{r})$. If we denote by $\Gamma(f)$ the restriction of the pairing Γ to $\{f\}\times\pi_q(\mathrm{SO}_p)$, then $\Gamma(f)$ is a homomorphism: $\pi_q(\mathrm{SO}_p)\to\Theta^{p+q}$. From Lemma 4.8, we have the following.

PROPOSITION 4.10. *The image of the homomorphism* L *is the factor group of* $\mathrm{Ker}\,\Gamma(f)$ *by* $\mathrm{Im}\,\alpha_f$.

We have defined the map Ψ from $\mathrm{Ker}\,\beta_f$, which is the subgroup of $\pi_q(\mathrm{SO}_p)$, to the set of $(p+q)$-dimensional homotopy spheres.

LEMMA 4.11. *The set of $(p+q)$-dimensional homotopy spheres which is the image of $x\in\mathrm{Ker}\,\beta_f$ by Ψ is one coset space of $\Theta^{p+q}/\mathrm{Im}\,\Gamma(f)$.*

PROOF. The change of image Ψ is due to the bundle map from T_0 to $T(x)$. Let l_1 and l_2 be maps from $T_0\cup\mathcal{E}_g^-$ to $T(x)\cup\mathcal{E}_g^-$ which are the identity on $\mathcal{E}_g^-$ and a bundle map on T_0, and let $\Sigma_1=\varphi(l_1)$ and $\Sigma_2=\varphi(l_2)$. By definition we have $\varphi(l_2^{-1}l_1)=\Sigma_1-\Sigma_2$. On the other hand $\varphi(l_2^{-1}l_1)=\mathrm{c}(f(l_2^{-1}l_1|\partial T_0)f^{-1})\in\mathrm{Im}\,\Gamma(f)$. Hence it follows that Σ_1 and Σ_2 belong to a coset class. Conversely let $\Sigma=\varphi(l)$ and let $\Sigma'=\Gamma(f)(r)$, where $r\in\pi_q(\mathrm{SO}_p)$. Let us define a diffeomophism l_0: $T_0\cup\mathcal{E}_g^-\to T_0\cup\mathcal{E}_g^-$ by the identity on $\mathcal{E}_g^-$ and by the bundle map corresponding to r. Then obviously $\varphi(ll_0)=\Sigma+\Sigma'$. Consequently $\Sigma+\Sigma'\in\Psi(x)$, which completes the proof.

By this lemma we can regard the map Ψ as a map

$$\Psi': \operatorname{Ker}\beta\to\Theta^{p+q}/\operatorname{Im}\Gamma(f),$$

this map Ψ' is clearly a homomorphism.

(4) The final step is the study of Ker L, for which we define a homomorphism M and investigate the image and the kernel of M.

Let T_0 be the tubular neighborhood of the canonical fibre S^q as before. An element of Ker L is a pseudo-diffeotopy class of diffeomorphisms of $\mathfrak{X}_f$ whose restriction on T_0 is concordant to the identity as embeddings of $S^q\times D^p$ in $\mathfrak{X}_f$. But by the pseudo-diffeotopy extension theorem (see § 0), we can choose a representative diffeomorphism of Ker L to be the one which keeps T_0 invariant. Consider $\mathfrak{X}_f$ as $D^q_+\times S^q\cup D^p_-\times S^q$. Then we can regard $T_0=D^p_+\times S^q$. Given x (resp. y) $\in$ Ker L, we can take d (resp. e) as a representative of x (resp. y) such that d (resp. e) keeps $D^p_+\times S^q$ (resp. $D^p_-\times S^q$) invariant. Then obivously $de=ed$. Hence we have the next lemma.

LEMMA 4.12. Ker L *is an abelian group.*

Now we define the homomorphism M. Let us denote by (λ1) the subgroup of Diff $\mathfrak{X}_f$ consisting of those which keep $D^p_+\times S^q$ invariant and whose pseudo-diffeotopy classes belong to Ker L. Given $h\in(\lambda 1)$, we define a diffeomorphism $m(h)$ of $S^{p+q}=S^{p-1}\times D^{q+1}\cup D^p\times S^q$ by

$$\begin{cases} m(h)(x,y)=(x,y) & (x,y)\in S^{p-1}\times D^{q+1}\\ m(h)(x,y)=h\,|\,D^p_-\times S^q(x,y) & (x,y)\in D^p\times S^q\,. \end{cases}$$

Since $\tilde{\pi}_0(\operatorname{Diff} S^{p+q})\approx\Theta^{p+q+1}$, we can consider $m(h)$ as to be an element of Θ^{p+q+1}. Let (λ2) be the subgroup of (λ1) consisting of elements of (λ1) which are pseudo-diffeotopic to the identity. Let us denote by G_f the subgroup of Θ^{p+q+1}, which is the image of (λ2) by m. Given $x\in$ Ker L, we can take a representative $l\in(\lambda 1)$. Difine a map

$$\mathrm{M}: \operatorname{Ker}\mathrm{L}\to\Theta^{p+q+1}/G_f$$

by $\mathrm{M}(x)=\{m(l)\}\in\Theta^{p+q+1}/G_f$. If we choose another representative $l'\in(\lambda 1)$ of x, then $l^{-1}l'\in(\lambda 2)$. Consequently $\{m(l^{-1}l')\}=0\in\Theta^{p+q+1}/G_f$ and the map M is well-defined. Obviously M is a homomorphism. In order to study the image of M, we define the homomorphism

$$\mathrm{N}: \Theta^{p+q+1}\to\operatorname{Ker}\mathrm{L}\,.$$

We know that $\Theta^{p+q+1}\approx\tilde{\pi}_0(\operatorname{Diff} D^{p+q}\operatorname{rel}\partial D)$, the group of pseudo-diffeotopy class of D^{p+q} which point-wisely fix the boundary ∂D, where the psuedo-diffeotopy equivalence also needs fixing $(\partial D)\times I$ (see Wall [**14**]). Given $\Sigma\in\Theta^{p+q+1}$, let

$r \in (\mathrm{Diff}\, D^{p+q} \,\mathrm{rel}\, \partial D)$ be the corresponding element. Define a diffeomorphism $\mathrm{N}(\Sigma)$ of $\mathcal{F}_f$ by

$$\begin{cases} \mathrm{N}(\Sigma)(x) = r(x) & x \in D^{p+q} = D^p_+ \times D^q_+ \\ \mathrm{N}(\Sigma)(x) = x & x \in \mathcal{F}_f - D^p_+ \times D^q_+ . \end{cases}$$

Then N is a well-defined homomorphism from Θ^{p+q+1} to Ker L. Let p: $\Theta^{p+q+1} \to \Theta^{p+q+1}/G_f$ be the natural projection. Obviously the following triangle commutes

$$\begin{tikzcd} & \Theta^{p+q+1} \arrow[dl, "\mathrm{N}"'] \arrow[dr, "p"] & \\ \mathrm{Ker\,L} \arrow[rr, "\mathrm{M}"'] & & \Theta^{p+q+1}/G_f \end{tikzcd}$$

Consequently the homomorphism M is an epimorphism.

Further the next proposition holds.

PROPOSITION 4.3. *The following sequence is exact,*

$$0 \longrightarrow \eta_*(\pi_p(\mathrm{SO}_{q+1})/\mathrm{Ker\,B}) \longrightarrow \mathrm{Ker\,L} \xrightarrow{\mathrm{M}} \Theta^{p+q+1}/G_f \longrightarrow 0$$

where we identify $\mathrm{Ker\,A} \in \tilde{\pi}_0(\mathrm{Diff}\,\mathcal{E}_f)$ *with* $\pi_p(\mathrm{SO}_{q+1})/\mathrm{Ker\,B}$.

