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A mean value theorem in adele geometry

Dedicated to Professor S. Iyanaga for his 60th birthday

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In this paper, we point out that one can test the validity of the mean value theorem ([**10**], [**11**]) for the adele transformation spaces attached to certain algebraic transformation spaces defined over the rationals by looking at the first two homotopy groups of the underlying complex manifolds.

Notation and conventions: As usual $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ are the integers, the rational numbers, the real numbers, the complex numbers, respectively. Further, we shall frequently use the following notation

$\boldsymbol{Q}_v$: the completion of $\boldsymbol{Q}$ at a valuation v of $\boldsymbol{Q}$.
$\boldsymbol{Z}_p$: the p-adic integers in $\boldsymbol{Q}_p$.
$\boldsymbol{F}_p$: the finite field with p elements.
S: any finite set of valuations of $\boldsymbol{Q}$ including $v=\infty$.
Ω: a universal domain.
G_m: the multiplicative group of Ω.
K^*: the multiplicative group of a field K.
$\bar{K}$: the algebraic closure of a field K.
$[E]$: the cardinality of a set E.

If O denotes a set of geometric objects, we shall denote by O_K the subset of O which is rational over a field K. For a topological space X, we shall denote by $L(X)$ the set of all $\boldsymbol{R}$-valued continuous functions on X with compact support. We shall use freely terminology and results in the first two chapters of [**12**].

§ 1. Three properties of a variety.

Let X be an algebraic variety defined over $\boldsymbol{Q}$. There is a finite set S such that for $p \notin S$ the variety $X^{(p)}$ over $\boldsymbol{F}_p$ obtained by the reduction modulo p has at least one rational point over $\boldsymbol{F}_p$ ([7]). We shall say that X is of type (C_1) if it has the following property:

(C_1) *the product* $\prod_{p \notin S}([X^{(p)}_{F_p}]/p^{\dim X})$ *is absolutely convergent.*

Clearly, this property is independent of the choice of S.

The set $\mathrm{Mor}(X, G_m)$ of morphisms of X in G_m forms a module under the addition: $(f+g)(x)=f(x)g(x)$. This module contains the constants as a submodule. Put $M(X)=\mathrm{Mor}(X, G_m)/G_m$. We shall say that X is of type (C_2) if it has the following property:

(C_2) *the module* $M(X)=0$.

Since the set $X_{\boldsymbol{C}}$ of complex points of X is an arcwise connected Hausdorff space, the homotopy groups $\pi_i(X_{\boldsymbol{C}})$, $i\geqq 1$, make sense. We shall say that X is of type (C_3) if it has the following property:

(C_3) *the fundamental group* $\pi_1(X_{\boldsymbol{C}})$ *is finite.*

In this paper, we shall mainly consider the case where X is a connected linear algebraic group or a homogeneous space for such a group. However, we give here an example of X which is, in general, not a homogeneous space and has all the properties: namely, let X be the hypersurface in the affine $(r+1)$-space defined by the polynomial $F(X)=\sum_{i=0}^{r} a_i X_i^q - b=0$, $a_i\neq 0$, $b\neq 0$ in $\boldsymbol{Q}$. Then, we see that X has all three properties, provided $r\geqq 3$. Actually, $X_{\boldsymbol{C}}$ is simply connected.

§ 2. Special algebraic group.

Let G be a connected linear algebraic group defined over $\boldsymbol{Q}$. We denote by $\hat{G}$ the module of rational characters of $G: \hat{G}=\mathrm{Hom}(G, G_m)$. This is a submodule of $\mathrm{Mor}(G, G_m)$. Clearly, $\hat{G}\cap G_m=0$. On the other hand, by a theorem of Rosenlicht ([**12**, Theorem 2.2.2]), we have $\mathrm{Mor}(G, G_m)=\hat{G}+G_m$, and hence an isomorphism:

$$M(G)\cong \hat{G}\,. \tag{1}$$

We shall show that for a connected linear algebraic group G the three properties (C_i), $i=1, 2, 3$, are equivalent each other.

We first recall the Levi-Chevalley decomposition over $\boldsymbol{Q}$:

$$G=UTS\,, \tag{2}$$

where U is the unipotent radical of G, $R=UT$ is the radical of G, $A=TS$ is reductive, T is a torus defined over $\boldsymbol{Q}$ which is the identity component of the center of A, S is a semi-simple group which is the derived group of A, $G=UA$ is a semi-direct product with U normal and A is isogenous to $T\times S$. If $G=UT'S'=UA'$ is another decomposition of the same type, A' is conjugate to A by an element of $U_{\boldsymbol{Q}}$ and so T, S are uniquely determined up to isomorphisms over $\boldsymbol{Q}$. ([**4**, 0.8]).

We now prove the implications: $(C_2)\Rightarrow(T=1)\Rightarrow(C_3)\Rightarrow(C_2)$. Assume that $\hat{G}=0$. Since A is a factor group of G, $\hat{A}$ is a subgroup of $\hat{G}=0$, hence $\hat{A}=0$.

Let f be the isogeny $T\times S\rightarrow A$ given by $(t, s)\rightarrow ts$. Then $\hat{f}$ has the finite cokernel, and hence $\hat{T}\times\hat{S}=\hat{T}$ is finite. But, $\hat{T}$ is $\boldsymbol{Z}$-free, which implies that $T=1$. Next, assume that $T=1$, i.e. $G=US$. Since $U_{\boldsymbol{C}}$ is homeomorphic to a complex vector space, we have $\pi_1(G_{\boldsymbol{C}})=\pi_1(S_{\boldsymbol{C}})$, where the latter group is finite by Weyl's theorem. Lastly, assume that $\hat{G}\neq 0$. Take a nontrivial character $\chi\in\hat{G}$. Let $(\operatorname{Ker}\chi)_0$ be the (algebraic and topological) identity component of $\operatorname{Ker}\chi$. Since $G_{\boldsymbol{C}}/(\operatorname{Ker}\chi)_0$ is a finite covering of $G_{\boldsymbol{C}}/\operatorname{Ker}\chi=\boldsymbol{C}^*$, the fundamental group of $G_{\boldsymbol{C}}/(\operatorname{Ker}\chi)_0$ is infinite. Passing to the homotopy sequence of the exact sequence

$$0\rightarrow(\operatorname{Ker}\chi)_0\rightarrow G_{\boldsymbol{C}}\rightarrow G_{\boldsymbol{C}}/(\operatorname{Ker}\chi)_0\rightarrow 0$$

we get the exact sequence $\rightarrow\pi_1(G_{\boldsymbol{C}})\rightarrow\pi_1(G_{\boldsymbol{C}}/(\operatorname{Ker}\chi)_0)\rightarrow 0$, which shows that $\pi_1(G_{\boldsymbol{C}})$ is infinite, and our implications are completed, in view of (1).

