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# On a special kind of Dirichlet series

Dedicated to Professor Shôkichi Iyanaga on his 60th birthday

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The purpose of the present paper is to introduce a very simple idea which leads to a special kind of zeta-function satisfying a functional equation of the usual form. At least according to the knowledge of the author, the zeta-function has never been investigated at any place, although the function lies entirely in the frame of the classical theory of number fields. For this purpose, we use two tools. One is the Eisenstein series in the sense of [**6**], [**7**], containing a character of a discontinuous group, and the other is a special kind of character, as studied in [**3**], of a discontinuous group of Hilbert's type. If a character of this kind, which has an intimate connection with non-congruence normal subgroups of arithmetically defined discontinuous groups, is used in the construction of an Eisenstein series, then we obtain Dirichlet series of a rather unfamiliar fashion from non-constant terms in the Fourier expansion of the Eisenstein series, and the functional equation of the zeta-function determined by the Dirichlet series follows immediately from the functional equation of the Eisenstein series. The coefficients of the Dirichlet series, i. e. numerators of individual terms in the usual notation, are in fact Gauss sums containing congruence characters that are not quadratic.

The space on which the Eisenstein series of our present interest are considered is, in general, a finite direct product of real, three dimensional hyperbolic space. On such a space we have a nice discontinuous operation of a modular group of Hilbert's type made up from a totally imaginary number field $F$. Since we need non-quadratic power residue symbols of the ground field $F$ in the definition of the character $\chi$, the field $F$ must contain at least four roots of unity. Hence, it is enough to observe only totally imaginary fields.

Since, however, we intend in this paper a rapid explanation of the idea, we shall treat here only the simplest case of $F=\boldsymbol{Q}(\sqrt{-1})$. As discontinuous groups, we take subgroups of $SL(2,\mathfrak{o})$ where $\mathfrak{o}$ is the ring of integers of $F$. The space, on which such groups operate, is the three dimensional hyperbolic space $H=SL(2,\boldsymbol{C})/SU(2)$, so that all results, for example in [**5**], about discontinuous groups of Picard's type are applicable.

In § 1, we construct a discontinuous group $\Gamma$ which is most adequate for our purpose, together with a character $\chi$ of $\Gamma$ containing the fourth power residue symbol of $F$. In § 2, we define six Eisenstein series $E_i(u, s, \chi)$, $(i=1, 2, \cdots, 6)$, corresponding to cusps of $\Gamma$, and we explicitly write down the functional equation of $E_i(u, s, \chi)$ in a vectorial form. Our main results containing the functional equations of the functions $\zeta_i(s, \chi, m)$ which come from non-constant terms in the Fourier expansion of $E_i(u, s, \chi)$ are stated in § 3. The functional equation, which takes a vectorial form in the beginning, can easily be reduced to an usual form, namely to a symmetry of one function with gamma-factors, if we use some linear combinations of $\zeta_i(s, \chi, m)$.

The regularity of $\zeta_i(s, \chi, m)$ at $s=5/4$ is one of the unsolved problems, while it is not difficult to show that $\zeta_i(s, \chi, m)$ is holomorphic for $\mathrm{Re}\, s>5/4$. The behavior of $\zeta_i(s, \chi, m)$ at $s=5/4$ is resemble to that of ordinary zeta- and $L$-functions at $s=1$. So far, however, the author did not obtain satisfactory results for $\zeta_i(s, \chi, m)$ at $s=5/4$, although many analytic properties of $\zeta_i(s, \chi, m)$ can be deduced from those of $E_i(u, s, \chi)$. Including this point, several further remarks and indications of some other results for which detailed reports will appear separately are stated in § 4.

Throughout this paper, the terminologies of [5] will be used without repeating definitions.

## § 1. The group $\Gamma$ and the character $\chi$.

Let $F=\boldsymbol{Q}(\sqrt{-1})$ be the Gauss' number field, and let $\mathfrak{o}$ be the ring of integers in $F$. We denote then by $\Gamma$ once for all the group generated by $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ and by all the elements $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathfrak{o})$ with

$$a \equiv d \equiv 1 \pmod{\lambda^3}, \qquad b \equiv c \equiv 0 \pmod 4, \tag{1}$$

where $\lambda$ is defined by

$$\lambda=1-i, \qquad (i=\sqrt{-1}).$$

If we identity the hyperbolic space $H=SL(2, \boldsymbol{C})/SU(2)$ with the space of all matrices of the form $u=\begin{pmatrix} z & -v \\ v & \bar{z} \end{pmatrix}$, $(z=x+iy \in \boldsymbol{C},\ \boldsymbol{R} \ni v>0)$, then $H$ can be considered as an upper half space in $\boldsymbol{R}^3$, and the natural operation of $\sigma \in G=SL(2, \boldsymbol{C})$ on $H$ takes the form

$$u \to (\tilde{a}u+\tilde{b})(\tilde{c}u+\tilde{d})^{-1},$$

where $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and we write $\tilde{w}=\begin{pmatrix} w & \\ & \bar{w} \end{pmatrix}$ for any $w \in \boldsymbol{C}$.

Throughout the paper the elements $\omega$ and $-\omega\in G$ will be identified, so that a matrix $\omega$ can be regarded as an abbreviation of $\omega$ mod $\{\pm 1\}$, i.e. the transformation of $H$ raised by $\omega$. The group $\Gamma$, for example, should also be understood to denote a transformation group.