PROOF. It is sufficient to show the exactness at Ker L. Suppose $x \in \mathrm{Ker\,L}$ is mapped by M to the zero element of Θ^{p+q+1}/G_f. Then we can take representative $l \in (\lambda 1) \subset \mathrm{Diff}\,\mathcal{F}_f$ of x such that $m(l) \in \mathrm{Diff}\,S^{p+q}$ is extendable to a diffeomorphism of D^{p+q+1}. Then $l|D^p_- \times S^q$ is extendable to a diffeomorphism of $D^p_- \times D^{q+1}$. Consequently we have $\mathrm{Ker\,L} \subset \eta_*(\mathrm{Ker\,A}) \subset \eta_*(\tilde{\pi}_0(\mathrm{Diff}\,\mathcal{E}_f))$. Conversely let y belong to $\eta_*(\mathrm{Ker\,A})$. Since $\mathrm{Ker\,A} \approx \pi_p(\mathrm{SO}_{q+1})/\mathrm{Ker\,B}$, we can take $h \in \mathrm{Diff}\,\mathcal{F}_f$ such that m keeps $D^p_+ \times S^q$ invariant and $h|D^p_- \times S^q$ is extendable to a diffeomorphism of $D^p_- \times D^{q+1}$. Then $m(h)$ is pseudo-diffeotopic to the identity. The proof finishes.

Let sf be the composition of f with the inclusion s: $\mathrm{SO}_{q+1} \to \mathrm{SO}_{q+2}$. We have defined in (2) the subgroup $\mathrm{Ker}\,\beta_{sf} \subset \pi_{q+1}(\mathrm{SO}_p)$, and the map Ψ: $\mathrm{Ker}\,\beta_{sf} \to \{\text{Subsets of } \Theta^{p+q+1}\}$. Concerning the group G_f, we have the next lemma.

LEMMA 4.14. *The group G_f is equal as sets to* $\bigcup_{x \in \mathrm{Ker}\,\beta_{sf}} \Psi(x)$, *where the union is extended over* $\mathrm{Ker}\,\beta_{sf}$. *Especially if the bundle is trivial, then G_f is trivial.*

PROOF. Let $\Sigma \in G_f$, then there exists $h \in (\lambda 2) \subset \mathrm{Diff}\,\mathcal{F}_f$ which keeps $D^p_+ \times S^q$ invariant such that $m(h) = \Sigma$ and h is pseudo-diffeotopic to the identity. By Proposition 4.13 there exists a diffeomorphism $\tilde{h}$ of $\mathcal{E}_f$ which is an extension of h. Consider $\mathcal{F}_{sf}$ as $\mathcal{E}^+_f \cup \mathcal{E}^-_f = \mathcal{E}_f \cup D^p_+ \times D^{q+1} \cup D^p_- \times D^{q+1}$. Define a map k of $\mathcal{F}_f - D^p_- \times D^{q+1} = \mathcal{E}_f \cup D^p_+ \times D^{q+1}$ by

$$\begin{cases} k|\mathcal{E}_f=\tilde{h} \\ k|D^p_+\times D^{q+1}=\text{identity}. \end{cases}$$

Then k can naturally be regarded as a mod one point diffeomorphism of $\mathcal{F}_{sf}$, the obstruction to the extension of k to be a diffeomorphism of $\mathcal{F}_{sf}$ being equal to $\Sigma\in\Theta^{p+q+1}\approx\tilde{\pi}_0(\text{Diff } S^{p+q})$. Let S^{q+1} be the canonical fibre of $\mathcal{F}_{sf}$ and T_0 be the canonical tubular neighborhood of S^{q+1}. Let $1+x\in\pi_{q+1}(S^{q+1}\oplus\pi_{q+1}(S^p)\approx\pi_{q+1}(\mathcal{F}_f)$ be the homotopy class of $k(S^{q+1})$ and let $T(x)$ be the tubular neighborhood of $k(S^{q+1})$ such that $T(x)\cap\mathcal{E}_f^-=D^p_+\times D^{q+1}$. By the Haefliger's theorem and the tubular neighborhood theorem we can choose a mod one point diffeomorphism k' of $\mathcal{F}_f$ whose extension obstruction is equal to Σ such that k' maps $\mathcal{E}_f^-$ onto itself and T_0 to $T(x)$ both by bundle maps. Further, we can choose a mod one point diffeomorphism k'' of $\mathcal{F}_f$ which has the same obstruction such that k'' keeps $\mathcal{E}_f^-$ invariant and maps T_0 to $T(x)$ by a bundle map. Then by definition $\Sigma=\varphi(k'')\in\Psi(x)$. Conversely, let Π be an element of Θ^{p+q+1} such that $\Pi\in\Psi(y)$ for $y\in\text{Ker }\beta_{sf}$. Then there exists a mod one point diffeomorphism n of $\mathcal{F}_{sf}$ mapping the homotopy class $1+0\in\pi_{q+1}(\mathcal{F}_{sf})$ to $1+y\in\pi_{q+1}(\mathcal{F}_{sf})$ whose obstruction to the extension is equal to Π. By pseudo-diffeotopy we can regard n such that the restriction of n on $\mathcal{E}_f\cup D^p_+\times D^{q+1}=\mathcal{F}_{sf}-D^p_-\times D^{q+1}$ is a diffeomorphism onto itself and $n|D^p_+\times D^{q+1}$ is the identity. Then $n|\partial\mathcal{E}_f$ belongs to $(\lambda 1)\in\text{Diff }\mathcal{F}_f$ and it is easy to see that $m(n|\partial\mathcal{E}_f)=\Pi$. The diffeomorphism $n|\partial\mathcal{E}_f$ can be extendable to Diff $\mathcal{E}_f$, but it may not be pseude-diffeotopic to the identity. Since $\text{Ker A}\approx\pi_p(\text{SO}_{q+1})/\text{Ker }\beta_f$ by § 1, there exists a bundle map r of $\mathcal{E}_f$ keeping $D^p_+\times D^{q+1}\subset\mathcal{E}_f$ invariant such that $nr|\partial\mathcal{E}_f$ is pseudo-diffeotopic to the identity. Then $nr|\partial\mathcal{E}_f$ belongs to $(\lambda 2)\subset\text{Diff }\mathcal{F}_f$. Further

$$m(nr|\partial\mathcal{E}_f)=m(n|\partial\mathcal{E}_f)+m(r|\partial\mathcal{E}_f)=m(n|\partial\mathcal{E}_f)=\Pi\,.$$

Consequently Π is contained in G_f. If the bundle is trivial then $\text{Im }\Psi=0$ and $G_f=0$. The proof is complete.

In case the bundle $\mathcal{F}_f$ is trivial, then Ker A is isomorphic to $\pi_p(\text{SO}_{q+1})$ by Proposition 1.4 and η_* is injective by Lemma 3.2. Further by the above Lemma 4.14, the group G_f is trivial. Consequently the homomorphism $\text{N}:\Theta^{p+q+1}\to\text{Ker L}$ is a right inverse of $\text{M}:\text{Ker L}\to\Theta^{p+q+1}$ and the following sequence splits.

$$0\longrightarrow\pi_p(\text{SO}_{q+1})\longrightarrow\text{Ker L}\xrightarrow{\text{M}}\Theta^{p+q+1}\longrightarrow 0\,.$$

Hence we have the next proposition.

PROPOSITION 4.15. *If the bundle* $\mathcal{F}_f$ *is trivial, then* $\text{Ker L}\subset\tilde{\pi}_0(\text{Diff } S^p\times S^q)$ *is isomorphic to* $\Theta^{p+q+1}\oplus\pi_p(\text{SO}_{q+1})$.

