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Conformal transformations in complete product Riemannian manifolds

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This paper is a direct continuation of a previous one [7], which we shall refer to for terminologies and notations. In the previous paper, one of the present authors determined the structure of complete Riemannian manifolds admitting a concircular scalar field, and furthermore the structure of complete product Riemannian manifolds admitting a nonisometric conformal vector field under an assumption relative to dimension of manifolds. Here and hereafter we say a vector field to be *isometric* or *conformal* if it generates a one-parameter group of isometric or conformal transformations, respectively. A vector field is said to be *complete* if it generates a global one-parameter group of transformations.

After preliminaries are stated in § 1, we shall study in § 2 the structure of manifolds, named pseudo-hyperbolic spaces in [7], in more details. In § 3, the expression of a concircular scalar field in a space form will be obtained. In § 4, we shall consider complete product Riemannian manifolds admitting a conformal vector field and obtain the equations satisfied by the associated scalar field in a simpler way than that in [7]. As a consequence, the structure of manifolds having such properties is determined without assumption relative to dimension. The purpose of § 5 and of the present paper is to prove the following

MAIN THEOREM. *If a complete reducible Riemannian manifold admits a complete nonisometric conformal vector field, then the manifold is locally Euclidean and the vector field is homothetic.*

This is a generalization of Tanaka's theorem [5] for manifolds with parallel Ricci tensor and of Tachibana's [4] for compact manifolds. Our method of proof is elementary and different from theirs.

§ 1. Preliminaries.

In this paper we shall always deal with connected Riemannian manifolds with positive definite metric, and suppose that manifolds and quantities are differentiable of class C^∞.

Let M be an n-dimensional Riemannian manifold. Greek indices κ, λ, μ, ν run on the range $1, \cdots, n$. Denote by $g_{\mu\lambda}$ the metric tensor of M, by $\left\{ {\kappa \atop \mu\lambda} \right\}$ the Christoffel symbol and by ∇ covariant differentiation with respect to $\left\{ {\kappa \atop \mu\lambda} \right\}$. A *concircular scalar field* (abbrev. *C-field*) ρ is by definition a scalar field satisfying the equations

$$\nabla_\mu \nabla_\lambda \rho = \partial_\mu \partial_\lambda \rho - \left\{ {\kappa \atop \mu\lambda} \right\} \partial_\kappa \rho = \phi g_{\mu\lambda}, \tag{1.1}$$

where ϕ is a scalar field. If the equations are in particular of the form

$$\nabla_\mu \nabla_\lambda \rho = (-k\rho + b) g_{\mu\lambda} \tag{1.2}$$

with constant coefficients k and b, then ρ is called a *special concircular scalar field* (abbrev. *SC-field*) and k the *characteristic constant* of ρ. If $\phi = 0$, the field ρ is said to be *parallel.* A point P is called a *stationary* or *ordinary* point of ρ whether the gradient vector field $\rho_\lambda = \partial_\lambda \rho$ vanishes at P or not.

Along any geodesic curve with arc length u, the scalar field ϕ and ρ are differentiable functions of u and the equations (1.1) turn to the ordinary differential equation

$$\frac{d^2\rho}{du^2} = \phi \tag{1.3}$$

and (1.2) to

$$\frac{d^2\rho}{du^2} = -k\rho + b\,. \tag{1.4}$$

According to the signature of k, we put

$$k = \begin{cases} \text{I)} & 0\,, \\ \text{II)} & -c^2\,, \\ \text{III)} & c^2\,, \end{cases}$$

where c is a positive constant. Choosing suitably the arc length u and supposing $b = 0$ in the case of $k \neq 0$ without loss of generality, the solution of (1.4) is given by one of

$$\rho(u) = \begin{cases} \text{I, A)} & au & (b=0) \\ \text{I, B)} & \frac{1}{2}bu^2 + a & (b \neq 0) \\ \text{II, A}_0) & a \exp cu\,, & \\ \text{II, A}_-) & a \sinh cu\,, & \\ \text{II, B)} & a \cosh cu\,, & \\ \text{III)} & a \cos cu\,, & \end{cases} \tag{1.5}$$

where a is an arbitrary constant.

The ρ-curves, the trajectories of the vector field $\rho^{\kappa}=\rho_{\lambda}g^{\lambda\kappa}$, are geodesics. When we denote by u^1 their arc length, suitably chosen, and by prime ordinary derivatives of functions of u^1, there is an adapted coordinate system $(u^{\kappa})=(u^1, u^{\alpha})$ in a neighborhood of an ordinary point of ρ such that the metric form of M is given by

$$ds^2=(du^1)^2+(\rho'(u^1))^2\overline{ds^2}, \tag{1.6}$$

where $\overline{ds^2}$ is the metric form of an $(n-1)$-dimensional Riemannian manifold M^{n-1}. Making Greek indices α, β, γ run on the range $2, \cdots, n$, we shall put $\overline{ds^2}=f_{\gamma\beta}du^{\gamma}du^{\beta}$ as occasion demands.