$\Gamma$ has a fundamental domain, whose volume with respect to the invariant measure

$$d\mu(u)=\frac{dxdydv}{v^3}$$

of $H$ is finite.

Denote now by $\left(\frac{c}{d}\right)$ the power residue symbol of degree four in $F$[1], and put

$$\chi(\sigma)=\begin{cases}\left(\frac{c}{d}\right), & (c\neq 0\,,\ d\equiv 1 \pmod 4)\,,\\ \left(\frac{c}{d}\right)i\,, & (c\neq 0\,,\ d\equiv 1+\lambda^3 \pmod 4)\,,\\ 1\,, & (c=0)\,,\end{cases}$$

for $\sigma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma$ with (1). Then we have $\chi(\sigma\sigma')=\chi(\sigma)\chi(\sigma')$ for two such transformations $\sigma$, $\sigma'$. This assertion is slightly different from the theorem of [3], but the proof is the same. Since

$$\chi\left(\begin{pmatrix} & 1\\ -1 & \end{pmatrix}\sigma\begin{pmatrix} & -1\\ 1 & \end{pmatrix}\right)=\chi\left(\begin{pmatrix}d & -c\\ -b & a\end{pmatrix}\right)=\chi\left(\begin{pmatrix}a & c\\ b & d\end{pmatrix}\right)^{-1}=\chi(\sigma)\,,$$

we can uniquely extend the character $\chi$ to $\Gamma$ by the condition $\chi\left(\begin{pmatrix} & -1\\ 1 & \end{pmatrix}\right)=1$. We fix the character $\chi$ of $\Gamma$ once for all. For $\sigma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with

$$c\equiv -b\equiv 1 \pmod{\lambda^3}\,,\qquad a\equiv d\equiv 0 \pmod 4\,,$$

the value of $\chi$ is given by

$$\chi(\sigma)=\begin{cases}\left(\frac{-d}{c}\right), & (c\equiv 1 \pmod 4\,,\ d\neq 0)\,,\\ \left(\frac{-d}{c}\right)i\,, & (c\equiv 1+\lambda^3 \pmod 4\,,\ d\neq 0)\,,\\ 1\,, & (d=0)\,.\end{cases}$$

An important thing which we have to do next is the determination of all the essential cusps, in the sense of [5], of $\Gamma$ with respect to $\chi$. If the $z$-plane determined by $v=0$ in $\boldsymbol{R}^3$ is identified with $\boldsymbol{C}$, then every cusp of $\Gamma$ is either $\infty$ or a number in $F$, and it is clear that such cusps are all equivalent with

1) For the definition and for all properties of power residue symbols, see [1].

respect to $SL(2, \mathfrak{o})$. Therefore, if we denote by $\hat{\Gamma}_\infty$ the group of all $\sigma=\begin{pmatrix} a & b \\ & d \end{pmatrix}$ $\in SL(2, \mathfrak{o})$, then the equivalence classes are in one to one correspondence with double cosets of the form $\Gamma\sigma\hat{\Gamma}_\infty$, $(\sigma\in SL(2, \mathfrak{o}))$. It is easy to see that we can take representatives for the double cosets always from the finite set of matrices of the form $\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}$, $\mathfrak{o}\ni x \bmod 4$. Let $\kappa$ be a cusp of $\Gamma$, let $\Gamma_\kappa$ be the group of all $\gamma\in\Gamma$ with $\gamma\kappa=\kappa$, and let $B$ be the subgroup of $G$ consisting of all $\begin{pmatrix} x_{11} & x_{12} \\ & x_{22} \end{pmatrix}\in G$. Then

$$\Gamma_\kappa=\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}B\begin{pmatrix} 1 & \\ -x & 1 \end{pmatrix}\cap\Gamma$$

for $\kappa=1/x$, and we see that the elements of $\Gamma_\kappa$ are all of the form

$$\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}\begin{pmatrix} \varepsilon & n \\ & \bar{\varepsilon} \end{pmatrix}\begin{pmatrix} 1 & \\ -x & 1 \end{pmatrix}=\begin{pmatrix} \varepsilon-nx, & n \\ \varepsilon x-\bar{\varepsilon}x-nx^2, & \bar{\varepsilon}+nx \end{pmatrix}$$

with $x\in\mathfrak{o}$, $n\in\mathfrak{o}$, $\varepsilon^4=1$. If here $(n, 2)=1$, then $\varepsilon-nx\equiv\bar{\varepsilon}-nx\equiv 0 \pmod 4$; hence $\varepsilon-\bar{\varepsilon}\equiv 2nx \pmod 4$. This means $\varepsilon=\pm i$, and $x\equiv i \pmod 2$. So, $n$ must be divisible by 4, and $\varepsilon=\pm 1$, whenever $x\not\equiv i \pmod 2$. Thus we have verified that

$$\Gamma_\kappa=\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix}\Gamma_\infty\begin{pmatrix} 1 & \\ -x & 1 \end{pmatrix}, \qquad (x\not\equiv i \pmod 2),$$

and that all such $\Gamma_\kappa$ are isomorphic to $\Gamma_\infty$, i.e. the group, isomorphic to $\boldsymbol{Z}\times\boldsymbol{Z}$, of all $\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}$ with $\mathfrak{o}\ni n\equiv 0 \pmod 4$. In order that a cusp $1/x$ is essential, it is necessary that

$$\left(\frac{-nx^2}{1+nx}\right)=\left(\frac{x}{1+nx}\right)=1$$

holds for all $n\equiv 0 \pmod 4$. If one cheques all 16 representatives mod 4 making use of the properties of the power residue symbol, then there will remain the following six values of $x$ corresponding to non-equivalent essential cusps $1/x$ of $\Gamma$ with respect to $\chi$:

$$x_1=1, \quad x_2=1+\lambda^3, \quad x_3=\lambda^2, \quad x_4=2, \quad x_5=\lambda^3, \quad x_6=0.$$

We denote the cusp $1/x_i$ by $\kappa_i$, and $\Gamma_{\kappa i}$ by $\Gamma_i$.