At this stage we prove Proposition 4.1 which states that $\text{Im A}=\text{Im A}'$.

PROOF OF PROPOSITION 4.1. It is clear that $\text{Im A}\subset\text{Im A}'$. We will prove

$\mathrm{Im}\,A \subset \mathrm{Im}\,A'$. Suppose that there exists an orientation preserving diffeomorphism h of $\mathcal{F}_f$ which maps a generator of $H_p(\mathcal{F}_f) \approx \boldsymbol{Z}$ to the other generator. The obstruction to the extension of h to a diffeomorphism of $\mathcal{E}_f$ lies in $\mathrm{Im}\,K$, $\mathrm{Im}\,L$ and $\mathrm{Im}\,M$. Consequently there exists $g \in \mathrm{Ker}\,A'$ such that the composition hg is extendable to $\mathrm{Diff}\,\mathcal{E}_f$. Since hg is orientation preserving and is mapped by A nontrivially it follows $\mathrm{Im}\,A \supset \mathrm{Im}\,A'$, which completes the proof.

We know by Proposition 1.2 that $\mathrm{Im}\,A \approx X$. Since $p<q$, we have

$$X \approx \begin{cases} \boldsymbol{Z}_2 & \text{if } p \not\equiv 3 \pmod 4 \\ 0 & \text{if } p \equiv 3 \pmod 4). \end{cases}$$

Combining these results we have the next formula.

THEOREM 4.16. *For $p<q<2p-q$, the order of $\tilde{\pi}_0(\mathrm{Diff}\,\mathcal{F}_f)$ is equal to the order of the direct sum group*

$$X \oplus \eta_*(\pi_p(SO_{q+1})/\mathrm{Ker}\,B) \oplus \mathrm{Ker}\,\Psi' \oplus \mathrm{Ker}\,\Gamma(f)/\mathrm{Im}\,(F_f+\partial) \oplus \Theta^{p+q+1}/\mathrm{Im}\,\Psi\,.$$

Especially

THEOREM 4.17. *For $p<q<2p-4$, the order of $\tilde{\pi}_0(\mathrm{Diff}\,S^p \times S^q)$ is equal to the order of the direct sum group*

$$\boldsymbol{Z}_2 \oplus \pi_p(SO_{q+1}) \oplus \pi_q(SO_{p+1}) \oplus \Theta^{p+q+1}\,.$$

PROOF. The theorem follows from the fact that the order of $\pi_q(SO_p)/\mathrm{Im}\,\partial \oplus \mathrm{Ker}\,\partial$ is equal to the order of $\pi_q(SO_{p+1})$.

§5. Classification of manifolds (1).

Let M^{p+q+1} be a $(p+q+1)$-dimensional oriented differentiable manifold such that

$$\left\{\begin{array}{l} M\text{: closed and simply-connected} \\ H_i(M) = \begin{cases} \boldsymbol{Z} & \text{for } 0,\ p,\ q+1,\ p+q+1 \\ 0 & \text{otherwise} \end{cases} \\ \pi_p(SO_{q+1}) = 0 \\ \qquad p<q<2p-4 \end{array}\right\} \qquad (*)$$

We consider couples (M, γ), where M is such a manifold and γ is a generator of $H_p(M)$; we identify (M, γ) and (M', γ') if there exists a diffeomorphism of M on M', preserving orientation, and carrying γ on γ'. By this identification we get the set $\mathcal{M}_{p+q+1}$ of diffeomorphism classes of manifolds satisfying (*) with a preferred basis.

Since $p<q$, we can embed S^p in M such that it is the generator γ of $H_p(M^{p+q+1}) \approx \boldsymbol{Z}$. Let $t \in \pi_{p-1}(SO_{q+1})$ be the characteristic class of the embedded sphere S^p. In our case, any two homotopic embeddings are isotopic. Con-

sequently we have a map P from $\mathcal{M}_{p,q+1}$ to $\pi_{p-1}(\mathrm{SO}_{q+1})$ defined by $\mathrm{P}\{(M, \gamma)\}=t$. Obviously the map P is surjective. Denote by ${}_t\mathcal{M}_{p,q+1}$ the set of manifolds which are mapped by P to t.

We now give a group structure on ${}_t\mathcal{M}_{p,q+1}$. Given $(M_1, \gamma_1), (M_2, \gamma_2) \in {}_t\mathcal{M}_{p,q+1}$, let us fix $\mathcal{E}_f$, a D^{q+1} bundle over S^p whose characteristic class is $t \in \pi_{p-1}(\mathrm{SO}_{p+1})$. Let $c: S^p \to \mathcal{E}_f$ be the zero cross section. Since $\partial\mathcal{E}_f = \mathcal{S}_f$ admits a cross section, the reflection R of O_{q+1} on a fibre is extendable to the reflection of whole $\mathcal{E}_f$, which shows that $\mathcal{E}_f$ is orientation reversing diffeomorphic to itself. Consequently we can choose embeddings

$$i_1: \mathcal{E}_f \to M_1 \quad \text{orientation preserving}$$

$$i_2: \mathcal{E}_f \to M_2 \quad \text{orientation reversing}$$

such that $(ic)_*[S^p]=\gamma_\nu$ $(\nu=1, 2)$. We define $(M_1 \underset{(p)}{\#} M_2, \gamma_1 \# \gamma_2)$, *the connected sum of* (M_1, γ_1) *and* (M_2, γ_2) *along the p-cycle.* The manifold $M_1 \underset{(p)}{\#} M_2$ is made from disjoint sum $(M_1 - \mathrm{Int}\, i_1(\mathcal{E}_f)) \cup (M_2 - \mathrm{Int}\, i_2(\mathcal{E}_f))$ by identifying $i_1(x)$ and $i_2(x)$ for $x \in \mathcal{S}_f = \partial\mathcal{E}_f$ and given a differentiable structure by the canonical way. The orientation is taken so as to be compatible with that of M_1 and M_2. Let $c': S^p \to \partial\mathcal{E}_f \subset \mathcal{E}_f$ be a cross section which is homotopic to c as maps $S^p \to \mathcal{E}_f$. Let us define $\gamma_1 \# \gamma_2 \in \mathrm{H}_2(M_1 \underset{(p)}{\#} M_2) \approx \boldsymbol{Z}$ by the class $(i_1 c')_*[S^p]$. Then the couple $(M_1 \underset{(p)}{\#} M_2, \gamma_1 \# \gamma_2)$ also belongs to ${}_t\mathcal{M}_{p,q+1}$.

LEMMA 5.1. *The connected sum along the cycle operation is well-defined and associative up to orientation and basis preserving diffeomorphism.*

PROOF. We will show that this operation does not depend on the choice of embedding i_1. Let i_1' be another embedding. Then by Haefliger [4], since $(i_1 c)_*[S^p]=(i_2 c)_*[S^p]$ and $p<q$, we can consider i_1' after an isotopy such that $i_1' = i_2 c$. Further, by the tubular neighborhood theorem, we can suppose that $i_1: \mathcal{E}_f \to M$ and $i_1': \mathcal{E}_f \to M$ differ only by a bundle map of $\mathcal{E}_f$. Since $\pi_p(\mathrm{SO}_{q+1}) = 0$, the similar argument as in § 1 shows that they are isotopic as embeddings of $\mathcal{E}_f$. Consequently the definition does not depend on the choice of i_1, similarly of i_2. The associativity is obvious.