If M admits an SC-field ρ, then, by substituting (1.5) into (1.6) and transferring constant factors into the metric of M^{n-1}, we can express the metric form ds^2 of M as

$$ds^2=\begin{cases} \text{I, A)} & (du^1)^2+\overline{ds^2}, \\ \text{I, B)} & (du^1)^2+(u^1)^2\overline{ds^2}, \\ \text{II, A}_0\text{)} & (du^1)^2+(\exp 2cu^1)\overline{ds^2}, \\ \text{II, A}_-\text{)} & (du^1)^2+(\cosh cu^1)^2\overline{ds^2}, \\ \text{II, B)} & (du^1)^2+(\sinh cu^1)^2\overline{ds^2}, \\ \text{III)} & (du^1)^2+(\sin cu^1)^2\overline{ds^2}, \end{cases} \tag{1.7}$$

in the respective cases. If we rewrite the coefficients of $\overline{ds^2}$ in these forms by $\tau^2(u^1)$, the Christoffel symbol $\left\{ {\kappa \atop \mu\lambda} \right\}$ of M has components

$$\begin{gathered} \left\{ {1 \atop 11} \right\}=\left\{ {\alpha \atop 11} \right\}=\left\{ {1 \atop 1\beta} \right\}=0, \quad \left\{ {1 \atop \gamma\beta} \right\}=-\tau\tau' f_{\gamma\beta}, \\ \left\{ {\alpha \atop 1\beta} \right\}=\frac{\tau'}{\tau}\delta^{\alpha}_{\beta}, \quad \left\{ {\alpha \atop \gamma\beta} \right\}=\overline{\left\{ {\alpha \atop \gamma\beta} \right\}}, \end{gathered} \tag{1.8}$$

where $\overline{\left\{ {\alpha \atop \gamma\beta} \right\}}$ is the Christoffel symbol of the metric tensor $f_{\gamma\beta}$ in M^{n-1}.

One of the authors proved in [7, Theorem 2] that

THEOREM A. *If a complete Riemannian manifold M of dimension $n\geqq 2$ admits an SC-field ρ having the characteristic constant k, then M is one of the following manifolds:*

I, A) *the direct product $I\times M^{n-1}$ of a straight line I with an $(n-1)$-dimensional complete Riemannian manifold M^{n-1},*

I, B) *a Euclidean space,*

II, A) *a pseudo-hyperbolic space of zero or negative type,* II, B) *a hyperbolic space of curvature* $k=-c^2$,

III) *a spherical space of curvature* $k=c^2$.

A pseudo-hyperbolic space of zero or negative type has been defined in [7] as a complete Riemannian manifold which is diffeomorphic to the product $I\times M^{n-1}$ and whose metric form is given by II, A_0) or II, A_-) of (1.7) respectively. A pseudo-hyperbolic space is of negative constant sectional curvature $-c^2$ if and only if the manifold M^{n-1} is locally Euclidean in Case II, A_0) or of negative constant sectional curvature $-c^2$ in Case II, A_-), [7, Lemma 3.1]. The SC-field ρ has no stationary point in Cases I, A) and II, A), one corresponding to $u^1=0$ in Cases I, B) and II, B), and two corresponding to $u^1=0$ and $u^1=\pi/c$ in Case III). The metric forms in (1.7) are valid in the whole manifold M except these stationary points.

§ 2. Manifolds admitting functionally independent C-fields.

First we prove the following

LEMMA. *If a C-field* ρ *satisfies the equation* (1.2) *at ordinary points, so does it at stationary points, that is,* ρ *is an SC-field in the whole manifold M.*

PROOF. If ϕ vanishes identically, then ρ has no stationary point as is easily seen. Hence we suppose that ϕ dose not identically vanish. Let Q be a stationary point of ρ, $\rho_\lambda(Q)=0$. Suppose first $\phi(Q)\neq 0$. Then we have $\partial_\mu\partial_\lambda\rho(Q)=\phi(Q)g_{\mu\lambda}(Q)$ and see that the matrix $(\partial_\mu\partial_\lambda\rho)$ is regular in a neighborhood of Q. Hence Q is an isolated stationary point. By the comparison of (1.1) with (1.2) and the continuity of ϕ and ρ, we have also $\phi=-k\rho+b$ at Q.

Suppose next $\phi(Q)=0$. Then $\rho_\lambda(Q)=0$ and $\nabla_\mu\rho_\lambda(Q)=0$. Join Q with an ordinary point P neighboring to Q by the geodesic curve, and let Q be renewedly the stationary point which is met first on the geodesic curve from P and at which ϕ vanishes. Then we have the equation (1.3) along the geodesic curve and the equation (1.4) along the arc PQ. Since $\rho'(u)$ and $\rho''(u)$ vanish at Q, the constants k and b vanish in order that (1.4) is compatible with (1.3) at Q, and consequently ϕ vanishes identically. This is a contradiction.

Q. E. D.

By virtue of this lemma, we can restate more precisely Theorem 1 of [8] as follows:

THEOREM 1. *If two C-fields are functionally independent of each other, then they are SC-fields in the whole manifold and have a characteristic constant in common.*

Now we are going to determine the structure of a complete Riemannian manifold M admitting m functionally independent C-fields $\rho_{(i)}$ $(i=1,\cdots,m)$, $m\leqq n$. Let k be the characteristic constant in common with $\rho_{(i)}$.