Next, we prove the equivalence: $(C_1)\Leftrightarrow(T=1)$. Choose S so that for $p\notin S$ we have $G^{(p)}=U^{(p)}T^{(p)}S^{(p)}$ where $U^{(p)}$, $T^{(p)}$, $S^{(p)}$ remain connected unipotent, torus, connected semi-simple, respectively. Denote by $\mu_p(X)$ the factor belonging to p in the infinite product introduced at the beginning of § 1. Since the number of points rational over a finite field is unchanged under the isogeny, we have

$$\mu_p(G)=\mu_p(U)\mu_p(T)\mu_p(S)\qquad(p\notin S).$$

In order to prove the proposed equivalence, it is enough to show that U and S have the property (C_1) but $T(\neq 1)$ has not that property. As for U, the matter is trivial, because U is isomorphic to an affine space as a variety. As for S, the assertion is a consequence of the vanishing of the first and second Betti numbers of the maximal compact subgroup of $S_{\boldsymbol{C}}$ and the formula due to Chevalley and Steinberg on the number $[S^{(p)}_{F_p}]$. Finally, for T, the expression of $[T^{(p)}_{F_p}]$ in terms of the Galois module structure of $\hat{T}$ shows that $T\,(\neq 1)$ has not the property (C_1). (For more details, see [8, Appendix II].) Summarizing these arguments, we get

THEOREM 2.1. *Let G be a connected linear algebraic group defined over $\boldsymbol{Q}$. Let $G=UTS$ be a Levi-Chevalley decomposition over $\boldsymbol{Q}$ given by* (2). *Then, the four properties* (C_1), (C_2), (C_3) *and* $(T=1)$ *are equivalent each other.*

DEFINITION 2.2. A connected linear algebraic group G defined over $\boldsymbol{Q}$ will be called a *special* algebraic group if it has any one of the four equivalent properties in Theorem 2.1.

Thus, G is special if and only if it is a semi-direct product of a normal unipotent subgroup and a connected semi-simple subgroup, defined over $\boldsymbol{Q}$.

§ 3. Special homogeneous space.

A pair (G, X) of a connected algebraic group G and an algebraic variety X is called a homogeneous space defined over $\boldsymbol{Q}$, if G acts transitively on X and $\boldsymbol{Q}$ is the field of definition for G, X and the action. As usual, the action will be written as $(g, x) \to gx$, $g \in G$, $x \in X$. For an $x \in X$, denote by G_x the isotropy group of x. If $x, x' \in X$ are related by $x' = gx$, $g \in G$, we have $G_{x'} = gG_xg^{-1}$; hence, over Ω, algebraic (and topological when $\Omega = \boldsymbol{C}$) properties of G_x is independent of the choice of $x \in X$. When we fix an $x \in X$, we denote by ι, κ the injection $G_x \to G$ and the projection $G \to X$ given by $\iota(g) = g$, $g \in G_x$, $\kappa(g) = gx$, $g \in G$. Thus, we get the exact sequence

$$0 \to G_x \to G \to X \to 0\,, \tag{3}$$

where the point x is distinguished in X. If G_x is connected, (3) induces the exact sequence

$$0 \to \pi_2(X_{\boldsymbol{C}}) \to \pi_1(G_{x,\boldsymbol{C}}) \to \pi_1(G_{\boldsymbol{C}}) \to \pi_1(X_{\boldsymbol{C}}) \to 0\,, \tag{4}$$

because π_2 (connected Lie group) $= 0$.

If, in particular, $X = G$, G acts on itself by the left multiplication and $(G. G)$ can be identified with G. In the following, all definitions for (G, X) will be consistent with this identification.

DEFINITION 3.1. A homogeneous space (G, X) defined over $\boldsymbol{Q}$ will be called a *special* homogeneous space if it satisfies the following conditions:

(i) $X_{\boldsymbol{Q}}$ is non-empty,

(ii) G and G_ξ, for some $\xi \in X_{\boldsymbol{Q}}$, are both special algebraic groups.

From the above remark, (ii) is independent of the choice of $\xi \in X_{\boldsymbol{Q}}$. For $\xi, \eta \in X_{\boldsymbol{Q}}$, G_ξ and G_η are $\boldsymbol{Q}$-forms of each other. In case $X = G$, this definition is identical with Definition 2.2, because $e \in G_{\boldsymbol{Q}}$ and $G_e = \{e\}$.