We have denoted by $\mathcal{S}_{sf}$ the manifold which is an S^{q+1} bundle over S^p with the characteristic map being the image of f by the inclusion $s: \mathrm{SO}_{q+1} \to \mathrm{SO}_{q+2}$. Let $\gamma_0 \in \mathrm{H}_p(\mathcal{S}_{sf})$ be the generator represented by the image of a cross section. Denote by $-M$, the orientation reversed manifold of M. The identity map $i: M \to -M$ is the orientation reversing diffeomorphism. Let us define $-\gamma \in \mathrm{H}_p(-M) \approx \boldsymbol{Z}$ by $-\gamma = i_*(\gamma)$. Then we have naturally

LEMMA 5.2. *The pair* $(\mathcal{S}_{sf}, \gamma_0)$ *serves as the identity element and the pair* $(-M, -\gamma)$ *is the inverse element of* (M, γ).

Consequently the set ${}_t\mathcal{M}_{p,q+1}$ forms a group by the connected sum along the cycle operation. Let us suppose that p and q satisfy the conditions $p<q<2p-4$ and $\pi_p(\mathrm{SO}_{q+1})=0$. Given an element $k\in \mathrm{Ker}\, \mathrm{A}'\subset\tilde{\pi}_0(\mathrm{Diff}\ \mathcal{T}_f)$, take two copies $\mathcal{E}_f^1$ and $\mathcal{E}_f^2$ of the disk bundle $\mathcal{E}_f$ and attach them by a representative of k. An orientation is chosen to be compatible with $\mathcal{E}_f^1$. Obviously the constrcuted manifold satisfies the condition (*). A generator of p-dimensional homology group is fixed to be the one represented by the zero cross section of $\mathcal{E}_f^1$. This manifold does not depend on the choice of representative. Hence we have a well-defined map

$$\mathrm{Q}: \mathrm{Ker}\, \mathrm{A}'\to {}_t\mathcal{M}_{p,q+1}\,.$$

It is easy to see that this is a homomorphism.

PROPOSITION 5.3. *The homomorphism* Q *is an isomorphism.*

PROOF. First we will prove the surjectivity. Suppose that (M, γ) belongs to ${}_t\mathcal{M}_{p,q+1}$. Then there exists an orientation preserving embedding $j: \mathcal{E}_f\to M$ such that $(jc)_*[S^p]=\gamma$. We have

$$\mathrm{H}_i(M-\mathrm{Int}\, j(\mathcal{E}_f))=\begin{cases} \boldsymbol{Z} & \text{for } i=0,\ p \\ 0 & \text{otherwise.}\end{cases}$$

Let $i_*: \mathrm{H}_p(M-\mathrm{Int}\, j(\mathcal{E}_f))\to\mathrm{H}_p(M)$ be the isomorphism induced by the inclusion. Since $\mathcal{E}_f$ is orientation reversing diffeomorphic to itself, there exists an orientation reversing embedding

$$k: \mathcal{E}_f\to M-\mathrm{Int}\, j(\mathcal{E}_f)$$

such that $i_*(kc)_*[S^p]=\gamma$. Since the boundaries $\partial(M-j(\mathcal{E}_f))$ and $\partial(k(\mathcal{E}_f))$ are h-cobordant, by Smale's theorem [11], $M-j(\mathcal{E}_f)$ and $k(\mathcal{E}_f)$ are diffeomorphic. Consequently M can be regarded to be made from $\mathcal{E}_f\cup\mathcal{E}_f$ attached by an orientation preserving diffeomorphism $g: \mathcal{T}_f\to\mathcal{T}_f$. It is easy to see that g belongs to Ker A$'$ and the surjectivity follows. Next we prove the injectivity. Suppose that $x\in\mathrm{Ker}\, \mathrm{A}'$ is mapped by Q trivially. Let $k\in\mathrm{Diff}\ \mathcal{T}_f$ be a representative of x and let (M, γ) be the pair representing Q(x). Then there exists an orientation preserving diffeomorphism from M to $\mathcal{T}_{sf}$ mapping γ to γ_0. We can regard $M=\mathcal{E}_f^1\underset{k}{\cup}\mathcal{E}_f^2$ and $\mathcal{T}_{sf}=\mathcal{E}_f^1\underset{i}{\cup}\mathcal{E}_f^2$. Then there exists a diffeomorphism from M to $\mathcal{T}_{sf}$ whose restriction on $\mathcal{E}_f^2$ is the identity, since $\pi_p(\mathrm{SO}_{q+1})=0$. Consequently $k\in\mathrm{Diff}\ \mathcal{T}_f$ is extendable to Diff $\mathcal{E}_f$. But since $0=\mathrm{Ker}\, \mathrm{A}'\subset\tilde{\pi}_0(\mathrm{Diff}\ \mathcal{T}_f)$ in our case, k is pseudo-diffeotopic to the identity, which completes the proof.

Combining Theorem 4.16 and Proposition 5.3 we have the next theorem.

THEOREM 5.4. *The number of differentiable structures of manifolds satisfying* (*), *modulo diffeomorphisms preserving orientation and the preferred basis of p-dimensional homology group, is equal to*

$$\sum_{\{f\}\in\pi_{p-1}(\mathrm{SO}_{q+1})} \sharp\,\{\mathrm{Ker}\,\Psi'\oplus\mathrm{Ker}\,\Gamma(f)/\mathrm{Im}\,(F_f+\partial)\oplus\Theta^{p+q+1}/\mathrm{Im}\,\Psi\}\,,$$

where $\sharp$ *denotes the order and the summation is extended over representatives of all homotopy classes.*

Especially;

PROPOSITION 5.5. *If* $p\equiv 5, 6 \pmod 8$, *the number of differentiable manifolds satisfying* (*), *modulo diffeomorphisms preserving orientation and the preferred basis of p-dimensional homology group, is equal to*

$$\sharp(\pi_q(\mathrm{SO}_{p+1})\oplus\Theta^{p+q+1})\,.$$

§ 6. Classification of manifolds (2).

In this section we treat $(p+q+1)$-dimensional oriented differentiable manifold M^{p+q+1} which satisfies the next conditions

$$\left\{\begin{array}{l} M\text{: closed and simply connected} \\ \mathrm{H}_i(M)=\left\{\begin{array}{ll} \boldsymbol{Z} & \text{for } 0,\ p,\ q+1,\ p+q+1 \\ 0 & \text{otherwise} \end{array}\right. \\ \pi_q(M)=\pi_q(S^p) \\ \quad p<q<2p-4 \end{array}\right\} \qquad (**)$$

Remark that we abandon the condition $\pi_p(\mathrm{SO}_{q+1})=0$, but in its place we put the condition on the q-dimensional homotopy group of manifold. We consider couples (M, γ), where M is such a manifold and γ is a generator of $\mathrm{H}_p(M)$; we identify (M, γ) and (M', γ') if there exists a diffeomorphism of M on M', preserving orientation, and carrying γ on γ' as before. We denote by $\mathscr{\tilde M}_{p,q+1}$, the set of diffeomorphism class of manifolds satisfying (**), with a preferred basis. Let $\Sigma^{p+q+1}\in\Theta^{p+q+1}$. Then the identity map of $M-D^{p+q+1}$ is extendable to a homomorphism h from M to $M\sharp\Sigma^{p+q+1}$. We denote again by γ the image of γ by $h_*\colon \mathrm{H}_p(M)\to\mathrm{H}_p(M\sharp\Sigma^{p+q+1})$. Then we call that (M, γ) and (M', γ') are orientation and basis preserving diffeomorphic modulo one point if there exists some $\Sigma^{p+q+1}\in\Theta^{p+q+1}$ such that $(M\sharp\Sigma^{p+q+1}, \gamma)=(M', \gamma')\in\mathscr{\tilde M}_{p,q+1}$. We denote by $\mathscr{N}_{p,q+1}$ the factor set of $\mathscr{\tilde M}_{p,q+1}$ obtained by identifying the couples which are orientation and basis preserving diffeomorphic modulo one point.