I) The case of $k=0$. If there is an SC-field with $b\neq 0$ in (1.2), then the manifold M itself is a Euclidean space by Theorem A, I, B). If all the fields $\rho_{(i)}$ are parallel, i. e.,

$$\nabla_\mu \nabla_\lambda \rho_{(i)}=0\,, \tag{2.2}$$

then we take u^1 as the arc length, suitably chosen, of the ρ-curves of $\rho_{(1)}$. By Theorem A, I, A), M is the direct product $I\times M^{n-1}$. The metric form of M is given by I, A) of (1.7) and the components of the Christoffel symbol vanish all except $\left\{ {\alpha \atop \gamma\beta} \right\}$ which is equal to $\overline{\left\{ {\alpha \atop \gamma\beta} \right\}}$ formed from the metric tensor $f_{\gamma\beta}$ of M^{n-1}. Then the equations (2.2) are decomposed into

$$\partial_1\partial_1\rho_{(i)}=0\,,\quad \partial_1\partial_\beta\rho_{(i)}=0\,,\quad \nabla_\gamma\nabla_\beta\rho_{(i)}=0 \qquad (i=2,\cdots,m)\,. \tag{2.3}$$

From the first equations, we may put

$$\rho_{(i)}=a_{(i)}u^1+\beta_{(i)} \qquad (i=2,\cdots,m)\,, \tag{2.4}$$

$a_{(i)}$ and $\beta_{(i)}$ being functions on M^{n-1}. Moreover, from the second equations of (2.3), it follows that $a_{(i)}$ are constants, and hence $\beta_{(i)}$ should be functionally independent of one another on M^{n-1}. From the third equations of (2.3), we see $\beta_{(i)}$ satisfying

$$\nabla_\gamma\nabla_\beta\beta_{(i)}=\bar{\nabla}_\gamma\bar{\nabla}_\beta\beta_{(i)}=0 \qquad (i=2,\cdots,m)$$

where $\bar{\nabla}$ indicates covariant differentiation with respect to $\overline{\left\{ {\alpha \atop \gamma\beta} \right\}}$ in M^{n-1}. Thus the part M^{n-1} admits $m-1$ parallel scalar fields $\beta_{(i)}$ functionally independent of one another. Applying the above discussions repeatedly, we see that the manifold M is the product $E^m\times M^{n-m}$ of an m-dimensional Euclidean space E^m with an $(n-m)$-dimensional complete Riemannian manifold M^{n-m} and the m fields $\rho_{(i)}$ are regarded as functions on E^m. In a separate coordinate system (u^i, u^p), of which (u^i) belong to E^m and (u^p) to M^{n-m}, the metric form of M is expressed in the form

$$ds^2=ds_0^2+\overline{ds^2}\,,$$

ds_0^2 and $\overline{ds^2}$ being the metric forms of E^m and M^{n-m} respectively. In the remaining of this paragraph, Latin indices i, j, k run on the range $1,\cdots,m$, and p, q, r on the range $m+1,\cdots,n$ unless otherwise stated. The Christoffel symbol of M is decomposed into those of the parts E^m and M^{n-m}. If we denote by ∇^0 covariant differentiation in E^m, the equations (2.2) are reduced to

$$\nabla_k^0\nabla_j^0\rho_{(i)}=0\,,$$

which mean that the C-fields $\rho_{(i)}$ are regarded as parallel scalar fields in the Euclidean part E^m.

II) The case of $k=-c^2$. We have the equations

(2.5) $$\nabla_\mu \nabla_\lambda \rho_{(i)} = c^2 \rho_{(i)} g_{\mu\lambda} \qquad (i=1, \cdots, m).$$

B) If one of the fields, say $\rho_{(1)}$, is given by II, B) of (1.5) along the ρ-curves of $\rho_{(1)}$, then M itself is a hyperbolic space of curvature $-c^2$ by Theorem A, II, B).

A_0) If $\rho_{(1)}$ is given by II, A_0) of (1.5), then M is diffeomorphic to $I\times M^{n-1}$ and the metric form of M is given by II, A_0) of (1.7) in an adapted coordinate system (u^1, u^α). Then the nontrivial components of the Christoffel symbol are

$$\left\{ {1 \atop \gamma\beta} \right\} = -c(\exp 2cu^1) f_{\gamma\beta}, \qquad \left\{ {\alpha \atop 1\beta} \right\} = c\delta_\beta^\alpha, \qquad \left\{ {\alpha \atop \gamma\beta} \right\} = \overline{\left\{ {\alpha \atop \gamma\beta} \right\}}.$$

Consequently the equations (2.5) are decomposed into

(2.6) $$\begin{aligned} &\nabla_1 \nabla_1 \rho_{(i)} = \partial_1 \partial_1 \rho_{(i)} = c^2 \rho_{(i)}, \\ &\nabla_1 \nabla_\beta \rho_{(i)} = \partial_1 \partial_\beta \rho_{(i)} - c\partial_\beta \rho_{(i)} = 0, \\ &\nabla_\gamma \nabla_\beta \rho_{(i)} = \bar{\nabla}_\gamma \bar{\nabla}_\beta \rho_{(i)} + c(\exp 2cu^1) f_{\gamma\beta} \partial_1 \rho_{(i)} \\ &\qquad = c^2 (\exp 2cu^1) \rho_{(i)} f_{\gamma\beta} \qquad (i=2, \cdots, m). \end{aligned}$$