## § 2. Eisenstein series $E_i(u, s, \chi)$.

For each $i=1, 2, \cdots, 6$, put

$$\sigma_i=\begin{pmatrix} 1 & \\ x_i & 1 \end{pmatrix}\begin{pmatrix} 2 & \\ & 2^{-1} \end{pmatrix}.$$

Then we see $\sigma_i\infty=\kappa_i$, and that $\sigma_i^{-1}\Gamma_i\sigma_i$ is isomorphic to $\mathfrak{o}$ by $\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\leftrightarrow b$. Hence, the euclidean measure of a period parallelogram $\sigma_i^{-1}\Gamma_i\sigma_i\backslash \boldsymbol{C}$ is 1 for all $i$. Now we define as in (18) of [**5**] the Eisenstein series $E_i(u, s, \chi)$ by

$$E_i(u, s, \chi)=\sum_{\sigma\in\Gamma_i\backslash\Gamma}\overline{\chi}(\sigma)v(\sigma_i^{-1}\sigma u)^s, \qquad (\mathrm{Re}\, s>2),$$

where $v(u)$ stands for $v$ for $u=\begin{pmatrix}z & -v\\ v & \bar{z}\end{pmatrix}\in H$. Since the different of $F$ is 1/2, the module $\Gamma_{j,\infty}^*$ in (19) of [**5**] is $\dfrac{1}{2}\mathfrak{o}$. Hence $E_i(\sigma_j u, s, \chi)$ has a Fourier expansion of the form[2)]

$$E_i(\sigma_j u, s, \chi)=\sum_{m\in\mathfrak{o}}a_{ij,m}(v, s, \chi)e^{\pi i tr\widetilde{m}\widetilde{z}} \tag{2}$$

with

$$a_{ij,m}(v, s, \chi)=\int_0^1\int_0^1 E_i(\sigma_j u, s, \chi)e^{-\pi i tr\widetilde{m}\widetilde{z}}dxdy\,.$$

In particular, it follows form (24), (25) of [**5**] that $a_{ij,0}(v, s, \chi)$ is given by

$$a_{ij,0}(v, s, \chi)=\begin{cases} v^s+\varphi_{ii}(s, \chi)v^{2-s}, & (i=j),\\ \varphi_{ij}(s, \chi)v^{2-s}, & (i\neq j),\end{cases} \tag{3}$$

with

$$\varphi_{ij}(s, \chi)=\frac{\pi}{s-1}\cdot\frac{1}{2}\sum_{\substack{c\neq 0\\ \begin{pmatrix}* & *\\ c & d\end{pmatrix}\in\sigma_i^{-1}\Gamma\sigma_j}}\frac{1}{|c|^{2s}}\Big(\sum_{d \bmod c}\overline{\chi}_{ij}(c, d)\Big), \tag{4}$$

and with

$$\chi_{ij}(c, d)=\chi\Big(\sigma_i\begin{pmatrix}a & b\\ c & d\end{pmatrix}\sigma_j^{-1}\Big),$$

where the summation is extended over all pairs $(c, d)$ of numbers in $\mathfrak{o}$ such that there exists in $\sigma_i^{-1}\Gamma\sigma_j$ an element of the form $\pm\begin{pmatrix}* & *\\ c & d\end{pmatrix}$, $d$ being counted mod $c$.

We now propose to observe the matrix $\Phi(s, \chi)=(\varphi_{ij}(s, \chi))$ more closely, and to write down the functional equation for $E_i(u, s, \chi)$ in a explicit form. Put for the sake of simplicity

$$\sigma_i=\begin{pmatrix}1 & \\ x & 1\end{pmatrix}\begin{pmatrix}2 & \\ & 2^{-1}\end{pmatrix}, \qquad \sigma_j=\begin{pmatrix}1 & \\ y & 1\end{pmatrix}\begin{pmatrix}2 & \\ & 2^{-1}\end{pmatrix},$$

and assume

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2) As far as no confusion is possible, we use the letter $i$ both for suffix and for $\sqrt{-1}$ in this paper.

$$\sigma_i\begin{pmatrix} a & b \\ c & d \end{pmatrix}\sigma_j^{-1}=\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\in\Gamma\,.$$

Then $a$, $b$, $c$, $d$ must satisfy either the condition

$$a\equiv\eta \pmod 4\,,\qquad b\in\mathfrak{o}\,,$$

$$\frac{c}{4}\equiv\eta(-x+y) \pmod 4\,,\qquad d\equiv\eta \pmod 4\,,$$

or the condition

$$a\equiv-\eta y \pmod 4\,,\qquad 4b\equiv-\eta \pmod 4\,,$$

$$\frac{c}{4}\equiv\eta(1+xy) \pmod 4\,,\qquad d\equiv\eta x \pmod 4\,,$$

according to $d'\equiv 1 \pmod{\lambda^3}$ or $c'\equiv 1 \pmod{\lambda^3}$, where $\eta$ stands for 1 or $1+\lambda^3$ in both cases. Conversely, if these conditions are satisfied, and if moreover $ad-bc=1$, then $\sigma_i\begin{pmatrix} a & b \\ c & d \end{pmatrix}\sigma_j^{-1}$ belongs to $\Gamma$.