Let h be an element of Diff $\mathscr{F}_f$. We say that h represents manifold (M, γ) if M is constructed from $\mathscr{E}_f^1\cup\mathscr{E}_f^2$ with the attaching map h and γ is represented by the cross section of $\mathscr{E}_f^1$.

Let h_1 and h_2 belong to Diff $\mathscr{F}_f$ and let (M_1, γ_1) and (M_2, γ_2) be manifolds represented by h_1 and h_2 respectively. The next lemma is easy to see.

LEMMA 6.1. *Suppose that the pseudo-diffeotopy class of* $h_1^{-1}h_2$ *belongs to*

Ker L, *then the manifolds* (M_1, γ_1) *and* (M_2, γ_2) *are orientation and basis preserving diffeomophic modulo one point.*

As in § 5, we have a well-defined map

$$\tilde{\mathrm{P}}: \tilde{\mathcal{M}}_{p,q+1} \to \pi_{p-1}(\mathrm{SO}_{q+1})$$

by taking the characteristic class of sphere generating γ. Let ${}_t\tilde{\mathcal{M}}_{p,q+1}$ be the subset of $\tilde{\mathcal{M}}_{p,q+1}$ which are mapped by $\tilde{\mathrm{P}}$ to $t \in \pi_{p-1}(\mathrm{SO}_{q+1})$, and let ${}_t\mathcal{N}_{p,q+1}$ be the subset of $\mathcal{N}_{p,q+1}$ which is the factor set of ${}_t\tilde{\mathcal{M}}_{p,q+1}$. Then the same argument as the proof on surjectivity of Proposition 5.3 shows that any element of ${}_t\tilde{\mathcal{M}}_{p,q+1}$ is represented by an element of $\mathrm{Ker}\,\mathrm{A} \in \tilde{\pi}_0(\mathrm{Diff}\,\mathcal{E}_f)$.

Further we have the next lemma.

LEMMA 6.2. *Suppose that* k *represent* $(M, \gamma) \in {}_t\tilde{\mathcal{M}}_{p,q+1}$, *where* $\{k\} \in \mathrm{Ker}\,\mathrm{A} \subset \tilde{\pi}_0(\mathrm{Diff}\,\mathcal{E}_f)$. *Then* $\{k\}$ *belongs to* Ker K.

PROOF. We regard M to be made from $\mathcal{E}_f^1 \cup \mathcal{E}_f^2$ by the attaching map k. Decompose $\mathcal{E}_f^2$ as $D_+^p \times D^{q+1} \cup D_-^p \times D^{q+1}$. Let $N = M - \mathrm{Int}\,(D_-^p \times D^{q+1})$, which is equal to the manifold made from $\mathcal{E}_f^1 \cup D_+^p \times D^{q+1}$ by the attaching map $k \mid D_+^p \times \partial D^{q+1}$. This manifold N is homotopy equivalent to the complex made from S^p by attaching D^{q+1} by a map $g: \partial D^{q+1} = S^q \to S^p$, whose homotopy class is equal to $\mathrm{K}(k) \in \pi_q(S^p)$. But by the inclusion homomorphism $i^*: \pi_q(\mathcal{E}_f^1) \to \pi_q(N)$, the element $\mathrm{K}(k)$ is mapped to zero. Since $\pi_q(N) \approx \pi_q(M)$, $\pi_q(M) \approx \pi_q(S^p)$ by our condition and $\pi_q(\mathcal{E}_f^1) \approx \pi_q(S^p)$, it follows that $\mathrm{K}(k) = 0$.

On the otherhand the following lemma holds.

LEMMA 6.3. *The group* Ker L *is a normal subgroup of* Ker K.

PROOF. Let $l \in \mathrm{Ker}\,\mathrm{L}$ and $k \in \mathrm{Ker}\,\mathrm{K}$. Denote by T the tubular neighborhood of the fibre S^q. Then we can choose a representative g of l so that it keeps T fixed and h of k so that it maps T onto itself. Then $h^{-1}gh$ represents $k^{-1}lk$ and keeps T fixed. Consequently $k^{-1}lk$ belongs to Ker L, and the result follows.

Suppose that (M_1, γ_1) and (M_2, γ_2) belong to ${}_t\tilde{\mathcal{M}}_{p,q+1}$. Similarly as in § 5, we define $(M_1 \underset{(p)}{\#} M_2, \gamma_1 \# \gamma_2)$, the connected sum along the cycle of (M_1, γ_1) and (M_2, γ_2). The question is, when the condition $\pi_p(\mathrm{SO}_{q+1}) = 0$ is removed, the well-definedness of this operation.

LEMMA 6.4. *The connected sum along the cycle operation is well-defined and the associative up to orientation and basis preserving diffeomorphism modulo one point.*

PROOF. Suppose that (M_1, γ_1) and (M_2, γ_2) are represented by h_1 and h_2 respectively, where $h_1, h_2 \in \mathrm{Diff}\,\mathcal{F}_f$. Then if we take the canonical embedding of that representation $i_1: \mathcal{E}_f \to M_1$ and $i_2: \mathcal{E}_f \to M_2$ then $(M_1 \underset{(p)}{\#} M_2, \gamma_1 \# \gamma_2)$ is represented by $h_1 h_2$. But if we take other embeddings $i_1': \mathcal{E}_f \to M_1$ and $i_2': \mathcal{E}_f \to M_2$, then i_1 and i_1' (resp. i_2 and i_2') differ by an element d, where $\{d\} \in \mathrm{Ker}\,\mathrm{A}$

$\subset \mathrm{Diff}\,\mathcal{E}_f$ (resp. e, where $\{e\} \in \mathrm{Ker\,A} \subset \mathrm{Diff}\,\mathcal{E}_f$). Then $(M_1 \underset{(p)}{\#} M_2, \gamma_1 \# \gamma_2)$ is represented by $h_1\eta(d)\eta(e)h_2$, where η: $\mathrm{Diff}\,\mathcal{E}_f \to \mathrm{Diff}\,\mathcal{F}_f$ is the restriction. Since $\{h_1\}$ and $\{h_2\}$ belong to Ker K by Lemma 6.2 and since Ker L is a normal subgroup of Ker K by Lemma 6.3, we have $h_1\eta(d)\eta(e)h_2 = h_1h_2c$, where $\{c\} \in \mathrm{Ker\,L}$. Consequently the manifold $M_1 \underset{(p)}{\#} M_2$ made from i_1 and i_2 and the one made from i_1' and i_2' are orientation and basis preserving diffeomorphic modulo one point, by virtue of Lemma 6.1. This shows that the modulo one point diffeomorphism class of the operation does not depend on the choice of embeddings i_1 and i_2. If M_1 and M_1' are diffeomorphic modulo one point, then clearly $M_1 \underset{(p)}{\#} M_2$ and $M_1' \underset{(p)}{\#} M_2$ are diffeomorphic modulo one point and so on. The associativity is clear. The proof is complete.

As in the previous section we have;

LEMMA 6.5. *The pair* $(\mathcal{F}_{sf}, \gamma_0)$ *serves as the identity element and the pair* $(-M, -\gamma)$ *is the inverse element of* (M, γ).

Consequently the set ${}_t\mathcal{N}_{p,q+1}$ has the group structure. Let $p<q<2p-4$ and k belongs to Ker K. Then the couple represented by k belongs to ${}_t\mathcal{M}_{p,q+1}$. We define the map

$$\mathrm{R}: \mathrm{Ker\,K} \to {}_t\mathcal{N}_{p,q+1}$$

by taking the modulo one point diffeomorphism class of represented couple. Obviously R is well-defined and is a homomorphism. This map is an epimorphism by Lemma 6.2.