From the first equations we may put

(2.7) $$\rho_{(i)} = \beta_{(i)} \exp cu^1 + a_{(i)} \exp(-cu^1) \qquad (i=2, \cdots, m),$$

where $\beta_{(i)}$ and $a_{(i)}$ are functions on M^{n-1}. Moreover, substituting these expressions (2.7) into the second and the third of (2.6), we see that $a_{(i)}$ are constants and $\beta_{(i)}$ satisfy the equations

$$\bar{\nabla}_\gamma \bar{\nabla}_\beta \beta_{(i)} = 2c^2 a_{(i)} f_{\gamma\beta} \qquad (i=2, \cdots, m).$$

If one of the coefficients $a_{(i)}$ is not equal to zero, then by Theorem A, I, B) the part M^{n-1} is an $(n-1)$-dimensional Euclidean space E^{n-1}. Therefore M is diffeomorphic to an n-dimensional Euclidean space $E^n = I\times E^{n-1}$ and the metric form is given by

$$ds^2 = (du^1)^2 + (\exp 2cu^1)\overline{ds^2},$$

where $\overline{ds^2}$ is the Euclidean metric form of E^{n-1}. Thus the manifold M is a hyperbolic space of curvature $-c^2$.

If all the coefficients $a_{(i)}$ vanish, then the $m-1$ scalar fields $\beta_{(i)}$ are parallel and functionally independent of one another on M^{n-1}. By means of results in Case I), M^{n-1} is a product $E^{m-1}\times M^{n-m}$ and hence M is diffeomorphic to $E^m \times M^{n-m}$ and the metric form of M is given by

(2.8) $$ds^2 = (du^1)^2 + (\exp 2cu^1)(\overline{ds_0^2} + \overline{ds^2}),$$

where $\overline{ds_0^2}$ is the Euclidean metric form of E^{m-1}. If we put

$$ds_0^2 = (du^1)^2 + (\exp 2cu^1)\overline{ds_0^2},$$

then ds_0^2 is a hyperbolic one of curvature $-c^2$. The expression (2.8) can be rewritten in the form

$$ds^2 = ds_0^2 + \sigma^2 \overline{ds^2}$$

in a separate coordinate system (u^i, u^p), and σ is a function of u^i only. The components $\left\{ {i \atop k\,j} \right\}$ of the Christoffel symbol of M are identical with $\left\{ {i \atop k\,j} \right\}_0$ of the hyperbolic metric ds_0^2. Since $\beta_{(i)}$ $(i = 2, \cdots, m)$ are functions on E^{m-1}, the C-fields $\rho_{(i)}$ $(i = 1, \cdots, m)$ are dependent of the coordinates u^i only and the equations (2.5) are reduced to

$$\nabla_k^0 \nabla_j^0 \rho_{(i)} = c^2 \rho_{(i)} g_{kj} \qquad (i = 1, \cdots, m).$$

Therefore $\rho_{(i)}$ are regarded as SC-fields on an m-dimensional hyperbolic space M_0 of curvature $-c^2$.

A$_-$) If $\rho_{(1)}$ is given by II, A$_-$) of (1.5), then the metric form of M is given by II, A$_-$) of (1.7) in an adapted coordinate system and the nontrivial components of the Christoffel symbol are

$$\left\{ {1 \atop \gamma\beta} \right\} = -c(\sinh cu^1 \cosh cu^1) f_{\gamma\beta},$$

$$\left\{ {\alpha \atop 1\beta} \right\} = c(\tanh cu^1)\delta_\beta^\alpha, \qquad \left\{ {\alpha \atop \gamma\beta} \right\} = \overline{\left\{ {\alpha \atop \gamma\beta} \right\}}.$$

The equations (2.5) are decomposed into

$$\begin{aligned} &\nabla_1 \nabla_1 \rho_{(i)} = \partial_1 \partial_1 \rho_{(i)} = c^2 \rho_{(i)}, \\ &\nabla_1 \nabla_\beta \rho_{(i)} = \partial_1 \partial_\beta \rho_{(i)} - c(\tanh cu^1)\partial_\beta \rho_{(i)} = 0, \\ &\nabla_\gamma \nabla_\beta \rho_{(i)} = \overline{\nabla}_\gamma \overline{\nabla}_\beta \rho_{(i)} + c(\sinh cu^1 \cosh cu^1) f_{\gamma\beta} \partial_1 \rho_{(i)} \\ &\qquad = c^2 (\cosh cu^1)^2 \rho_{(i)} f_{\gamma\beta} \qquad (i = 2, \cdots, m). \end{aligned} \tag{2.9}$$

From the first equations we may put

$$\rho_{(i)} = a_{(i)} \sinh cu^1 + \beta_{(i)} \cosh cu^1 \qquad (i = 2, \cdots, m)$$

where $a_{(i)}$ and $\beta_{(i)}$ are functions on M^{n-1}. Substituting these expressions into the second and the third of (2.9), we see that $a_{(i)}$ are constants and $\beta_{(i)}$ satisfy the equations

$$\overline{\nabla}_\gamma \overline{\nabla}_\beta \beta_{(i)} = c^2 \beta_{(i)} f_{\gamma\beta} \qquad (i = 2, \cdots, m).$$

Therefore the $m-1$ fields $\beta_{(i)}$ are SC-fields functionally independent of one another on M^{n-1}.