On the other hand, the value of $\chi_{ij}(c, d)$ can be expressed by $c$ and $d$ because of

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}=\begin{pmatrix} a-4by & 4b \\ ax+\frac{1}{4}c-4bxy-dy\,, & 4bx+d \end{pmatrix}.$$

For example[3], if $d'\equiv 1 \pmod{\lambda^3}$, then we have

$$\chi_{ij}(c, d)=\left(\frac{c/4}{d}\right)i^{t(d)}\,, \tag{5}$$

if $c'\equiv 1 \pmod{\lambda^3}$, $(x, 2)=1$, then we have

$$\chi_{ij}(c, d)=\left(\frac{c/4}{d}\right)\left[\left(\frac{-1}{x}\right)i\right]^{t(c/4-dy)}\,, \tag{6}$$

and if $c'\equiv 1 \pmod{\lambda^3}$, $(x, 2)=1$, $(y, 2)=1$, then we have

$$\chi_{ij}(c, d)=\begin{cases} \left(\frac{d}{c/4}\right)(-i)^{t(c/4-dy)}\left(\frac{x}{1-xy}\right), & (x\neq 0)\,, \\ \left(\frac{d}{c/4}\right)(-i)^{t(c/4-dy)}\,, & (x=0)\,, \end{cases} \tag{7}$$

where $t(m)$, $(\mathfrak{o}\ni m\equiv 1 \pmod{\lambda^3})$ is defined by

$$t(m)=\begin{cases} 0\,, & m\equiv 1 \pmod 4\,, \\ & \text{for} \\ 1\,, & m\equiv 1+\lambda^3 \pmod 4\,, \end{cases}$$

3) Every result stated here is proved by a direct, elementary calculation similar to the proof of the theorem in [3].

and in all cases $c \neq 0$ and $d \neq 0$ are assumed. Thus, the sum $\sum\limits_{d \bmod c}$ in (4) is, up to an elementary factor, reduced to a sum of the form

$$\sum_{d \bmod c} \left(\frac{c}{d}\right), \qquad (c \equiv d \equiv 1 \pmod{\lambda^3}),$$

which is 0 unless $c$ is a fourth power, and is equal to $\varphi(c_0^4) = Nc_0^3 \cdot \varphi(c_0)$, if $c = c_0^4$. Here, $N$ means the norm with respect to $F/\boldsymbol{Q}$, and $\varphi$ is so-called Euler's function. Since

$$\sum_{c_0 \equiv 1(\lambda^3)} \frac{\varphi(c_0^4)}{Nc_0^{4s}} = \left(1 - \frac{1}{2^{4s}}\right) \frac{\zeta(4s-4)}{\zeta(4s-3)}$$

holds with Dedekind's zeta-function $\zeta(s)$ of $F$, the Dirichlet series in (4) coincides with $\zeta(4s-4)/\zeta(4s-3)$ up to an elementary factor. An explicit form of the matrix $\Phi(s) = (\varphi_{ij}(s, \chi))$, which the author obtained by a direct computation was as follows:

$$\Phi(s, \chi) = \left(-I \frac{2^{2-4s}}{1-2^{4s-3}} + M \frac{2^{1-4s}(1-2^{4s-4})}{1-2^{4s-3}}\right) \frac{\pi}{s-1} \frac{\zeta(4s-4)}{\zeta(4s-3)}, \tag{8}$$

where $I$ is the unit matrix, and

$$M = \begin{pmatrix} & & & 2 & & 2 \\ & & 1+i & -1-i & 1+i & 1+i \\ & 1-i & -1 & -1 & 1 & 1 \\ 2 & -1+i & -1 & 1 & -1 & 1 \\ & 1-i & 1 & -1 & -1 & 1 \\ 2 & 1-i & 1 & 1 & 1 & 1 \end{pmatrix}^{4)}$$

It follows from the general theory of Eisenstein series that every $E_i(u, s, \chi)$ has an analytic continuation in the whole $s$-plane, and is a single valued, meromorphic function. Furthermore, if we put

$$\begin{pmatrix} E_1(u, s, \chi) \\ \vdots \\ E_6(u, s, \chi) \end{pmatrix} = \mathcal{E}(u, s, \chi),$$

then, again by the general theory, we have the functional equation

$$\mathcal{E}(u, s, \chi) = \Phi(s, \chi) \mathcal{E}(u, 2-s, \chi) \tag{9}$$

for our Eisenstein series[5].

4) That this matrix must be hermitian is a consequence of (4).

5) For all of these results, see the literature of [5]. In our case, the proofs of the analytic continuation, of the single-valuedness, and of the functional equation can be simplified, because the constant terms in the Fourier expansion of Eisenstein series are known functions.