PROPOSITION 6.6. *The kernel of the homomorphism* $\mathrm{R}: \mathrm{Ker\,K} \to {}_t\mathcal{N}_{p,q+1}$ *is* Ker L.

PROOF. By Lemma 6.1, Ker L is contained in Ker R. Conversely let $r \in \mathrm{Ker\,K}$ be mapped to the unit element of ${}_t\mathcal{N}_{p,q+1}$. Then there exists modulo one point diffeomorphism d between $\mathcal{F}_{sf}$ and $M = \mathcal{E}_f^1 \underset{r}{\cup} \mathcal{E}_f^2$. We regard $\mathcal{F}_{sf} = \mathcal{E}_f^1 \underset{i}{\cup} \mathcal{E}_f^2$. We can assume as before that d maps $\mathcal{E}_f^2 \subset M$ onto $\mathcal{E}_f^2 \subset \mathcal{F}_{sf}$. Then the composition of r with $(d|\mathcal{E}_f^2)$ is modulo one point extendable to $\mathrm{Diff}\,\mathcal{E}_f$. That is $r\eta(d|\mathcal{E}_f^2) \in \mathrm{Ker\,L}$. Since $\eta(d|\mathcal{E}_f^2)$ belongs to Ker L, r belongs to Ker L. The proof finishes.

COROLLARY 6.8. *The group* ${}_t\mathcal{N}_{p,q+1}$ *is isomorphic to* Im L. *Consequently it is isomorphic to the factor group of* $\mathrm{Ker}\,\Gamma(f)$ *by* $\mathrm{Im}\,\alpha_f = \mathrm{Im}\,(F_f+\partial)$.

PROOF. The former half is obvious. The latter half follows from Propsition 4.7.

Consequently we have the following theorem.

THEOREM 6.9. *The number of differentiable manifolds satisfying* (**), *up to modulo one point diffeomorphisms preserving orientation and basis, is equal to*

$$\sum_{\{f\} \in \pi_{p-1}(SO_{q+1})} \# \{\mathrm{Ker}\,\Gamma(f)/\mathrm{Im}\,(F_f+\partial)\}\,.$$

In particular

PROPOSITION 6.10. *If* $p \equiv 3, 5, 6, 7 \pmod 8$, *the number of differentiable manifolds satisfying* (**), *up to modulo one point diffeomorphisms preserving orientation and basis, is equal to*

$$\# \{\mathrm{Im}\, s_* : \pi_q(\mathrm{SO}_p) \to \pi_q(\mathrm{SO}_{p+1})\}\,,$$

where s_* *is induced by the inclusion* $s: \mathrm{SO}_p \to \mathrm{SO}_{p+1}$.

We call as the inertia group of (M, γ) the subset of Θ^{p+q+1} consisting of exotic spheres whose operation does not change the orientation and basis preserving diffeomorphism class of (M, γ). To know the order of $\mathcal{M}_{p,q+1}$ it remains the investigation of inertia groups for such couples. Given a pair $(M, \gamma) \in \tilde{\mathcal{M}}_{p,q+1}$, let $h \in \mathrm{Diff}\, \mathcal{F}_f$ represent it. Then $\mathrm{L}(h) \in \pi_q(\mathrm{SO}_p)$. We have defined the pairing $\Gamma: \pi_p(\mathrm{SO}_{q+1}) \times \pi_q(\mathrm{SO}_{p+1}) \to \Theta^{p+q+1}$ and denoted by $\Gamma(s_*\mathrm{L}(h))$ the homomorphism: $\pi_p(\mathrm{SO}_{q+1}) \to \Theta^{p+q+1}$ defined by restricting the pairing Γ to $\pi_p(\mathrm{SO}_{q+1}) \times s_*(\mathrm{L}(h))$, where $s: \mathrm{SO}_p \subset \mathrm{SO}_{p+1}$. Further G_f was defined in § 4 as a subgroup of Θ^{p+q+1}. We have proved in Lemma 4.14 that $G_f = \bigcup_{x \in \mathrm{Ker}\, \beta_{sf}} \Psi(x)$. The next proposition follows.

PROPOSITION 6.11. *Let* $h \in \mathrm{Diff}\, \mathcal{F}_f$ *represent* $(X, \gamma) \in {}_t\tilde{\mathcal{M}}_{p,q+1}$. *Then the inertia group of* (X, γ) *is equal to the group generated by* $G_f \cup \mathrm{Im}\, \Gamma(s_*\mathrm{L}(h))$.

PROOF. Suppose that (X, γ) and $(X \# \Sigma, \gamma)$ are orientation and basis preserving diffeomorphic, where $\Sigma \in \Theta^{p+q+1}$. The element $\mathrm{N}(\Sigma) \in \mathrm{Diff}\, \mathcal{F}_f$ was defined in § 4 by the diffeomorphism different from the identity only on $D^p_+ \times D^q_+$. The manifold $X \# \Sigma$ is represented by $h\mathrm{N}(\Sigma)$. Then the argument like that of Proposition 5.3 or 6.6 shows that, since X and $X \# \Sigma$ are diffeomorphic, there exists $g_1, g_2 \in \mathrm{Diff}\, \mathcal{E}_f$ whose pseudo-diffeotopy classes $\{g_1\}$, $\{g_2\}$ belong to Ker A such that

$$\{\eta(g_1) h \eta(g_2)\} = \{h\mathrm{N}(\Sigma)\} \in \pi_0(\mathrm{Diff}\, \mathcal{F}_f)\,.$$

Then $\{h^{-1}\eta(g_1) h \eta(g_2)\} = \{\mathrm{N}(\Sigma)\}$. By the commutativity of the next diagram

$$\begin{tikzcd}
 & & \Theta^{p+q+1} \arrow[d, "\mathrm{N}"'] \arrow[dr, "p"] & & \\
0 \arrow[r] & \eta_*(\mathrm{Ker\,A}) \arrow[r] & \mathrm{Ker\,L} \arrow[r, "\mathrm{M}"] & \Theta^{p+q+1}/G_f \arrow[r] & 0
\end{tikzcd} \tag{1}$$

we have $M(h^{-1}\eta(g_1)h) = \{p(\Sigma)\} \in \Theta^{p+q+1}/G_f$. We have defined in § 1 the homomorphism $\mathrm{B}: \pi_p(\mathrm{SO}_{q+1}) \to \mathrm{Ker\,A} \subset \tilde{\pi}_0(\mathrm{Diff}\, \mathcal{E}_f)$. Let $r: S^p \to \mathrm{SO}_{q+1}$ be a C^∞-map such that $\mathrm{B}(r) = g_1$. Define $\tilde{r} \in \mathrm{Diff}\, S^p \times S^q$ by

$$\tilde{r}(x, y) = (x, r(x)y)\,.$$

We can assume that $\tilde{r}$ keeps $D^p_+ \times S^q$ fixed. We write $\mathrm{Diff}\,(X \,\mathrm{rel}\, Y)$ for the group of the diffeomorphisms of X which keeps the submanifold Y of X invariant. Let $r' \in \mathrm{Diff}\,(D^p \times S^q \operatorname{rel} \partial D^p \times S^q)$ be the restriction of $\tilde{r}$ on $D^p_- \times S^q$. We have defined in §4 the homomorphism $\mathrm{L}: \mathrm{Ker}\,\mathrm{K} \to \pi_q(\mathrm{SO}_p)/\mathrm{Im}\,\alpha_f$. Let $t: S^q \to \mathrm{SO}_p$ be a C^∞-map which represent $\mathrm{L}(h)$. Define $t' \in \mathrm{Diff}\, D^p \times S^q$ by