THEOREM 3.2. *Let (G, X) be a special homogeneous space defined over $\boldsymbol{Q}$. Then the variety X has all three properties (C_i), $i = 1, 2, 3$.*

PROOF. Taking the reduction modulo p of (3), for $x = \xi \in X_{\boldsymbol{Q}}$, we get the exact sequence

$$0 \to G_\xi^{(p)} \to G^{(p)} \to X^{(p)} \to 0\,,$$

for almost all p. Since $G_\xi^{(p)}$ is connected for almost all p, we get, by Lang's theorem, the exact sequence

$$0 \to G_{\xi,\boldsymbol{F}_p}^{(p)} \to G_{\boldsymbol{F}_p}^{(p)} \to X_{\boldsymbol{F}_p}^{(p)} \to 0\,,$$

and hence $[G_{\xi,\boldsymbol{F}_p}^{(p)}][X_{\boldsymbol{F}_p}^{(p)}] = [G_{\boldsymbol{F}_p}^{(p)}]$ for $p \notin S$, for sufficient large S. Then, the absolute convergence of the product

$$\prod_{p \notin S} ([X_{\boldsymbol{F}_p}^{(p)}]/p^{\dim X})$$

follows from the absolute convergence of those products for G_ξ and G and the relation $\dim G-\dim G_\xi=\dim X$, which proves (C_1) for X. Next, since $\kappa: G\to X$ is surjective, the induced map $\mathrm{Mor}(X, G_m)\to\mathrm{Mor}(G, G_m)$ is injective; but $\mathrm{Mor}(G, G_m)=G_m$ and hence $\mathrm{Mor}(X, G_m)=G_m$, which proves (C_2) for X. Finally, since $\pi_1(G_{\boldsymbol{C}})$ and $\pi_1(G_{\xi,\boldsymbol{C}})$ are finite, we see from (4) that $\pi_1(X_{\boldsymbol{C}})$ and $\pi_2(X_{\boldsymbol{C}})$ are also finite, which proves (C_3) for X.

Here are three typical examples of special homogeneous spaces.

(Ex I) $G=\mathrm{SL}(n)$, $X=\Omega^n-\{0\}$, $n\geqq 2$. G acts on X as linear transformations. For $\xi=\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}\in X_{\boldsymbol{Q}}$,

$$G_\xi=\left(\begin{array}{c|c}1 & \Omega^{n-1}\\\hline 0 & \mathrm{SL}(n-1)\end{array}\right)=US,\qquad U=\left(\begin{array}{c|c}1 & \Omega^{n-1}\\\hline 0 & 1\end{array}\right),\qquad S=\left(\begin{array}{c|c}1 & 0\\\hline 0 & \mathrm{SL}(n-1)\end{array}\right),$$

a semi-direct product with U normal. Since $\pi_1(G_{\boldsymbol{C}})=\pi_1(G_{\xi,\boldsymbol{C}})=0$, we get $\pi_1(X_{\boldsymbol{C}})=\pi_2(X_{\boldsymbol{C}})=0$ by (4).

(Ex II) Let s, t be non-singular symmetric matrices in $\boldsymbol{Q}$ of degree m, n, $m-n\geqq 3$.

$$G=0^+(s)=\{g\in\mathrm{SL}(m),\ {}^tgsg=s\},$$

$$X=\{x\in\Omega_{m,n},\ {}^txsx=t\},$$

where $\Omega_{m,n}$ denotes the set of all (m, n)-matrices over Ω. The action of G on X is the matrix multiplication. We assume that $X_{\boldsymbol{Q}}$ is non-empty so that we can take a $\xi\in X_{\boldsymbol{Q}}$. Then, G_ξ is isomorphic with $0^+(u)$, where u is a suitable non-singular symmetric matrix in $\boldsymbol{Q}$ of degree $m-n$; hence G_ξ is semi-simple and (G, X) is special. Over $\boldsymbol{C}$, we have

$$G_{\boldsymbol{C}}\approx 0^+(m,\boldsymbol{C})=0^+(m,\boldsymbol{R})\times R^\mu,\qquad \mu=m(m-1)/2,$$

$$G_{\xi,\boldsymbol{C}}\approx 0^+(m-n,\boldsymbol{C})=0^+(m-n,\boldsymbol{R})\times\boldsymbol{R}^\nu,\qquad \nu=(m-n)(m-n-1)/2.$$

By a result of Mostow, we know the existence of the exact sequence

$$0\to\boldsymbol{R}^{\mu-\nu}\to X_{\boldsymbol{C}}\to V_{m,n}=0^+(m,\boldsymbol{R})/0^+(m-n,\boldsymbol{R})\to 0 \tag{5}$$

(cf. [**1**, p. 424]). Since $m-n\geqq 3$, $\pi_1(V_{m,n})=\pi_2(V_{m,n})=0$. Hence, taking the homotopy sequence of (5), we get $\pi_1(X_{\boldsymbol{C}})=\pi_2(X_{\boldsymbol{C}})=0$.

(Ex III) $G=\mathrm{SL}(n)$, $X=\{x\in\Omega_n,\ {}^tx=x,\ \det x=1\}$, $n\geqq 3$, where Ω_n denotes the set of all square matrices of degree n. The action is given by $(g, x)\to gx{}^tg$. If we take $\xi=1_n\in X$, we have $G_\xi=0^+(n)$. Since $\pi_1(0^+(n))=\boldsymbol{Z}/(2)$, we get, from (4), $\pi_1(X_{\boldsymbol{C}})=0$ and $\pi_2(X_{\boldsymbol{C}})=\boldsymbol{Z}/(2)$.

§4. Adelization.

Let X be an algebraic variety defined over $\boldsymbol{Q}$. For each valuation v ($=\infty$ or p) of $\boldsymbol{Q}$, we get a locally compact space X_v consisting of points of X rational over the completion $\boldsymbol{Q}_v$. If $v=p$, X_p contains a compact set $X_{\boldsymbol{Z}_p}$. For an S, put $X_S=\prod_{v\in S} X_v \times \prod_{p\notin S} X_{\boldsymbol{Z}_p}$. The adele space X_A is the union $\bigcup_S X_S$ together with its inductive limit topology with respect to S, and is locally compact. We identify $X_{\boldsymbol{Q}}$ as a subset of X_A by the diagonal imbedding. If X is quasi-affine, i.e. if X is $\boldsymbol{Q}$-isomorphic with a $\boldsymbol{Q}$-open subset of an affine variety defined over $\boldsymbol{Q}$, $X_{\boldsymbol{Q}}$ is discrete in X_A. (As for more details on adele spaces, see [**12**, Ch. I]).