## § 3. Main results.

We now observe non-constant terms $a_{ij,m}(v, s, \chi)$, $(m \neq 0)$, in the Fourier expansion (2) of Eisenstein series. Without any loss of generality, we may restrict ourselves to the case of $j=6$. The formula (35) of [5] then implies that (9) turns out a functional equation

$$(\pi|m|)^s \Gamma(s)^{-1} \begin{pmatrix} \varphi_1(s, \chi, m) \\ \vdots \\ \varphi_6(s, \chi, m) \end{pmatrix} = \Phi(s, \chi)(\pi|m|)^{2-s}\Gamma(2-s)^{-1} \begin{pmatrix} \varphi_1(2-s, \chi, m) \\ \vdots \\ \varphi_6(2-s, \chi, m) \end{pmatrix} \tag{10}$$

where $\varphi_i(s, \chi, m)$ is given by

$$\varphi_i(s, \chi, m) = \sum_{c \neq 0} \frac{\tau_{i,m}(c)}{Nc^s} \tag{11}$$

according to (31) of [5] with

$$\tau_{i,m}(c) = \frac{1}{2} \sum_{\substack{d \bmod c \\ \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \sigma_i^{-1}\Gamma\sigma_6}} \bar{\chi}_{i6}(c, d) e^{\pi i tr\, md/c}. \tag{12}$$

Here, the trace means the ordinary trace with respect to $F/\boldsymbol{Q}$. From these facts one can already recognize that the functions $\varphi_1, \cdots, \varphi_6$ including Gauss sums satisfy a functional equation of a vectorial form.

To make the situation more clear, we shall show several precise formulas. First of all, we intend to give a more concrete expression for $\tau_{i,m}(c)$ in (11). Let $\mathrm{ord}_\lambda\, c$, $(c \in \mathfrak{o})$, denote the natural number $l$ such that $c \equiv 0 \pmod{\lambda^l}$, $c \not\equiv 0 \pmod{\lambda^{l+1}}$, and put

$$\tau_m(c, \delta) = \sum_{\substack{d \bmod c \\ d \equiv \delta\ (4)}} \left(\frac{c/4}{d}\right)^{-1} e^{\pi i tr\, md/c}$$

for $\mathrm{ord}_\lambda\, c \geqq 4$, $\neq 5$, $\mathrm{ord}_\lambda\, \delta = 0$, $c/4 \neq \pm i$, and

$$\tau'_m(c, \delta) = \sum_{\substack{d \bmod c \\ d \equiv \delta\ (4)}} \left(\frac{d}{c/4}\right)^{-1} e^{\pi i tr\, md/c}$$

for $\mathrm{ord}_\lambda\, c = 0$[6]. Then a direct calculation using (5), (6), (7), and (12) shows that

$$\tau_{1,m}(c) = \tau_m(c, 1) - i\tau_m(c, 1+\lambda^3)$$
$$+ (-i)^{t(-c/4)} \{\tau_m(-c, 1) + \tau_m(-c, 1+\lambda^3)\}$$

for $c/4 \equiv -1 \pmod{\lambda^3}$,

6) If $(c/4, d) = 1$, then the residue symbol in these two sums must be understood to represent 0 as usual.

$$\tau_{2,m}(c)=\tau_m(c,1)-i\tau_m(c,1+\lambda^3)$$
$$+i^{t(-c/4)}\{\tau_m(-c,1)+\tau_m(-c,1+\lambda^3)\}$$

for $c/4\equiv -1 \pmod{\lambda^3}$, that $\tau_{3,m}(c)$ for $c/8\equiv i \pmod 2$, $\tau_{4,m}(c)$ for $c/8\equiv 1 \pmod 2$, $\tau_{5,m}(c)$ for $\mathrm{ord}_\lambda\, c=7$, and $\tau_{6,m}(c)$ for $\mathrm{ord}_\lambda\, c\geqq 8$ are all equal to $\tilde{\tau}_m(c)+\tilde{\tau}_m(-c)$ with $\tilde{\tau}_m(c)=\tau_m(c,1)-i\tau_m(c,1+\lambda^3)$, and that, whenever $c/4\equiv 1 \pmod{\lambda^3}$, $\tau_{i,m}(c) =i^{t(c/4)}\tau'_m(c,x_i)$ for $i=3,4,5,6$. On the other hand, we have always a trivial relation $\tau_{i,m}(c)=\tau_{i,m}(-c)$. All other values of $\tau_{i,m}(c)$ which were not listed above are 0. Thus we have seen that the numerator $\tau_{i,m}(c)$ on the right hand side of (12) is a linear combination of ordinary Gauss sums containing congruence characters of order 4.

What we have to do next is to bring the functional equation (10) into a usual form. Making use of the functional equation

$$\pi^{1-2s}\frac{\Gamma(s)}{\Gamma(1-s)}\zeta(s)=\zeta(1-s) \tag{13}$$

of Dedekind's zeta-function, one gets

$$\frac{\zeta(4s-4)}{\zeta(4s-3)}=\frac{\zeta(4(2-s)-3)}{\zeta(4s-3)}\pi^{-9+8s}(4-4s)\frac{\Gamma(4-4s)}{\Gamma(4s-4)}.$$