$$t'(x, y) = (t(y)x, y).$$

Then $t'^{-1}r't'$ belongs to $\mathrm{Diff}\,(D^p \times S^q \operatorname{rel} \partial D^p \times S^q)$. Let us define $k \in \mathrm{Diff}\,\mathcal{F}_f$ by the identity on $D^p_+ \times S^q$ and by $t'^{-1}r't'$ on $D^p_- \times S^q$. Then k belongs to $(\lambda 1)$ and k is pseudo-diffeotopic to $h^{-1}\eta(g_1)h$. We have defined $m: (\lambda 1) \to \Theta^{p+q+1}$, where $m(k)$ represents $\mathrm{M}(h^{-1}\eta(g_1)h)$. Hence there exists $\Sigma' \in G_f$ such that

$$m(k) = \Sigma + \Sigma'.$$

Now we show that $m(k)$ is contained in $\mathrm{Im}\,\Gamma(s_*\mathrm{L}(h))$. Let us define a map $\mathrm{T}: (\mathrm{Diff}\, D^p \times S^q \operatorname{rel} \partial D^p \times S^q) \to \mathrm{Diff}\, S^p \times S^q$ by the canonical extension. That is, if $f \in \mathrm{Diff}\,(D^p \times S^q \operatorname{rel} \partial D^p \times S^q)$, we define $\mathrm{T}(f) \in \mathrm{Diff}\, S^p \times S^q$ by the identity on $D^p_+ \times S^q$ and by f on $D^p_- \times S^q$. Then the next diagram commutes

$$\begin{tikzcd}
(\lambda 1) \arrow[r, "m_1"] & (\mathrm{Diff}\, D^p_- \times S^q \operatorname{rel} \partial D^p_- \times S^q) \arrow[r, "\mathrm{T}"] \arrow[d, "m_2"] & \mathrm{Diff}\, S^p \times S^q \arrow[d, "\mathrm{c}"] \\
& \mathrm{Diff}\, S^{p+q} \arrow[r, "m_3"] & \Theta^{p+q+1}
\end{tikzcd} \tag{2}$$

where m_1 is defined by the restriction, m_2 by the canonical extension and m_3 by the natural projection. The map $m: (\lambda 1) \to \Theta^{p+q+1}$ is equal to the composition $m_3 m_2 m_1$. Thus we have

$$m(k) = \mathrm{c}\mathrm{T}(t'^{-1}r't').$$

But we have $\mathrm{T}(t'^{-1}r't') = \widetilde{st}^{-1}\tilde{r}\widetilde{st}$, where $\widetilde{st} \in \mathrm{Diff}\, S^p \times S^q$ is defined by $\widetilde{st}(x, y) = (st(y)x, y)$ and s is the inclusion $\mathrm{SO}_q \to \mathrm{SO}_{q+1}$. Consequently

$$m(k) = \mathrm{c}(\widetilde{st}^{-1}\tilde{r}\widetilde{st}).$$

Since we have defined $\Gamma(\{r\}, s_*\mathrm{L}(h))$ by $\mathrm{c}(\widetilde{st}^{-1}\tilde{r}\widetilde{st})$, we have $m(k) \in \mathrm{Im}\,\Gamma(s_*\mathrm{L}(h))$. Hence Σ lies in $G_f \cup \mathrm{Im}\,\Gamma(s_*\mathrm{L}(h))$.

Conversely let $\Sigma_1 \in G_f$. Then $\mathrm{N}(\Sigma_1)$ is mapped by M trivially. By the exact sequence of the diagram (1), we have $\{\mathrm{N}(\Sigma_1)\} = \eta_*(g_1)$ for some $\{g_1\} \in \mathrm{Ker}\,\mathrm{A} \subset \tilde{\pi}_0(\mathrm{Diff}\,\mathcal{E}_f)$. Then $\{h\mathrm{N}(\Sigma_1)\} = \{h\}\eta_*(g_1)$. Recall that (X, γ) and $(X \# \Sigma, \gamma)$ are orientation and basis preserving diffeomorphic if and only if there exists $g, g' \in \mathrm{Diff}\,\mathcal{E}_f$ whose pseudo-diffeotopy class $\{g\}$, $\{g'\}$ belong to Ker A, such that $\{\eta(g)h\eta(g')\} = \{h\mathrm{N}(\Sigma)\}$. Hence it follows that Σ_1 belongs to the inertia group of (X, γ). Next suppose Σ_2 belongs to $\mathrm{Im}\,\Gamma(s_*\mathrm{L}(h))$. Then there exists or C^∞-map $r_2: S^p \to \mathrm{SO}_{q+1}$ such that

$$\Sigma_2 = \mathrm{c}(\widetilde{st}^{-1}\tilde{r}_2\widetilde{st}'),$$

where $t\colon S^q \to SO_p$ represents $L(h)$. Let $g_2 \in \mathrm{Diff}\,\mathcal{E}_f$ represent $B(r_2) \in \mathrm{Ker}\,A$. By the commutative diagram (2), we have

$$M(h^{-1}\eta(g_2)h) = p(\Sigma_2) \in \Theta^{p+q+1}/G_f\,.$$

By the diagram (1), it follows that there exists $g_3 \in \mathrm{Diff}\,\mathcal{E}_f$ such that $g_3 \in \mathrm{Ker}\,A$ and

$$\{N(\Sigma_2)^{-1}h^{-1}\eta(g_2)h\} = \{\eta(g_3)\}\,.$$

Consequently $\eta(g_2)h\eta(g_3^{-1}) = hN(\Sigma_2)$ and the couples (X, γ) and $(X \# \Sigma_2, \gamma)$ are orientation and basis preserving diffeomorphic. Hence the group generated by G_f and $\Gamma(s_* L(h))$ belongs to the inertia group of X. The proof is complete.

By this proposition we can compute the order of ${}_t\widetilde{\mathcal{M}}_{p,q+1}$ and hence $\widetilde{\mathcal{M}}_{p,q+1}$.

THEOREM 6.12. *The number of differentiable manifolds satisfying* (**), *modulo diffeomorphisms preserving orientation and the preferred basis of p-dimensional homology group, is equal to*

$$\sum_{\{f\}\in\pi_{p-1}(SO_{q+1})} \ \sum_{l\in \mathrm{Ker}\,\Gamma(f)/\mathrm{Im}\,(F_f+\delta)} \#[\Theta^{p+q+1}/\{G_f \cup \Gamma(sl)\}]\,.$$

Especially,

PROPOSITION 6.13. *If* $p \equiv 3, 5, 6, 7 \pmod 8$, *the number of differentiable manifolds satisfying* (**), *modulo diffeomorphisms preserving orientation and the preferred basis, is equal to*

$$\sum_{l\in \mathrm{Im}\, s_*\colon \pi_q(SO_p)\to\pi_q(SO_{p+1})} \#[\Theta^{p+q+1}/\Gamma(l)]\,.$$

PROPOSITION 6.14. *If* $p \equiv 5, 6 \pmod 8$, *the number of differentiable manifolds, modulo diffeomorphism preserving orientation and preferred basis is equal to*

$$\#[\{\mathrm{Im}\, s_*\colon \pi_q(SO_p) \to \pi_q(SO_{p+1})\} \oplus \Theta^{p+q+1}]\,.$$

§ 7. Computation for diffeotopy groups.

Concerning the relation between diffeotopy and pseudo-diffeotopy, the following theorem due to Cerf [3] is known.