Now, suppose that X is non-singular and admits a gauge form ω defined over $\boldsymbol{Q}$, i.e. an everywhere holomorphic never zero differential form of highest degree defined over $\boldsymbol{Q}$. For each v, ω induces a measure ω_v on X_v and we have, for some S,

$$\int_{X_{\boldsymbol{Z}_p}} \omega_p = [X^{(p)}_{\boldsymbol{F}_p}]/p^{\dim X}, \qquad p \notin S .$$

If X has the the property (C_1), the formal product $\prod_v \omega_v$ well-defines a measure on X_A. If, in addition, X has the property (C_2), that measure on X_A is the unique one as is seen by the product formula in $\boldsymbol{Q}$. We shall call this measure the canonical measure on X_A and denote by dX_A. (As for more details on measures on X_A, see [**12**, Ch. II].)

THEOREM 4.1. *Let (G, X) be a special homogeneous space defined over $\boldsymbol{Q}$. Then, the variety X is quasi-affine; hence $X_{\boldsymbol{Q}}$ is discrete in X_A. Furthermore, X admits the canonical measure dX_A which is independent of G and its action on X which make X into a homogeneous space.*

PROOF. Since $\hat{G}_\xi=0$, $\xi\in X_{\boldsymbol{Q}}$, the first assertion is a consequence of [**9**, Theorem 3]. Secondly, since G and G_ξ, $\xi\in X_{\boldsymbol{Q}}$, have the property (C_2), they are unimodular and hence X admits a G-invariant gauge form ω defined over $\boldsymbol{Q}$. In view of the properties (C_1), (C_2) of X, the measure $\omega_A=\prod_v \omega_v$ must be the dX_A, q.e.d.

§5. Orbits of G_A in X_A.

Let (G, X) be a special homogeneous space defined over $\boldsymbol{Q}$. For a fixed $\xi\in X_{\boldsymbol{Q}}$, the exact sequence

$$0 \longrightarrow G_\xi \xrightarrow{\iota} G \xrightarrow{\kappa} X \longrightarrow 0 \tag{6}$$

induces the exact sequence

$$(6)_A \qquad 0 \longrightarrow G_{\xi,A} \xrightarrow{\iota_A} G_A \xrightarrow{\kappa_A} X_A .$$

Although κ_A is not necessarily surjective, we have the following

LEMMA 5.1. *κ_A is open.*

PROOF. The statement is equivalent to say that $\kappa_v : G_v \to X_v$ is open for all v and $\kappa_{Z_p} : G_{Z_p} \to X_{Z_p}$ is surjective for almost all p, where the former is a standard property of v-adic transformation groups and the latter is a special case of a well-known fact [3, Remarque, p. 161].

LEMMA 5.2. *For any $x \in X_A$, the orbit $G_A x$ is open in X_A.*

In fact, since $G_A x = \bigcup_S G_S x$, it is enough to show that $G_S x$ is open in X_A for a sufficiently large S, or equivalently that $G_v x_v$ is open in X_v for $v \in S$ and $G_{Z_p} x_p = X_{Z_p}$ for $p \notin S$, where $x = (x_v)$. Now, the former is, again, a standard property of v-adic transformation groups. As for the latter, take S large so that $x_p \in X_{Z_p}$ for $p \notin S$ and $G_{Z_p}\xi = X_{Z_p}$ for $p \notin S$ (cf. Lemm 5). Then, one takes a $g_p \in G_{Z_p}$ such that $x_p = g_p \xi$ and one has $G_{Z_p} x_p = G_{Z_p} \xi = X_{Z_p}$, for $p \notin S$, q. e. d.

One can generalize Lemma 5.2 as follows.

LEMMA 5.3. *For any subset $E \subset X_A$, $G_A E$ is open and closed in X_A.*

In fact, since $G_A E = \bigcup_{x \in E} G_A x$, it is open by Lemma 5.2. On the other hand, $X_A = G_A E + (X_A - G_A E)$ is a disjoint sum of G_A-stable subsets, hence $X_A = G_A E + G_A E'$, with $E' = X_A - G_A E$. Since $G_A E'$ is open, $G_A E$ is closed, q. e. d.

Later on, we shall be interested in the case where $E = X_{\boldsymbol{Q}}$. Here, we give a sufficient condition in order that $G_A X_{\boldsymbol{Q}} = X_A$.

DEFINITION 5.4. A special homogeneous space (G, X) over $\boldsymbol{Q}$ is said to be of type (W) if, for any field K containing $\boldsymbol{Q}$, the sequence

$$0 \longrightarrow G_{\xi,K} \xrightarrow{\iota_K} G_K \xrightarrow{\kappa_K} X_K \longrightarrow 0 , \qquad \xi \in X_{\boldsymbol{Q}},$$

is exact. (Obviously, this property is independent of the choice of ξ.)

Spaces in (Ex I), (Ex II) are of type (W); the matter is trivial for (Ex I), and the property (W) for (Ex II) is nothing but the Witt theorem.

LEMMA 5.5. *If (G, X) is of type (W), then κ_A is surjective, i. e. $G_A \xi = X_A$, for any $\xi \in X_{\boldsymbol{Q}}$, and, a fortiori, $G_A X_{\boldsymbol{Q}} = X_A$.*

In fact, take any $x = (x_v) \in X_A$. By Lemma 5.1 and by the definition of the adele, there is an S such that, for $p \notin S$, we have $x_p = g_p \xi$ for some $g_p \in G_{Z_p}$. On the other hand, by the condition (W), there is a $g_v \in G_v$ such that $x_v = g_v \xi$, for $v \in S$. Putting $g = (g_v)$, we get $x = g\xi$, which proves our assertion.

REMARK 5.6. We shall show that $G_A X_{\boldsymbol{Q}} \neq X_A$ for (G, X) in (Ex III). In view of Lemma 5.5, this implies that the space in (Ex III) is not of type (W).