So, if we put

$$\zeta_i(s,\chi,m)=\zeta(4s-3)\varphi_i(s,\chi,m), \tag{14}$$

then (10) is transformed into

$$\pi^{-3s}|m|^s\frac{\Gamma(4(s-1))}{\Gamma(s-1)}\begin{pmatrix}\zeta_1(s,\chi,m)\\ \vdots\\ \zeta_6(s,\chi,m)\end{pmatrix} \tag{15}$$
$$=M(s)\pi^{-3(2-s)}|m|^{2-s}\frac{\Gamma(4(1-s))}{\Gamma(1-s)}\begin{pmatrix}\zeta_1(2-s,\chi,m)\\ \vdots\\ \zeta_6(2-s,\chi,m)\end{pmatrix}$$

with

$$M(s)=4\Big(-I\frac{2^{2-4s}}{1-2^{4s-3}}+M\frac{2^{1-4s}(1-2^{4s-4})}{1-2^{4s-3}}\Big). \tag{16}$$

Since the Eisenstein series is single valued and meromorphic, so is $\zeta_i(s,\chi,m)$, too. Moreover, the coefficients in the Dirichlet series expansion of $\zeta_i(s,\chi,m)$ are, by definition (14), again linear combinations of Gauss sums, and (15) is already a functional equation in a vectorial form of our zeta-functions $\zeta_i(s,\chi,m)$.

At the end of this §, we propose to observe what happens after the diagonalization of the matrix $M(s)$. Let $q$ be a characteristic root of the hermitian matrix $M$, and put

$$g(s, q) = 4\Big(-\frac{2^{2-4s}}{1-2^{4s-3}} + q\,\frac{2^{1-4s}(1-2^{4s-4})}{1-2^{4s-3}}\Big).$$

Then, (15) entails

$$M(s)M(2-s) = I\,, \tag{17}$$

and consequently

$$g(s, q)g(2-s, q) = 1\,.$$

Hence, $q$ is either 4 or $-2$. This means that the characteristic roots of $M$ consist of two 4 and four $-2$. Since

$$g(s, 4) = 2^{4-4s}\,, \qquad g(s, -2) = -\frac{2^{3-4s}(1-2^{5-4s})}{1-2^{3-4s}}\,,$$

and since

$$g(s, q) = (1+g(s, q))(1+g(2-s, q))^{-1}\,,$$

the diagonalization of $M(s)$ by a unitary matrix gives rise to six functional equations of the form

$$\xi(s) = \xi(2-s) \tag{18}$$

with

$$\xi(s) = \pi^{-3s}|m|^{s}\frac{\Gamma(4(s-1))}{\Gamma(s-1)}Z(s, \chi, m)\,,$$

where we put either

$$Z(s, \chi, m) = (1+2^{4-4s})^{-1}\zeta_i^*(s, \chi, m)$$

or

$$Z(s, \chi, m) = \Big(1-\frac{2^{3-4s}(1-2^{5-4s})}{1-2^{3-4s}}\Big)^{-1}\zeta_i^*(s, \chi, m)\,,$$

$\zeta_i^*(s, \chi, m)$ being a linear combination of $\zeta_i(s, \chi, m)$, corresponding to the diagonalization of $M$ or $M(s)$ in (15), (16) by a constant unitary matrix. The function $Z(s, \chi, m)$ is also meromorphic and single valued on the whole $s$-plane, and can be expressed by a Dirichlet series whose coefficients are linear combinations of Gauss sums containing the fourth power residue symbol. Besides, the functional equation (18) has an usual form, except that the gamma factor

$$\frac{\Gamma(4(s-1))}{\Gamma(s-1)} = (2^5\pi^3)^{-\frac{1}{2}}2^{8(s-1)}\Gamma\Big(s-\frac{3}{4}\Big)\Gamma\Big(s-\frac{1}{2}\Big)\Gamma\Big(s-\frac{1}{4}\Big)$$

is somewhat eccentric.

## § 4. Further remarks.

1. All of our previous arguments can be transfered to the case of general algebraic groups at least in principle. A comparatively easy example of the

higher dimensional cases, which is nevertheless of considerable number theoretical importance, is the one where a direct product of finite copies of $SL(2, \boldsymbol{C})$ and a discontinuous group of Hilbert's type with respect to a totally imaginary number field $F$ are treated. Of course, each component of the direct product must correspond to an infinite place of $F$. In this case, all results of this paper hold quite similarly. The explicit form, like (8), of the matrix $M(s)$ is, however, rather difficult to obtain, and it may not be possible to diagonalize the matrix $M(s)$ by a constant unitary matrix. Same situations occur even in the case as was carried out in this paper, whenever one uses smaller subgroups of $\Gamma$ instead of itself. But, such a situation causes no essential difficulty. For, even in the general case, one can prove the formula (17) first, to get $I+M(s) = M(s)(I+M(2-s))$, and then one can deduce from (15) the invariance of

$$(I+M(s))^{-1}\pi^{-3s}|m|^{s}\frac{\Gamma(4(s-1))}{\Gamma(s-1)}\begin{pmatrix}\zeta_1(s, \chi, m)\\ \vdots \\ \zeta_6(s, \chi, m)\end{pmatrix}$$

under $s\to 2-s$, which furnishes a set of functional equations of a usual form[7].