When we take the coordinate u^2 as the arc length of ρ-curves of $\beta_{(2)}$ in M^{n-1}, there occur three cases where $\beta_{(2)}$ is given by the form of II, B), II, A$_0$) or II, A$_-$) of (1.5). In the first case, it follows from the previous discussions that M is diffeomorphic to $I\times E^{n-1}$ and the metric form is expressed in the form

$$ds^2=(du^1)^2+(\cosh cu^1)^2\overline{ds^2}\,, \tag{2.10}$$

where $\overline{ds^2}$ is a hyperbolic metric form of curvature $-c^2$, and the form (2.10) itself is a hyperbolic one of curvature $-c^2$. In the second case, the previous discussions in A_0) imply that M is diffeomorphic to $I\times E^{m-1}\times M^{n-m}$ and the metric form can be expressed in the form

$$ds^2=(du^1)^2+(\cosh cu^1)^2(\overline{ds_0^2}+\bar{\sigma}^2\overline{ds^2})$$

or

$$ds^2=(du^1)^2+(\cosh cu^1)^2\overline{ds_0^2}+\sigma^2\overline{ds^2}\,,$$

where $\bar{\sigma}$ is a function of the coordinates $u^2,\cdots,u^m$ belonging to E^{m-1} and consequently σ is a function of u^i belonging to $E^m=I\times E^{m-1}$. Therefore the manifold M is diffeomorphic to the product $M_0\times M^{n-m}$ of a hyperbolic space M_0 with a manifold M^{n-m}, and the metric form is

$$ds^2=ds_0^2+\sigma^2\overline{ds^2}$$

in a separate coordinats system (u^i, u^p). By the same argument as that in Case A_0), we see that $\rho_{(i)}$ are regarded as SC-fields in the hyperbolic part M_0.

In the third case where $\beta_{(2)}$ is given by the form II, A_-) of (1.5), on the repeated way to obtain the metric form of M, we may consider only the case where the form II, A_-) of (1.5) appears successively. Finally we see that the manifold M is diffeomorphic to $E^m\times M^{n-m}$ and the metric form is expressed in the form

$$\begin{aligned} ds^2&=(du^1)^2+\cosh^2 cu^1[(du^2)^2+\cosh^2 cu^2[(du^3)^2+\cdots\\ &\qquad\cdots[(du^{m-1})^2+\cosh^2 cu^{m-1}[(du^m)^2+\cosh^2 cu^m\overline{ds^2}]]\cdots]]\\ &=(du^1)^2+\cosh^2 cu^1[(du^2)^2+\cosh^2 cu^2[(du^3)^2+\cdots\\ &\qquad\cdots[(du^{m-1})^2+\cosh^2 cu^{m-1}(du^m)^2]\cdots]]+\sigma^2\overline{ds^2}\,, \end{aligned} \tag{2.12}$$

where σ is certain function of u^i. Since $(du^{m-1})^2+\cosh^2 cu^{m-1}(du^m)^2$ is a two-dimensional hyperbolic metric of curvature $-c^2$, the metric form (2.12) deleted the last term off is by induction a hyperbolic metric of curvature $-c^2$. Thus the manifold M has the same structure as that of the second case, and $\rho_{(i)}$ are SC-fields in the hyperbolic part M_0.

III) The case of $k=c^2$. The manifold M is a spherical space of curvature c^2 by means of Theorem A, III).

Summerizing the above results, we can state that

THEOREM 2. *If a complete Riemannian manifold M of dimension $n\geqq 2$ admits C-fields $\rho_{(i)}$ functionally independent of one another, then either the* *manifold M itself or a part M_0 is a space form of curvature k, that is,* I) *a Euclidean space for $k=0$,* II) *a hyperbolic space for $k<0$, or* III) *a spherical space for $k>0$. In addition, $\rho_{(i)}$ are SC-fields having k as the characteristic constant in common and are regarded as functions on the space form.*

The theorem enables us to pay our considerations to space forms as far as we are concerned with *SC*-fields.

§ 3. *SC*-fields and isometric vector fields in space forms.

For the sake of later use, we shall seek for the expressions of *SC*-fields and isometric vector fields in space forms.

It is well known that, in a space form M of curvature k, there is a coordinate system (x^h) in which the metric form is expressed in the form

$$ds^2 = \frac{\sum(dx^h)^2}{\left\{1+\frac{k}{4}\sum(x^h)^2\right\}^2}. \tag{3.1}$$

In this paragraph, Latin indices h, i, j, k run on the range $1, \cdots, n$, and the summation convention is also applied to repeated indices unless otherwise is stated. We put

$$R^2 = \sum(x^h)^2, \qquad S = 1+\frac{k}{4}R^2.$$

For $k=-c^2$, the manifold M is diffeomorphic to the ball $R<2/c$ and the expression (3.1) is valid there. For $k=c^2$, the coordinate system (x^h) covers the manifold M except the north pole corresponding to $R=\infty$, and the metric form (3.1) tends to zero as R tends to the infinity.

