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A Deformation Theorem on Conformal Mapping.

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We will prove the following deformation theorem on conformal mapping.

Theorem 1. *Let D be a simply connected domain on the z-plane, which contains $z=0$ and is contained in $|z|<M$. Let E be a continuum, which contains $z=0$ and is contained in D, such that a disc of radius ρ about any point of E is contained in D. If we map D conformally on $|w|<1$ by $w=w(z)$, $z=z(w)$ $(w(o)=o))$, then the image of E is contained in $|w|<1-k<1$, where $k=k\left(\frac{\rho}{M}\right)$ depends on $\frac{\rho}{M}$ only.*

We can take

$$k=\frac{\rho}{4M}e^{-a\frac{M^2}{\rho^2}}\left(a=\frac{64\pi}{\sqrt{3}}\log\frac{32}{9}<100\right).$$

Proof. We cover the z-plane by a net of regular triangles Δ_i of sides $\frac{\rho}{4}$, whose vertices are $z_{m,n}=m\frac{\rho}{4}e^{\frac{\pi i}{3}}+n\frac{\rho}{4}(m,\ n=0,\ \pm1,\ \pm2,\cdots\cdots)$. It is easily seen that if Δ_i contains a point of E, then a disc of radius $\frac{3\rho}{4}$ about a vertex ζ_i of Δ_i is contained in D, so that Δ_i is contained in D and $w(z)$ is regular and schlicht in $|z-\zeta_i|<\frac{3\rho}{4}$. Let $\Delta_1, \Delta_2,\cdots\cdots,\Delta_N$ be the triangles which contain points of E, where $z=0$ is a vertex of Δ_1, then since the area of Δ_i is $\frac{\sqrt{3}\rho^2}{64}$ and is contained in $|z|<M$,

$$N<\mu=\frac{64\pi}{\sqrt{3}}\frac{M^2}{\rho^2}. \tag{1}$$

Let z_0 be any point of E and let z_0 be contained in Δ_{n0} $(n_0\leqq N)$, then since E is a continuum, there exists a chain of triangles:

$$\Delta_1,\ \Delta_2,\cdots\cdots,\ \Delta_{n0} \qquad (n_0\leqq N),$$

where Δ_i, Δ_{i+1} have a common side, so that $|\zeta_i-\zeta_{i+1}|=\frac{\rho}{4}$ and each Δ_i con-

tains a point of E.
Since $z(w)$ $(z(o)=o)$ is regular and $|z(w)|<M$ in $|w|<1$, we have by Schwarz's lemma

$$\left|\left(\frac{dz}{dw}\right)_0\right|\leqq M, \text{ or } \left|\left(\frac{dw}{dz}\right)_0\right|\geqq\frac{1}{M} \quad . \tag{3}$$

Now

$$w(z)=\lambda z+\cdots\cdots \quad (|\lambda|\geqq\frac{1}{M}) \tag{4}$$

is regular and schlicht in $|z|<\frac{3\rho}{4}$, hence by putting $z=\frac{3\rho}{4}t$

$$F(t)=\frac{w\left(\frac{3\rho}{4}t\right)}{\frac{3\lambda\rho}{4}}=t+\cdots\cdots$$

is regular and schlicht in $|t|<1$, so that by Koebe's theorem,

$$|F'(t)|\geqq\frac{1-|t|}{(1+|t|)^3} \quad (|t|<1),$$

or

$$|w'(z)|\geqq|\lambda|\frac{1-\frac{4|z|}{3\rho}}{\left(1+\frac{4|z|}{3\rho}\right)^3} \qquad (|z|<\frac{3\rho}{4}).$$

Hence for a vertex ζ_2 $(|\zeta_2|\frac{\rho}{4})$ of Δ_2,

$$|w'(\zeta_2)|\geqq|\lambda|\frac{9}{32} \ .$$

Repeating the same process at $\zeta_3,\cdots\cdots\zeta_{n0}$, we have

$$|w'(\zeta_{no})|\geqq|\lambda|\left(\frac{9}{32}\right)^{n_0-1}.$$

Since z_0 lies in Δ_{n0} and $|z_0-\zeta_{n0}|\leqq\frac{\rho}{4}$, if we apply again Koebe's theorem in $|z-\zeta_{on}|<\frac{3}{4}\varrho$, we have

$$|w'(z_0)| \geqq |\lambda| \left(\frac{9}{32}\right)^{n_0} \geqq \left(\frac{9}{32}\right)^{N} \frac{1}{M}. \tag{5}$$

Since

$$w - w_0 = a(z - z_0) + \cdots\cdots \quad \left(w_0 = w(z_0),\ |a| \geqq \left(\frac{9}{32}\right)^{N} \frac{1}{M}\right) \tag{6}$$

is regular and schlicht in $|z - z_0| < \rho$, if we put $z - z_0 = \rho\zeta$,

$$F(\zeta) = \frac{w - w_0}{a\rho} = \zeta + \cdots\cdots$$

is regular and schlicht in $|\zeta| < 1$, so that by Koebe's theorem, the image of $|\zeta| < 1$ contains a disc of radius $\frac{1}{4}$, so that the disc $|w - w_0| < \frac{a\rho}{4}$ and hence the disc $|w - w_0| < \frac{\rho}{4M}\left(\frac{9}{32}\right)^{N}$ is contained in $|w| < 1$, or $|w_0| <$ $1 - \frac{\rho}{4M}\left(\frac{9}{32}\right)^{N} \leqq 1 - \frac{\rho}{4M}\left(\frac{9}{32}\right)^{\mu} \qquad \left(\mu = \frac{64\pi}{\sqrt{3}} \frac{M^2}{\rho^2}\right)$.

Since z_0 is arbitrary, the image of E is contained in $|w| < 1 - k < 1$, where

$$k = \frac{\rho}{4M}\left(\frac{9}{32}\right)^{\mu} = \frac{\rho}{4M} e^{-a\frac{M^2}{\rho^2}} \quad \left(a = \frac{64\pi}{\sqrt{3}} \log \frac{32}{9} < 100\right). \text{ q.e.d.}$$

Similarly we can prove:

Theorem 2. *Let D be a simply connected Riemann surface, whose projection on the z-plane lies in $|z| < M$, such that D has no branch points and $z = 0$ belongs to D, and $|D|$ be its area. Let E be a continuum contained in D, such that E contains $z = 0$, and a disc of radius ρ about any point of E is contained in D. If we map D conformally on $|w| < 1$ by $w = w(z)$ $(w(o) = o)$, then the image of E is contained in $|w| < 1 - k < 1$, where*

$$k = \frac{\rho}{4M} e^{-a\frac{|D|}{\rho^2}} \left(a = \frac{64}{\sqrt{3}} \log \frac{23}{9}\right).$$

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