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# On the Stability of the linear Transformation in Banach Spaces.

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Let $E$ and $E'$ be Banach spaces, and $f(x)$ be a transformation from $E$ into $E'$, which is "*approximately linear*". Ulam proposed the problem: *When does a linear transformation near an "approximately linear" transformation exist?* This was solved by D. H. Hyers[1]. The object of this paper is to generalize Hyer's theorem.

In generalizing the definition of Hyers, we shall call a transfor mation $f(x)$ from $E$ into $E'$ "*approximately linear*", when there exists $K(\geqq 0)$ and $p(0\leqq p<1)$ such that

$$\|f(x+y)-f(x)-f(y)\|\leqq K(\|x\|^p+\|y\|^p)$$

for any $x$ and $y$ in $E$.

Let $f(x)$ and $\varphi(x)$ be transformations from $E$ into $E'$. These are called "*near*", when there exists $K(\geqq 0)$ and $p(0\leqq p>1)$ such that

$$\|f(x)-\varphi(x)\|\leqq K\|x\|^p$$

for any $x$ in $E$.

**Theorem.** *If $f(x)$ is an approximately linear transformation from $E$ into $E'$, then there is a linear transformation $\varphi(x)$ near $f(x)$. And such $\varphi(x)$ is unique.*

Proof. By the assumption there are $K_0(\geqq 0)$ and $p(0\leqq p<1)$ such that

$$\|f(2x)/2-f(x)\|\leqq K_0\|x\|^p. \tag{1}$$

We shall now prove that

$$\|f(2^n x)/2^n-f(x)\|\leqq K_0\|x\|^p\sum_{i=0}^{n-1}2^{i(p-1)} \tag{2}$$

for any integer $n$. The case $n=1$ holds by (1). Assuming the case

1) D. H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci., vol 27, No. 4 (1941), p. 222-4.

$n$, we shall prove the case $n+1$. Replacing $x$ by $2x$ in (2), we get

$$\|f(2^n.\,2x)/2^n-f(2x)\| \leqq K_0\|x\|^p 2^p \sum_{i=0}^{n-1} 2^{i(p-1)}.$$

That is,

$$\|f(2^{n+1}x)/2^n-f(2x)\| \leqq K_0\|x\|^p \sum_{i=1}^{n} 2^{i(p-1)}.$$

By (1)

$$\|f(2^{n+1}x)/2^{n+1}-f(x)\| \leqq \|f(2^{n+1}x)/2^{n+1}-f(2x)/2\| + \|f(2x)/2-f(x)\| \leqq K_0\|x\|^p \sum_{i=0}^{n} 2^{i(p-1)}$$

Thus we get (2) for the case $(n+1)$. Hence (2) holds for any $n$.

Since $0 \leqq p<1$, $\sum_{i=0}^{\infty} 2^{i(p-1)}$ converges to $2/(2-2^p)$. Therefore (2) becomes

(3) $$\|f(2^n x)/2^n-f(x)\| \leqq K\|x\|^p,\ K=2K_0/(2-2^p).$$

Let us consider the sequence $(f(2^n x)/2^n)$. We have

$$\|f(2^m x)/2^m-f(2^n x)/2^n\| = \|f(2^m x)/2^{m-n}-f(2^n x)\|/2^n < K\|2^n x\|^p/2^n = 2^{n(p-1)}K\|x\|^p \to 0\ (n\to\infty).$$

Since $E'$ is complete, the sequence in consideration converges. We put

$$\varphi(x) \equiv \lim f(2^n x)/2^n.$$

Then $\varphi(x)$ is linear. For, by the approximate linearity of $f(x)$

$$\|f(2^n(x+y))-f(2^n x)-f(2^n y)\| \leqq K_0(\|2^n x\|^p+\|2^n y\|^p) = 2^{np}K_0(\|x\|^p+\|y\|^p).$$

Dividing both sides by $2^n$ and letting $n\to\infty$, we get

$$\varphi(x+y)=\varphi(x)+\varphi(y),$$

which shows that $\varphi$ is linear. In (3), letting $n\to\infty$, we get

(4) $$\|\varphi(x)-f(x)\| \leqq K\|x\|^p,$$

which shows that $\varphi(x)$ is near $f(x)$.

It remains to prove the unicity of $\varphi(x)$. Let $\psi(x')$ be another linear transformation near $f(x)$. Then there exist $K'(\leqq 0)$ and $p'(0\leqq p'<1)$ such that

$$\|\psi(x)-f(x)\|\leqq K'\|x\|^{p}. \tag{5}$$

By (4) we have

$$\|\varphi(x)-\psi(x)\|\leqq K\|x\|^{p}+K'\|x\|^{p'}.$$

By the linearity of $\varphi$ and $\psi$,

$$\begin{aligned}\|\varphi(x)-\psi(x)\| &= \|\varphi(nx)-\psi(nx)\|/n\\ &\leqq (K\|nx\|^{p}+K'\|nx\|^{p'})/n\\ &= K\|x\|^{p}/n^{1-p}+K'\|x\|^{p'}/n^{1-p'}\rightarrow 0\ (n\rightarrow\infty).\end{aligned}$$

Therefore $\varphi(x)=\psi(x)$.

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