2. The Dirichlet series in (11) are absolutely convergent for $\operatorname{Re} s>3/2$, because it is easy to see that $\tau_{i,m}(c)$ is $O(\sqrt{Nc})$ as $|c|\to\infty$. On the other hand, put $E(u, s, \chi)=E_6(\sigma_6 u, s, \chi)$, $\varphi_{66}(s, \chi)=\varphi(s, \chi)$, and define $\tilde{E}(u, s, \chi)$ by

$$\tilde{E}(u, s, \chi)=\begin{cases} E(u, s, \chi), & v\leqq Y, \\ & \text{for} \\ E(u, s, \chi)-v^{s}, & v>Y, \end{cases}$$

for a sufficiently large positive number $Y$. Furthermore, let $\mathfrak{D}$ be a fundamental domain of $\Gamma$ that has a similar form to the domain by (7) of [5]. Then, we have

$$\int_{\mathfrak{D}}|\tilde{E}(u, s, \chi)|^2 d\mu(u)=\frac{Y^{2S-2}}{2S-2}+\frac{\varphi(\bar{s}, \bar{\chi})Y^{2it}-\varphi(s, \chi)Y^{-2it}}{2it}$$

for $s=S+it$, $S>1$; this is an important evalution formula related to the analytic continuation of Eisenstein series. It follows from the formula that $\int_{\mathfrak{D}}|\tilde{E}(u, s, \chi)|^2 d\mu(u)$ is finite at $s=s_0$ with $\operatorname{Re} s_0>1$ whenever $\varphi(s, \chi)$ is regular at $s=s_0$, and that

$$\int_{\mathfrak{D}}\left|\frac{\tilde{E}(u, s, \chi)}{\varphi(s, \chi)}\right|^2 d\mu(u)$$

is finite at $s=s_0$, ($\operatorname{Re} s_0>1$), even if $s=s_0$ is a pole of $\varphi(s, \chi)$. The same assertion holds for all other $E_i(u, s, \chi)$, too. Hence, we can conclude by using

7) Not vectorial form.

(30) of [5] that $\varphi_i(s, \chi, m)$ is holomorphic in the half plane determined by $\operatorname{Re} s > 5/4$, and that $\varphi_i(s, \chi, m)$ has possibly a simple pole at $s = 5/4$. Thus, the meaning of $s = 5/4$ for our $\varphi_i(s, \chi, m)$ or $\zeta_i(s, \chi, m)$ is similar to that of $s = 1$ for ordinary zeta- and $L$-functions. But, it should be noted that $\varphi_i(s, \chi, m)$ is absolutely convergent only for $\operatorname{Re} s > 3/2$. To explain the nature of functions as $\zeta_i(s, \chi, m)$ satisfactorily, it might be necessary to discover some kind of modular forms, or theta series, and something like Mellin's transformation which supplies, for example, a nice integral representation of our functions.

3. The fact that $\varphi_i(s, \chi, m)$ is regular at $s = 3/2$ has some basic meanings for the distribution of arguments of Gauss sums. Namely, we can apply the property of the function to a problem such as so called Kummer's conjecture*) explained in [2]. Although the author did not try to obtain the final result yet, it is sure that we can attack the problem by means of $\varphi_i(s, \chi, m)$ in a similar, standard way as in the theory of prime numbers. For example, put $\tau(c) = \tau_{6,1}(4c)$, so especially

$$\tau(c) = (-i)^{t(c)} \sum_{d \bmod c} \left(\frac{d}{c}\right)^{-1} e^{\pi i t r d/c} \tag{19}$$

for $c \equiv 1 \pmod{\lambda^3}$. Then, by virtue of ordinary Tauberian theorems, the regularity of $\varphi_6(s, \chi, 1)$ at $s = 3/2$ yields

$$\sum_{Nc<Y} \frac{\tau(c)}{\sqrt{Nc}} = o(Y),$$

which is a necessary condition for the distribution of the arguments of Gauss sums to be uniform. A same kind of result can be obtained also for Gauss sums containing congruence characters of arbitrary orders defined in a number field involving correspondingly many roots of unity.

4. The behavior of $\zeta_i^*(s, \chi, m)$ at $s = 3/4$ has an important connection with the reciprocity law, because the factor $\dfrac{\pi}{s-1} \dfrac{\zeta(4s-4)}{\zeta(4s-3)}$ of $\Phi(s, \chi)$ becomes 0 at $s = 3/4$[8]. For example, if the functional equation (18) is satisfied by $\xi(s)$ such that $Z(s, \chi, m) = (1+2^{4-4s})^{-1}\zeta_i^*(s, \chi, m)$, and if $E^*(u, s, \chi)$ is the corresponding linear combination of $E_i(u, s, \chi)$, then the function

$$\Theta(u, \chi) = E^*\left(u, \frac{3}{4}, \chi\right) \tag{20}$$

has the property

$$\Theta(\sigma u, \chi) = \chi(\sigma)\Theta(u, \chi), \tag{21}$$

*) This problem itself was recently solved by A. I. Vinogradov, Izv. Akad. Nauk SSSR, Ser. Math., **31** (1967), 123–148. The author was told about that by G. Beyer after having prepared the manuscript.

8) This is the only real zero, and the only zero in $\operatorname{Re} s \leqq 1$ of the factor.

for every $\sigma \in \Gamma$, and has simple constant terms in the Fourier expansion at cusps. In fact, all such constant terms are constant multiples of $v^{\frac{3}{4}}$. On the other hand, if all what we have done in this paper are repeated using the quadratic residue symbol instead of biquadratic, then the function corresponding to $\zeta_i^*(s, \chi, m)$ becomes essentially Hecke's $L$-function with the congruence character determined by $F(\sqrt{m})/F$, and the latter has a simple pole at $s=3/2$ if and only if $m$ is a square. Therefore, the function corresponding to $\Theta$ in (20) becomes an ordinary theta function in this case. More precisely speaking, we get the function induced on $H$ by a theta function on Siegel's upper half space of degree two, under the embedding

$$\begin{pmatrix} z & -v \\ v & \bar{z} \end{pmatrix} \to \begin{pmatrix} x+iv, & -y \\ -y, & -x+iv \end{pmatrix}, \qquad (z=x+iy),$$

of $H$ into the Siegel space. A similar, related fact can be found in the equality between a classical theta function $\sum e^{\pi i m^2 z}$ and an Eisenstein series defined on the upper half plane[9]. Thus, the function $\Theta$ in (20) is regarded as a generalization of theta functions.