The metric tensor of M is equal to

$$g_{ji} = (1/S^2)\delta_{ji},$$

and the Christoffel symbol is equal to

$$\begin{aligned}\left\{{h \atop j\ i}\right\} &= -\frac{1}{S}(S_j\delta_{ih}+S_i\delta_{jh}-S_h\delta_{ji}) \\ &= -\frac{k}{2S}(x^j\delta_{ih}+x^i\delta_{jh}-x^h\delta_{ji}),\end{aligned}$$

where we have put $S_i=\partial_i S = kx^i/2$. Substituting these expressions into (1.2), we obtain the equations

$$\partial_j\rho_i+\frac{1}{S}(S_j\rho_i+S_i\rho_j-\delta_{ji}S_h\rho_h) = \frac{1}{S^2}(-k\rho+b)\delta_{ji}$$

or

$$\partial_j\partial_i(S\rho) = \left[\frac{k}{2}\rho+\frac{k}{2}x^h\rho_h-\frac{1}{S}(k\rho-b)\right]\delta_{ji}. \tag{3.2}$$

If we put

$$2C=\frac{k}{2}\rho+\frac{k}{2}x^h\rho_h-\frac{1}{S}(k\rho-b), \tag{3.3}$$

then (3.2) are decomposed into the two following types:

$$\partial_j\partial_i(S\rho)=0 \qquad (j\neq i),$$
$$\partial_i\partial_i(S\rho)=2C \qquad (\text{not summed in } i).$$

The first equations mean that, for each value of i, $\partial_i(S\rho)$ is a function of the corresponding coordinate x^i only. Then the second imply that C is a constant and the solution has the expression

$$\rho=\frac{1}{S}(CR^2+B_hx^h+A),$$

where A and B_h are arbitrary constants. Substituting this into (3.3), we see that

$$C=-\frac{k}{4}A+\frac{b}{2}.$$

Hence the SC-field ρ has the expression

$$\rho=\frac{1}{S}\Big[A\Big(1-\frac{k}{4}R^2\Big)+B_hx^h+\frac{b}{2}R^2\Big]$$

in the coordinate system (x^h).

Next let W be an isometric vector field and denote by (w^h) its components in the coordinate system (x^h). The Killing equations for W are

$$\pounds(W)g_{ih}=w^k\partial_kg_{ih}+(\partial_iw^k)g_{kh}+(\partial_hw^k)g_{ik}=0,$$

where $\pounds(W)$ indicates Lie differentiation with respect to W, see [**9**]. These equations are reduced to

$$\partial_iw^h+\partial_hw^i=2\frac{w^kS_k}{S}\delta_{ih}. \tag{3.4}$$

with respect to the coordinate system (x^h) in the space form M. Putting $T=w^kS_k/S$, $T_i=\partial_iT$ and differentiating (3.4) in x^j, we have

$$\partial_j\partial_iw^h+\partial_j\partial_hw^i=2T_j\delta_{ih}.$$

Permuting the indices $j\ i\ h$ and taking the sum of the permutations $j\ i\ h$ and $i\ j\ h$ diminished by the permutation $h\ j\ i$, we get

$$\partial_j\partial_iw^h=T_j\delta_{ih}+T_i\delta_{jh}-T_h\delta_{ji}. \tag{3.5}$$

Differentiating again in x^k, exchanging the indices k and j, and equating the results, we obtain

$$(\partial_kT_i)\delta_{jh}-(\partial_kT_h)\delta_{ji}=(\partial_jT_i)\delta_{kh}-(\partial_jT_h)\delta_{ki}.$$

It follows that $\partial_jT_i=0$ for $n>2$ and hence T_i are constants, say $T_i=c_i$. Then the equations (3.5) are rewritten in

$$\partial_j\partial_i w^h = c_j\delta_{ih} + c_i\delta_{jh} - c_h\delta_{ji}\,. \tag{3.6}$$

For the indices $j\ i\ h$ different from one another, (3.6) are reduced to $\partial_j\partial_i w^h = 0$, which mean that, for each pair i, h $(i \neq h)$, $\partial_i w^h$ depends on the corresponding coordinates x^i and x^h only. Moreover, for $j = i \neq h$, we have

$$\partial_i\partial_i w^h = -c_h \qquad \text{(not summed in } i\text{)},$$

from which we may put

$$\partial_i w^h = -c_h x^i + \beta_{ih} \qquad (i \neq h),$$

β_{ih} depending on x^h only. Substituting these expressions into (3.6) with $j = h \neq i$, we have $d\beta_{ih}/dx^h = c_i$ (not summed in h) and hence put

$$\partial_i w^h = c_i x^h - c_h x^i + b_{ih} \qquad (i \neq h)$$

b_{ih} being arbitrary constants. Therefore we can write

$$w^h = (\sum_{i\neq h} c_i x^i) x^h - \frac{1}{2} c_h \sum_{i\neq h} (x^i)^2 + \sum_{i\neq h} b_{ih} x^i + \alpha_h\,, \tag{3.7}$$

where, for each value of h, α_h is a function of the corresponding coordinate x^h only. If the second derivative of (3.7) in x^h is substituted into (3.6) with $j = i = h$, we have $d^2\alpha_h/dx^{h2} = c_h$ and hence we put