THEOREM 7.1 (Cerf). *Let V be a compact C^∞-manifold of dimension $\geqq 9$, such that $\pi_1(V) = \pi_2(V) = 0$. Let $\mathcal{G}$ be the group of diffeomorphisms of $V \times [0, 1]$ which induce identity on $V \times \{0\}$, then $\pi_0(\mathcal{G}) = 0$.*

As a corollary, we have,

COROLLARY 7.2. *If V is a compact C^∞-manifold without boundary of dimension $\geqq 9$ such that $\pi_1(V) = \pi_2(V) = 0$. Then $\tilde{\pi}_0(\mathrm{Diff}\ V) \approx \pi_0(\mathrm{Diff}\ V)$.*

If we apply this to our case,

THEOREM 7.3. *If p, $q \geqq 3$, $p+q \geqq 9$, then*

$$\tilde{\pi}_0(\mathrm{Diff}\ \mathcal{F}_f) \approx \pi_0(\mathrm{Diff}\ \mathcal{F}_f)\,.$$

Therefore by Theorem 4.16, we can compute the order of $\pi_0(\mathrm{Diff}\ \mathcal{F}_f)$ for $p<q<2p-4$.

In analogy with the case $\tilde{\pi}_0(\mathrm{Diff}\ \mathcal{E}_f)$, we can compute $\pi_0(\mathrm{Diff}\ \mathcal{E}_f)$. We first define

$$\bar{\mathrm{A}}:\ \pi_0(\mathrm{Diff}\ \mathcal{E}_f) \rightarrow \boldsymbol{Z}_2\,,$$

and next we can define

$$\bar{\mathrm{B}}:\ \pi_p(\mathrm{SO}_{q+1}) \rightarrow \mathrm{Ker}\ \bar{\mathrm{A}}$$

by the same method as in §1. Analogous to Proposition 1.1, we have

PROPOSITION 7.4. *In case $p<2q-1$, $p+q \geqq 9$, $p, q \geqq 3$ the homomorphism* B *is epimorphic.*

PROOF. The first part of the proof is the same as that of Proposition 1.1. We will give a diffeotopy $\bar{Q}$ connecting k with the diffeomorphism k', which is equal to $k|T$ when restricted on T, and is equal to $(k|\partial T)\times$identity of $\mathcal{E}_f-T = \mathcal{F}_f \times I$. By Theorem 7.1, there exists a diffeotopy of $\mathcal{F}_f \times I$ connecting $(k|\partial T)$ identity and $k|\mathcal{F}_f \times I$. We can take this diffeotopy so that it is transversal near $\mathcal{F}_f \times 0$ (see Cerf [**2**, N°8, Proposition]). Hence this diffeotopy is extendable to whole $\mathcal{E}_f$ by the indentity map on T. Hence the proof is comlete.

We also have analogous one to Proposition 1.2. Likewise in correspondence with Theorem 1.3, we have:

THEOREM 7.5. *Let $3 \leqq p < 2q-1$, $p+q \geqq 9$, then the order of $\pi_0(\mathrm{Diff}\ \mathcal{E}_f)$ is equal to the order of the group*

$$\pi_p(\mathrm{SO}_{q+1})/\mathrm{Ker}\ \bar{\mathrm{B}} \oplus X\,.$$

The groups Ker B and Ker $\bar{\mathrm{B}}$ can be different. But $\eta_*(\pi_p(\mathrm{SO}_{q+1})/\mathrm{Ker\ B}) \approx \eta_*(\pi_p(\mathrm{SO}_{q+1})/\mathrm{Ker}\ \bar{\mathrm{B}})$ by Corollary 7.2.

The analogous one to Proposition 1.4 holds. Consequently

THEOREM 7.6. *If $3 \leqq p < 2q-1$, $p+q \geqq 9$, then $\tilde{\pi}_0(\mathrm{Diff}\ S^p \times D^{q+1}) \approx \pi_0(\mathrm{Diff}\ S^p \times D^{q+1}) \approx \pi_p(\mathrm{SO}_{q+1})$.*

Appendix

In this appendix, we will give an example that the pairing

$$\Gamma:\ \pi_{p-1}(\mathrm{SO}_{q+1}) \times \pi_q(\mathrm{SO}_p) \rightarrow \Theta^{p+q}$$

is not always the trivial map. This example was pointed out by Professor I. Tamura. The higher dimensional analogue can be found in [**9**]. The pairing $\Gamma(f, r)$ is defined by the equation $(f, r) = \mathrm{c}(\tilde{f}^{-1}\tilde{r}\tilde{f})$ (see § 4). Since $\mathrm{c}(\tilde{f}^{-1}) = \mathrm{c}(\tilde{f}) = \mathrm{c}(\tilde{r}) = 0$ by Lemma 3.1, it follows that the map C in some case is not the homomorphism. The group $G_f \subset \Theta^{p+q}$ was proved in § 6 to act as the inertia group of manifolds satisfying the condition (**). But we have the equation $G_f = \bigcup_{x \in \operatorname{Ker} \beta_{sf}} \Psi(x)$, by Lemma 4.14. Further by Lemma 4.11, the image of $x \in \operatorname{Ker} \beta_{sf}$ by the map Ψ is a coset space of $\Theta^{p+q}/\operatorname{Im} \Gamma(sf)$. Hence our example also shows the non-triviality of the inertia group.

We have

$$\pi_7(\mathrm{SO}_{12}) \approx \boldsymbol{Z} \qquad \pi_{11}(\mathrm{SO}_8) \approx \boldsymbol{Z} + \boldsymbol{Z}_2 .$$

Let a be a generator of $\pi_7(\mathrm{SO}_{12})$ and let b and c be generators of the free part and the torsion part of $\pi_{11}(\mathrm{SO}_8)$ respectively. Then we can write the homotopy class of the map $f: S^7 \to \mathrm{SO}_{12}$ and $r: S^{11} \to \mathrm{SO}_8$ by

$$\{f\} = ma, \qquad \{r\} = nb + pc .$$

THEOREM. *Suppose* $m \cdot n \not\equiv 0$ (mod 73), *then* $\Gamma(f, r)$ *is not the trivial element in* Θ^{p+q}.

PROOF. In § 4, we see that the diffeomorphism $\tilde{r}$ of T_0, with T_0 being the tubular neighborhood of the fibre S^{11} of $\mathcal{T}_f$, is extendable to a diffeomorphism d of $\mathcal{T}_f$ if and only if the diffeomorphism $\tilde{f}\tilde{r}\tilde{f}^{-1}$ of $S^7 \times S^{11}$ is extendable to a diffeomorphism of $D^8 \times S^{11}$. And by Lemma 4.8, it is equivalent to the equation $\mathrm{c}(\tilde{f}\tilde{r}\tilde{f}^{-1}) = 0$. Suppose such diffeomorphism d of $\mathcal{T}_f$ exists. Then attaching two copies of $\mathcal{E}_f$ by d, we obtain a closed smooth manifold

$$M = \mathcal{E}_f \bigcup_d \mathcal{E}_f .$$

The homology group of M are as follows;

$$\mathrm{H}_i(M) = \begin{cases} \boldsymbol{Z} & \text{for } i = 0,\ 8,\ 12,\ 20 \\ 0 & \text{otherwise} . \end{cases}$$

We can embed S^8 as the zero cross section of $\mathcal{E}_f$ and S^{12} by $D^{12} \cup D^{12}$, where D^{12} denotes the fibre of $\mathcal{E}_f$. Then such embedded spheres generate $\mathrm{H}_8(M)$ and $\mathrm{H}_{12}(M)$ respectively and the characteristic class of their tubular neighborhood are $\{f\}$ and $\{r\}$. According to Bott-Milnor [1], the Pontrjagin classes of the manifold M are given by

$$p_2(M) = 3!\,m , \qquad p_3(M) = 2 \cdot 5!\,n .$$

Let $[M]$ be the fundamental homology class of M. Then the index theorem [7] implies that

$$(5110\ p_5(M) - 336\ p_3(M) p_2(M))[M^5] = 0 .$$

Thus we have $m \cdot n = 0$ (mod 73). The proof finishes.

University of Tokyo

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