Because of the functional equation, the value concerning (20) of $\zeta_i^*(s, \chi, m)$ at $s=3/4$ is essentially the residue of $\zeta_i^*(s, \chi, m)$ at $s=5/4$ if 5/4 is a simple pole, and is 0 if 5/4 is no pole. The possibility for $\zeta_i^*(s, \chi, m)$ to have a multiple pole at 5/4 is excluded by the arguments in 2. of this §. In this way, we face to an important problem to investigate the regularity of our functions at $s=5/4$, and $s=5/4$ may well be compared with $s=1$ in the case of classical zeta- and $L$-functions. Since, however, the Dirichlet series which define our functions are absolutely convergent only for $\mathrm{Re}\, s > 3/2$, we must find some new way to get the behavior of the function at $s=5/4$. This seems rather difficult. In all the cases, concerning quadratic residue symbols, the both values like 3/2 and 5/4 above, the former being related to Gauss sums and the latter related to the reciprocity law, are the same.

Any way, if we could know more about $\zeta_i^*(s, \chi, m)$ at $s=3/4$, and could construct $\Theta(u, \chi)$ in such a form that (21) is directly to verify, then we have $\chi(\sigma)\chi(\sigma')=\chi(\sigma\sigma')$ for $\sigma, \sigma' \in \Gamma$, and can in turn prove the reciprocity law of the biquadratic residue symbol. This kind of knowledge would also supply a possibility to generalize the theory of unitary operators, as was done in [**9**], in such a direction where the topological covering groups constructed in [**4**] of some $p$-adic matric groups may play a similar role to the metaplectic group in [**9**].

5. At the very end, some miscellaneous facts should be mentioned. Our

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9) This is stated in [8] as a result of Maass. Maass' original work is in Abh. Math. Sem. Hamburg, 12 (1938), 133–162.

functions as $\zeta_i(s, \chi, m)$ has no Euler product. But, if $(c, c')=1$ and $c\equiv c'\equiv 1$ (mod $\lambda^3$), then the multiplicative relation

$$\tau(c)\tau(c')=\left(\frac{c}{c'}\right)\left(\frac{c'}{c}\right)\left(\frac{c, c'}{\lambda}\right)\tau(cc')$$

holds for quantities defined by (19), where $\left(\frac{c, c'}{\lambda}\right)$ is the norm residue symbol[10] of degree 4 in $\boldsymbol{Q}(\sqrt{-1})$. Therefore we can expect that there still exists an analogy of the Euler product, which may probably be made clear if we observe Hecke operators attached to a global covering group as mentioned in [**4**] of the adele group of $SL(2, F)$. The Gauss sums then will give rise to quantities concerning the representation of the group.

If $(m, c)=1$, $c\equiv 1$ (mod 4), then we have a well-known, elementary relation

$$\sum_{d \bmod c}\left(\frac{d}{c}\right)^{-1} e^{\pi itrmd/c}=\left(\frac{m}{c}\right)\sum_{d \bmod e}\left(\frac{d}{c}\right)^{-1} e^{\pi itrd/c},$$

which means that the coefficients of the Dirichlet series $\varphi_i(s, \chi, m)$ are obtained essentially by multiplying the congruence character $\chi_m(c)=\left(\frac{m}{c}\right)$ of $c$ to corresponding coefficients of $\varphi_i(s, \chi, 1)$. Hence, our Dirichlet series are similar in nature also to those Dirichlet series which were investigated in [**10**] in connection with modular forms.

By a similar argument as in 2. of this §, it is not hard to show that every $Z(s, \chi, m)$ is meromorphic in the whole $s$-plane, and possible singularities are simple poles at $s=5/4$ and 1. The distribution of the zeros of $Z(s, \chi, m)$ is, however, a little unnatural. From the functional equation (18), we see at once that $Z(s, \chi, m)$ has zeros at $s=1/2, 1/4$, and at $\frac{1}{4}(3-4n)$, $\frac{1}{4}(2-4n)$, $\frac{1}{4}(1-4n)$ for all integers $n\geqq 1$. At $s=1$, $Z(s, \chi, m)$ may have a pole, and 3/4 is a zero if and only if $s=5/4$ is not a pole. It is impossible that $Z\left(\frac{3}{4}, \chi, m\right)=0$ for all $m$ and $i$, because $E^*\left(u, \frac{3}{4}, \chi\right)$ as in (20) then becomes a function like $v^{\frac{3}{4}}$, and cannot be an automorphic function. By a similar reason, it cannot happen that $Z(-n, \chi, m)$ with a given integer $n\geqq 0$ are 0 for all $m$ and $i$. Thus, for a reasonable distribution of zeros, it may be preferable to multiply $Z(s, \chi, m)$ by a suitable function with zeros at $1, 0, -1, -2, \cdots$.

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10) See [**1**].

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