$$\alpha_h = \frac{1}{2} c_h (x^h)^2 + b_{hh} x^h + a_h \qquad \text{(not summed in } h\text{)}, \tag{3.8}$$

where b_{hh} and a_h are arbitrary constants. From (3.7) and (3.8), it follows that

$$w^h = c_i x^i x^h + \frac{1}{2} c_h R^2 + b_{ih} x^i + a_h\,. \tag{3.9}$$

Substituting these expressions of w^h into the Killing equations (3.4), we see that the coefficients have to satisfy the relations

$$b_{ih} + b_{hi} = 0 \quad \text{and} \quad c_i = \frac{k}{2} a_i\,,$$

and we have finally the expressions

$$w^h = a_i\Big[\Big(1 - \frac{k}{4} R^2\Big)\delta_{ih} + \frac{k}{2} x^i x^h\Big] + \frac{1}{2} b_{ji}(x^j\delta_{ih} - x^i\delta_{jh}) \tag{3.10}$$

as general solutions of (3.4).

Let K be the Lie algebra of isometric vector fields. The algebra K is spanned by $n(n+1)/2$ vector fields

$$\begin{aligned} &W_i\ :\ w^h_{(i)} = \Big(1 - \frac{k}{4} R^2\Big)\delta_{ih} + \frac{k}{2} x^i x^h\,, \\ &W_{ij}:\ w^h_{(ij)} = \delta_{ih} x^j - \delta_{jh} x^i\,. \end{aligned} \tag{3.11}$$

Let K_1 and K_2 be the vector subspaces of K spanned by n vectors W_i and by $n(n-1)/2$ vectors W_{ij} respectively. By computations, we see that $[K, K] \subset K_2$. It is seen for $n=1$ and 2 that the $n(n+1)/2$ vectors of (3.11) satisfy the Killing equations (3.4), that is, they are isometric vector fields, and span the Lie algebra K because $\dim K = n(n+1)/2$.

§ 4. Conformal vector fields in a product Riemannian manifold.

A conformal vector field $V=(v^\kappa)$, or an infinitesimal conformal transformation, in a Riemannian manifold M is characterized by the equations

$$\nabla_\mu v_\lambda + \nabla_\lambda v_\mu = 2\rho g_{\mu\lambda}, \tag{4.1}$$

where ρ is a scalar field and said to be *associated* with V. A concircular vector field is a conformal one, of which the associated scalar field is a C-field.

In this and next paragraphs, we shall consider a product Riemannian manifold, say, the product $M=M_1 \times M_2$ of two Riemannian manifolds M_1 and M_2. Let n_1 and n_2 be the dimensions of M_1 and M_2 respectively; $n_1+n_2=n$. A part, a submanifold isometrc to M_a, through a point P will be denoted by $M_a(P)$, $a=1, 2$. The orthogonal projection of a vector field V, restricted on a part $M_a(P)$, onto M_a is called the restriction of V on M_a and denoted by $V_{(a)}$.

Denote a separate coordinate system in M by (u^h, u^p), (u^h) belonging to M_1 and (u^p) to M_2. Latin indices h, i, j, k will run on the range $1, \cdots, n_1$ and p, q, r, s on the range $n_1+1, \cdots, n$. The metric tensor of $M=M_1 \times M_2$ has the form

$$(g_{\mu\lambda}) = \begin{pmatrix} g_{ji}(u^h) & 0 \\ 0 & g_{rq}(u^p) \end{pmatrix}$$

in such a coordinate system, and the nontrivial components of the Christoffel symbol of M is identical with those of the Christoffel symbols of M_1 and M_2. Therefore covariant differentiations in M along the parts M_1 and M_2 coincide with those in M_1 and M_2 and are commutative with each other. They are denoted by ∇ too, and distinguished with indices belonging to the parts.

We proved in [6] that a conformal vector field V in a product Riemannian manifold $M=M_1 \times M_2$ satisfies the equations

$$\nabla_j v_i + \nabla_i v_j = 2\rho g_{ji}, \qquad \nabla_q v_p + \nabla_p v_q = 2\rho g_{qp}, \tag{4.2}$$

and the associated scalar field ρ does

$$\nabla_j \nabla_i \rho = \phi g_{ji}, \qquad \nabla_q \nabla_p \rho = -\phi g_{qp}. \tag{4.3}$$

These equations mean that the restriction $V_{(a)}$ of V on each part $M_a(P)$ $(a=1, 2)$ through any point P defines a concircular vector field in M_a. Now we shall prove that

THEOREM 3. *Let M be a complete product Riemannian manifold $M_1 \times M_2$ but not a locally Euclidean manifold. Given a nonisometric conformal vector field V in M, the scalar field ρ associated with V satisfies the equations*

$$\nabla_j \nabla_i \rho = -k\rho g_{ji}, \qquad \nabla_q \nabla_p \rho = k\rho g_{qp}, \tag{4.4}$$

k being a nonvanishing constant.