We first recall briefly the Hilbert symbol and the Hasse-Witt symbol for quadratic forms (for details, see [5]). For $a, b \in \boldsymbol{Q}_v^*$, the Hilbert symbol $(a, b)_v$ takes values ± 1; it is 1 if and only if the equation $ax^2+by^2=1$ has a solution in $\boldsymbol{Q}_v$. For a quadratic from $q(x)=a_1x_1^2+\cdots+a_nx_n^2$, $a_i \in \boldsymbol{Q}_v^*$, the Hasse-Witt symbol is defined by

$$c(q)_v=(a_1, d_1)_v \cdots (a_n, d_n)_v, \ d_i=a_1\cdots a_i, \qquad 1 \leqq i \leqq n .$$

For any quadratic form $q(x)$ over $\boldsymbol{Q}_v$, $c(q)_v$ is well-defined by using its diagonal equivalent. If $q(x)$ and $q'(x)$ are equivalent, then $c(q)_v=c(q')_v$. For a quadratic form $q(x)$ over $\boldsymbol{Q}$, $c(q)_v=1$ for almost all v and we have

$$\prod_v c(q)_v = 1 . \tag{7}$$

Now, if $G_A X_{\boldsymbol{Q}}=X_A$ for (Ex III), then, for $x=(x_v) \in X_A$ defined by

$$x_\infty = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \\ & & & & -1 \end{pmatrix}, \qquad x_p = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \qquad \text{for all } p,$$

there must be a $\xi \in X_{\boldsymbol{Q}}$ such that $x=g\xi {}^t g$ with some $g \in G_A$. Identifying a quadratic form with the corresponding symmetric matrix, the above argument shows that ξ is equivalent to x_v for all v; hence $c(\xi)_v=c(x_v)_v$ for all v. Now, we have $c(x_\infty)_\infty=-1$ and $c(x_p)_p=1$ for all p, hence $\prod_v c(x_v)_v=-1$, which contradicts the product formula (7) applied for $q=\xi$. Therefore we must have $G_A X_{\boldsymbol{Q}} \neq X_A$.

§ 6. Global and local classes in $X_{\boldsymbol{Q}}$.

We are going to define two equivalence relations in the discrete set $X_{\boldsymbol{Q}}$ of X_A. Namely, let ξ, η be points of $X_{\boldsymbol{Q}}$. We shall say that ξ, η are *globally equivalent*, written $\xi \underset{\text{glob}}{\sim} \eta$, if we have $\eta=g\xi$ for some $g \in G_{\boldsymbol{Q}}$. Hence the global class containing ξ is the orbit $G_{\boldsymbol{Q}}\xi$. On the other hand, we shall say that ξ, η are *locally equivalent*, written $\xi \underset{\text{loc}}{\sim} \eta$, if we have $\eta=g_A\xi$ for some $g_A \in G_A$. Hence the local class containing ξ is $G_A\xi \cap X_{\boldsymbol{Q}}$.

LEMMA 6.1. *$\xi \underset{\text{loc}}{\sim} \eta \Longleftrightarrow \eta=g_v\xi$, $g_v \in G_v$, for all v.*

In fact, ($\Rightarrow$) is trivial. As for ($\Leftarrow$), choose S such that $\eta \in X_{\boldsymbol{Z}_p}$, $G_{\boldsymbol{Z}_p}\xi=X_{\boldsymbol{Z}_p}$ for $p \notin S$ (cf. Lemma 5.1). Then, for $p \notin S$, there is a $g'_p \in G_{\boldsymbol{Z}_p}$ such that $\eta=g'_p\xi$. For $v \in S$, put $g'_v=g_v$. Then, $g_A=(g'_v) \in G_A$ satisfies the condition $\eta=g_A\xi$ of local equivalence, q. e. d.

Since $G_{\boldsymbol{Q}} \subset G_A$, globally equivalent points are necessarily locally equivalent and a local class consists of a certain number of global classes. By the following Lemma, this number is always finite. For a local class Θ, we denote by $h(\Theta)$ the number of global classes in Θ. Denoting by ξ_i, $1 \leqq i \leqq h(\Theta)$ representatives of global classes in Θ, we have the relations:

$$(8) \qquad G_A\Theta = G_A\xi_i, \qquad \Theta = \bigcup_{i=1}^{h(\Theta)} G_{\boldsymbol{Q}}\xi_i \text{ (disjoint)} = G_A\xi_i \cap X_{\boldsymbol{Q}},$$

$$G_A X_{\boldsymbol{Q}} = \bigcup_{\Theta} G_A\Theta \qquad \text{(disjoint)}.$$

Let A be an algebraic group defined over $\boldsymbol{Q}$ or a Galois module over $\boldsymbol{Q}$. Using the standard notation in Galois cohomology, we put

$$h^p(A) = [H^p(\boldsymbol{Q}, A)], \qquad p \geqq 0,$$

$$I^p(A) = \operatorname{Ker}(H^p(\boldsymbol{Q}, A) \to \prod_v H^p(\boldsymbol{Q}_v, A)),$$

$$i^p(A) = [I^p(A)].$$

It is fundamental that $i^1(G)$ is finite if G is a linear algebraic group [**3**, 7.1, Théorème].

LEMMA 6.2. $h(\Theta) \leqq i^1(G_\xi)$, $\xi \in \Theta$.

PROOF. It is enough to show that there is an injection φ from the quotient $\Theta/_{\widetilde{\text{glob}}}$ into $I^1(G_\xi)$. Now, for any $\eta \in \Theta = G_A\xi \cap X_{\boldsymbol{Q}}$, there is a $g \in G_{\overline{\boldsymbol{Q}}}$ such that $\eta = g\xi$. Since ξ, η are invariant under the Galois group $\mathfrak{g} = \mathfrak{g}(\overline{\boldsymbol{Q}}/\boldsymbol{Q})$, $\eta_\sigma = g^{-1}g^\sigma$, $\sigma \in \mathfrak{g}$, define a cocycle $c(\eta)$ of $\mathfrak{g}$ in $G_{\xi,\overline{\boldsymbol{Q}}}$. Since $\eta = g_v\xi$, $g_v \in G_v$, for all v, the element $g_v^{-1}g = h_v$ is in $G_{\xi,\overline{\boldsymbol{Q}}_v}$, and hence, for $\sigma \in \mathfrak{g}_v = \mathfrak{g}(\overline{\boldsymbol{Q}}_v/\boldsymbol{Q}_v)$, one has $\eta_\sigma = g^{-1}g^\sigma = h_v^{-1}g_v^{-1}g_v^\sigma h_v^\sigma = h_v^{-1}h_v^\sigma$, which shows that the cocycle $c(\eta)$ is trivial locally everywhere. If $\eta' \underset{\text{glob}}{\sim} \eta$, then it is easy to see that $c(\eta) \sim c(\eta')$ (cohomologous); therefore we obtain a well-defined map φ: $\Theta/_{\widetilde{\text{glob}}} \to I^1(G_\xi)$ induced by $\eta \to c(\eta)$. It remains to show that φ is injective. In fact, suppose that $c(\eta) \sim c(\eta')$, i.e. $\eta'_\sigma = h^{-1}\eta_\sigma h^\sigma$ for some $h \in G_{\xi,\overline{\boldsymbol{Q}}}$. Putting $\eta = g\xi$, $\eta' = g'\xi$, $g, g' \in G_{\overline{\boldsymbol{Q}}}$, we get $g'h^{-1}g^{-1} = g'^\sigma h^{-\sigma}g^{-\sigma}$ for all $\sigma \in \mathfrak{g}$ and hence $g' = kgh$, $k \in G_{\boldsymbol{Q}}$, i.e. $\eta' \underset{\text{glob}}{\sim} \eta$, q.e.d.

DEFINITION 6.3. A special homogeneous space (G, X) defined over $\boldsymbol{Q}$ is said to be of type (H) if $h(\Theta) = 1$ for all Θ.

LEMMA 6.4. *If (G, X) is of type* (W), *then it is of type (H).*

In fact, since $X_{\boldsymbol{Q}} = G_{\boldsymbol{Q}}\xi$ for any $\xi \in X_{\boldsymbol{Q}}$, we have $G_A\xi \cap X_{\boldsymbol{Q}} = G_{\boldsymbol{Q}}\xi$ for all $\xi \in X_{\boldsymbol{Q}}$.

REMARK 6.5. The spaces in (Ex I), (Ex II) being of type (W), they are of type (H). The space in (Ex III), which is not of type (W) by Remark 5.6, is also of type (H); this is an immediate consequence of the well-known "Hasse principle" for the equivalence of quadratic forms over $\boldsymbol{Q}$. In view of Lemma 6.2, if $i^1(G_\xi) = 1$ for any $\xi \in X_{\boldsymbol{Q}}$, then (G, X) is of type (H); this is the case if G_ξ is simply connected and has no simple factor of type E_8 ([**6**]).

When we view a special algebraic group G as homogeneous space by the identification $G=(G, G)$, G is clearly of type (W), and hence of type (H).

§7. Tamagawa number of special algebraic groups.

Let G be a special algebraic group defined over $\boldsymbol{Q}$. Let dG_A be the canonical measure of the adele group G_A. Since $\hat{G}=0$, we have $(\hat{G})_{\boldsymbol{Q}}=0$ and hence, by a result of Borel and Harish-Chandra [2, 9.4, Theorem] the Tamagawa number

$$\tau(G)=\int_{G_A/G_{\boldsymbol{Q}}} dG_A$$

is well-defined. A. Weil has conjectured that

$$(\sharp) \qquad \pi_1(G_{\boldsymbol{C}})=0 \Rightarrow \tau(G)=1\,.$$

This has been proved for a large part of classical groups (Tamagawa, Weil), for some exceptional groups (Demazure, Mars) and for Chevalley groups (Langlands), but is not yet completely solved.

On the other hand, the author has determined $\tau(G)$ modulo $(\sharp)$, the relative theory, as an application of his determination of the Tamagawa number of algebraic tori. Since we need the result later on, we recall it here briefly. First of all, we have to introduce a Galois module structure on $\pi_1(G_{\boldsymbol{C}})$. As is well-known, there is a simply connected algebraic group $\tilde{G}$ defined over $\boldsymbol{Q}$ and an isogeny $f: \tilde{G} \to G$ defined over $\boldsymbol{Q}$, unique up to $\boldsymbol{Q}$-isomorphisms in the sense of algebraic coverings. (In [8] only semi-simple case is treated, but one can extend it to the case of special algebraic groups without difficulty.) Since $\operatorname{Ker} f$ is a $\boldsymbol{Q}$-closed finite commutative subgroup of $\tilde{G}$, it is acted by the Galois group $\mathfrak{g}=\mathfrak{g}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$. In view of the isomorphism

$$\operatorname{Ker} f \approx \pi_1(G_{\boldsymbol{C}})$$

obtained by $\operatorname{Ker} f \ni z \to f$ (a path joining z the idetity of $\tilde{G}$), we can transfer the $\mathfrak{g}$-module structure of $\operatorname{Ker} f$ to $\pi_1(G_{\boldsymbol{C}})$. From now on, we regard $\pi_1(G_{\boldsymbol{C}})$ as a $\mathfrak{g}$-module in this manner, which is unique up to $\mathfrak{g}$-isomorphisms. In this situation, the relative theory tells that the ratio $\tau(G)/\tau(\tilde{G})$ depends only upon the $\mathfrak{g}$-module $\pi_1(G_{\boldsymbol{C}})$. More precisely, we have

$$(9) \qquad \tau(G)/\tau(\tilde{G})=h^0(\widehat{\pi_1(G_{\boldsymbol{C}})})/i^1(\widehat{\pi_1(G_{\boldsymbol{C}})})\,,$$

where $\hat{M}=\operatorname{Hom}(M, (\bar{\boldsymbol{Q}})^*)$, the character module. From this we can deduce, for example, the following

LEMMA 7.1. *Let (G, X) be a special homogeneous space defined over $\boldsymbol{Q}$ and let $\tilde{G}$, $\tilde{G}_\xi$ be the universal covering groups (over $\boldsymbol{Q}$) of G, G_ξ, $\xi \in X_{\boldsymbol{Q}}$, respec-*

tively. If $\pi_1(X_C)=\pi_2(X_C)=0$, *then one has* $\tau(G)/\tau(\tilde{G})=\tau(G_\xi)/\tau(G_\xi)$.

In fact, since $\pi_1(X_C)=\pi_2(X_C)=0$, we get an isomorphism $\alpha:\pi_1(G_{\xi,C})\approx \pi_1(G_C)$ by the exact sequence (4). Now, one checks without difficulty that the map α is compatible with the $\mathfrak{g}$-module structures of $\pi_1(G_{\xi,C})$ and $\pi_1(G_C)$, i.e. a $\mathfrak{g}$-isomorphism. Our assertion then follows from (9), q.e.d.

§8. Tamagawa number of uniform special homogeneous spaces.

Let (G, X) be a special homogeneous space defined over $\boldsymbol{Q}$ and let dG_A, dX_A be the canonical measures on G_A, X_A, respectively (cf. Theorem 4.1). Since $G_AX_{\boldsymbol{Q}}$ is open and closed in X_A (Lemma 5.3), dX_A induces on this subset a measure which we denote again by dX_A. For a function $f\in L(G_AX_{\boldsymbol{Q}})$ and $g\in G_A$, the summation $\sum_{\xi\in X_{\boldsymbol{Q}}} f(g\xi)$ is actually a finite sum, because the support of f is compact and $X_{\boldsymbol{Q}}$ is discrete in X_A, and hence in $G_AX_{\boldsymbol{Q}}$. Furthermore, the summation viewed as a function on G_A can be regarded as a function on $G_A/G_{\boldsymbol{Q}}$ since it is invariant under the right multiplication by the elements of $G_{\boldsymbol{Q}}$. Thus the quantity

$$\tau(G)^{-1}\int_{G_A/G_{\boldsymbol{Q}}}\Big(\sum_{\xi\in X_{\boldsymbol{Q}}} f(g\xi)\Big)dG_A$$

represents the "mean value" of that function on $G_A/G_{\boldsymbol{Q}}$. We shall be interested in the connection of this mean value with the plain integral $\int_{G_AX_{\boldsymbol{Q}}} f(x)dX_A$.

DEFINITION 8.1. A special homogeneous space (G, X) is said to be *uniform* if there is a constant $\tau(G, X)$ such that

$$\int_{G_AX_{\boldsymbol{Q}}} f(x)dX_A=\tau(G, X)\tau(G)^{-1}\int_{G_A/G_{\boldsymbol{Q}}}\Big(\sum_{\xi\in X_{\boldsymbol{Q}}} f(g\xi)\Big)dG_A$$

for all $f\in L(G_AX_{\boldsymbol{Q}})$. When that is so, the number $\tau(G, X)$ will be called the *Tamagawa number* of (G, X). If, in particular, $\tau(G, X)=1$, we say that (G, X) has the *mean value property*.

REMARK 8.2. We check here that the above definition of $\tau(G, X)$ is compatible with the ordinary definition of $\tau(G)$ when $G=(G, G)$. Namely, for this case we have $G_AX_{\boldsymbol{Q}}=G_AG_{\boldsymbol{Q}}=G_A$. Then, by Fubini theorem, we have

$$\int_{G_A} f(x)dG_A=\int_{G_A/G_{\boldsymbol{Q}}}\Big(\sum_{\xi\in G_{\boldsymbol{Q}}} f(g\xi)\Big)dG_A$$

for all $f\in L(G_A)$, which shows that (G, G) is uniform with $\tau(G, G)=\tau(G)$. In our terminology, the Weil's conjecture (♯) can be restated as follows:

(♯♯) $\qquad \pi_1(G_C)=0 \Rightarrow (G, G)$ has the mean value property.

Using the notation in § 6, we have the following criterion for the uniformity of (G, X).

LEMMA 8.3. *A special homogeneous space* (G, X) *defined over* $\boldsymbol{Q}$ *is uniform if and only if the number* $\sum_{i=1}^{h(\Theta)}\tau(G_{\xi_i})$ *is independent of the local classes* Θ. *When that is so,* $\tau(G, X)$ *is given by* $\tau(G, X)=\tau(G)/(\sum_{i=1}^{h(\Theta)}\tau(G_{\xi_i}))$.

PROOF. For a $\xi\in X_{\boldsymbol{Q}}$, one has obviously the following two exact sequences, or fiberings:

$$0\rightarrow G_{\xi,\boldsymbol{Q}}\rightarrow G_{\boldsymbol{Q}}\rightarrow G_{\boldsymbol{Q}}\xi\rightarrow 0\,, \tag{$10_{\boldsymbol{Q}}$}$$

$$0\rightarrow G_{\xi,A}\rightarrow G_A\rightarrow G_A\xi\rightarrow 0\,. \tag{10_A}$$

In an obvious sense, $(10)_{\boldsymbol{Q}}$ forms a discrete subfibering of the fibering $(10)_A$. Since $G_A\xi$ is open and closed in X_A (Lemma 5.3), dX_A induces a measure on $G_A\xi$, which we denote again by dX_A. Then it is not difficult to see that the three measures $dG_{\xi,A}$, dG_A and dX_A are coherent with respect to the fibering $(10)_A$. In this situation, we know the following formula of Fubini type with respect to the pair of fiberings $((10)_A, (10)_{\boldsymbol{Q}})$:

$$\tau(G_\xi)\int_{G_A\xi}f(x)dX_A=\int_{G_A/G_{\boldsymbol{Q}}}(\sum_{\eta\in G_{\boldsymbol{Q}}\xi}f(g\eta))dG_A \tag{11}$$

for all $f\in L(G_A\xi)$ (cf. [**12**, Lemma 2.4.2]). Now take a local class Θ with representatives ξ_i, $1\leqq i\leqq h(\Theta)$, of global classes in it. Summing the formula (11) for all ξ_i, using (8), we get

$$(\sum_{i=1}^{h(\Theta)}\tau(G_{\xi_i}))\int_{G_A\Theta}f(x)dX_A=\int_{G_A/G_{\boldsymbol{Q}}}(\sum_{\eta\in\Theta}f(g\eta))dG_A \tag{12}$$

for all $f\in L(G_AX_{\boldsymbol{Q}})$. For simplicity, put

$$\tau(\Theta)=\sum_{i=1}^{h(\Theta)}\tau(G_{\xi_i})\,.$$

Then, if $\tau(\Theta)$ is independent of Θ, by summing up (12) over all Θ, we get

$$\tau(\Theta)\int_{G_AX_{\boldsymbol{Q}}}f(x)dX_A=\int_{G_A/G_{\boldsymbol{Q}}}(\sum_{\xi\in X_{\boldsymbol{Q}}}f(g\xi))dG_A \tag{13}$$

for all $f\in L(G_AX_{\boldsymbol{Q}})$. Hence we see that (G, X) is uniform with $\tau(G, X)=\tau(G)/\tau(\Theta)$. Conversely, suppose that (G, X) is uniform. Let $f\in L(G_AX_{\boldsymbol{Q}})$ be such that its support is contained in $G_A\Theta$ for a given Θ. We have then, by Definition 8.1 and (12),

$$\begin{aligned}\int_{G_A\Theta}f(x)dX_A&=\int_{G_AX_{\boldsymbol{Q}}}f(x)dX_A=\tau(G, X)\tau(G)^{-1}\int_{G_A/G_{\boldsymbol{Q}}}(\sum_{\xi\in X_{\boldsymbol{Q}}}f(g\xi))dG_A\\&=\tau(G, X)\tau(G)^{-1}\int_{G_A/G_{\boldsymbol{Q}}}(\sum_{\eta\in\Theta}f(g\eta))dG_A=\tau(G, X)\tau(G)^{-1}\tau(\Theta)\int_{G_A\Theta}f(x)dX_A\,,\end{aligned}$$

which implies that $\tau(\Theta)$ is equal to the constant $\tau(G)/\tau(G, X)$ independent of Θ, q. e. d.

COROLLARY 8.4. *If (G, X) is of type (W), then it is uniform.*

In fact, since (G, X) is of type (H) (Lemma 6.4), it is enough to show that $\tau(G_\xi)$ is independent of $\xi \in X_{\boldsymbol{Q}}$. However, since $\kappa_{\boldsymbol{Q}} : G_{\boldsymbol{Q}} \rightarrow X_{\boldsymbol{Q}}$ is surjective, we see that any two groups G_ξ, G_η, $\xi, \eta \in X_{\boldsymbol{Q}}$, are conjugate by an element of $G_{\boldsymbol{Q}}$, and hence $\tau(G_\xi) = \tau(G_\eta)$, q. e. d.

REMARK 8.5. The spaces in (Ex I), (Ex II) being of type (W), they are uniform. By using known values of Tamagawa numbers for special linear groups and special orthogonal groups, we get $\tau(G, X) = 1$, i. e. those spaces have the mean value property. For example, the sphere $X = \{x \in \Omega^n,\ x_1^2 + \cdots + x_n^2 = r,\ r \in \boldsymbol{Q}^*\}$, viewed as a homogeneous space for $0^+(n)$ has the Tamagawa number 1, for $n \geqq 4$. As for the space in (Ex III), we have $\tau(G, X) = 1/2$, because $G_\xi = 0^+(\xi)$ for all $\xi \in X_{\boldsymbol{Q}}$ and the space is of type (H). As we have seen before (the end of § 3), the first two homotopy groups of $X_{\boldsymbol{C}}$ vanish for (Ex I), (Ex II) but not for (Ex III). Therefore, we see that as far as those examples are concerned, the vanishing of the first two homotopy groups of $X_{\boldsymbol{C}}$ corresponds precisely to the validity of the mean value property of (G, X). In the next section, we shall look at these from more general point of view.

§ 9. Mean value theorem.

Let (G, X) be a special homogeneous space defined over $\boldsymbol{Q}$. We impose on X the condition

$$\pi_1(X_{\boldsymbol{C}}) = \pi_2(X_{\boldsymbol{C}}) = 0 . \tag{14}$$

If, in particular, $X = G$, (14) is reduced to a single condition $\pi_1(G_{\boldsymbol{C}}) = 0$, since $\pi_2(G_{\boldsymbol{C}}) = 0$ always. In view of (♯♯), (14) seems to be a natural requirement in order to expect the mean value property for (G, X). Now, let Θ be a local class of (G, X) and let ξ_i, $1 \leqq i \leqq h(\Theta)$, be representatives of global classes in Θ. By Lemma 8.3, we see that (G, X) has the mean value property if and only if $\tau(G) = \sum_{i=1}^{h(\Theta)} \tau(G_{\xi_i})$ for any Θ. Denoting by $\tilde{G}$ the universal covering group over $\boldsymbol{Q}$ of a special algebraic group G defined over $\boldsymbol{Q}$, the condition (14) implies that (G, X) has the mean value property if and only if $\tau(\tilde{G}) = \sum_{i=1}^{h(\Theta)} \tau(\tilde{G}_{\xi_i})$ for any Θ (cf. Lemma 7.1). Therefore, if we assume the validity of the Weil's conjecture (♯) for $\tilde{G}$ and $\tilde{G}_\xi$, for any $\xi \in X_{\boldsymbol{Q}}$, we see that (G, X) has the mean value property if and only if $h(\Theta) = 1$ for any Θ, i. e. (G, X) is of type (H). Summarizing, we get

THEOREM 9.1. *Let (G, X) be a special homogeneous space defined over $\boldsymbol{Q}$*

of type (H). *If Weil's conjecture holds for the universal covering groups* G, G_ξ, *for any* $\xi \in X_{\boldsymbol{Q}}$, *then one has:*

$$\pi_1(X_{\boldsymbol{C}}) = \pi_2(X_{\boldsymbol{C}}) = 0 \Rightarrow \tau(G, X) = 1,$$

i.e. (G, X) *has the mean value property.*

REMARK 9.2. If, in particular, $X = G$, the space (G, G) is of type (W), and hence type (H). Hence, as we remarked above, this Theorem is regarded as a direct extension of the Weil's conjecture for tne case of special homogeneous spaces.

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