The statement of this theorem was shown in the proof of [7, Theorem 5] under a stronger assumption that n_1 or $n_2 \geqq 3$. The reason was that we utilized properties of concircular vector fields in complete Riemannian manifolds of dimension $\geqq 3$ and it was rather complicated. We give here simpler

PROOF. If $n_1 = n_2 = 1$, the product $M = M_1 \times M_2$ is locally Euclidean. Consequently we may suppose that the dimension of one of the parts, say n_1, is $\geqq 2$. By the first equations of (4.3), the restrictions of ρ on the parts $M_1(P)$ through points P in M are C-fields in M_1. If, among the restrictions, there are C-fields functionally independent of one another, then they are SC-fields with the same characteristic constant by means of Theorem 1. Hence we may put

$$\phi = -k\rho + \beta, \tag{4.5}$$

where β is a function depending on points of M_2.

If all the restrictions are functionally dependent of one another, then we take one of the restrictions, say $\rho_1 = \rho | M_1(P)$ for a point P, and put

$$\rho = \alpha\rho_1 + \gamma, \tag{4.6}$$

by means of Lemma 1 of [8], where α and γ are functions in M_2. Let ϕ_1 be the restriction of ϕ on $M_1(P)$. Then we have

$$\nabla_j \nabla_i \rho_1 = \phi_1 g_{ji}. \tag{4.7}$$

By substituting (4.6) and (4.7) into the equations of (4.3), we have

$$\phi = \alpha\phi_1, \tag{4.8}$$

and the equations

$$(\nabla_q \nabla_p \alpha)\rho_1 + (\nabla_q \nabla_p \gamma) = -\alpha\phi_1 g_{qp}.$$

Differentiating covariantly these equations with respect to u^i belonging to M_1, we have

$$(\nabla_q \nabla_p \alpha)\nabla_i \rho_1 = -\alpha(\nabla_i \phi_1) g_{qp}.$$

Since ρ_1 and ϕ_1 are functions in M_1 and α and g_{qp} are functions in M_2, we may put

$$\nabla_i \phi_1 = -k \nabla_i \rho_1 \tag{4.9}$$

and

(4.10) $$\nabla_q \nabla_p \alpha = k\alpha g_{qp}$$

with a constant coefficient k. It follows from (4.9) that ϕ_1 is of the form

(4.11) $$\phi_1 = -k\rho_1 + a\,,$$

a being a constant, and from (4.8) and (4.11) that ϕ is also written in the form (4.5).

Next we show that β in the expression (4.5) vanishes. In the case of $k \neq 0$, by substituting (4.5) into the first of (4.3), we have

$$\rho g_{ji} = \frac{1}{k}(\beta g_{ji} - \nabla_j \nabla_i \rho)\,.$$

From these equations and the first of (4.2), we obtain the equations

(4.12) $$\nabla_j\Big(v_i + \frac{1}{k}\rho_i\Big) + \nabla_i\Big(v_j + \frac{1}{k}\rho_j\Big) = \frac{2}{k}\beta g_{ji}\,.$$

Since β is constant on $M_1(P)$ through any point P, these equations mean that the components $v_i + \rho_i/k$ define altogether a homothetic vector field on $M_1(P)$. It is known [**3**, **10**], however, that if a complete Riemannian manifold of dimension $\geqq 2$ admits a nonisometric homothetic vector field then the manifold is locally euclidean. Therefore either β should vanish identically or M_1 should be locally Euclidean. However, in the later case, M is locally decomposed into three parts, and it is known [**6**] that, in such a manifold, the scalar field associated with a conformal vector field is always parallel, that is, $k=0$; it is a contradiction.

In the case of $k=0$ and $\beta \neq 0$ at a point P, the restriction $\rho_1 = \rho \,|\, M_1(P)$ is an SC-field with $k=0$ and $b=\beta(P)$ in (1.2). By means of Theorem A, I, B), M_1 is a Euclidean space. In the case of $k=0$ and $\beta=0$, ρ_1 is a parallel scalar field and M_1 is a product manifold by means of Theorem A, I, A). In either of these two cases, M consists of at least three irreducible parts and a conformal vector field in M is reduced to an isometric one, unless M is locally Euclidean by [**7**, Theorem 6.1]. Q. E. D.

Now we define a vector field $Z = (z^\kappa)$ in M by putting

(4.13) $$z^h = g^{hi}\rho_i\,, \qquad z^p = -g^{pq}\rho_q$$

with respect to a separate coordinate system (u^h, u^p). Then we know [**7**, Theorem 7.1] that Z is a conformal vector field satisfying the equations

$$\pounds(Z) g_{\mu\lambda} = \nabla_\mu z_\lambda + \nabla_\lambda z_\mu = -2k\rho g_{\mu\lambda}\,,$$

and a conformal vector field V with associated scalar field ρ is decomposed into

(4.14) $$V = -\frac{1}{k} Z + W_1 + W_2\,,$$

where W_1 and W_2 are isometric vector fields in M_1 and M_2 respectively. This fact can be easily proved from the equations (4.12) with $\beta=0$ and the analogue in M_2.

Without loss of generality, we may assume $k=-c^2<0$. Since M_1 and M_2 admit SC-fields, the part M_1 is a pseudo-hyperbolic space and M_2 a spherical space of curvature c^2. By means of the notice following Theorem 2, M_1 contains a hyperbolic part M_0 of curvature $-c^2$ and Z is regarded as a vector field along the produdt $M_0\times M_2$ in $M=M_1\times M_2$. Hence the properties of conformal vector fields in M are clarified by those of isometric vector fields and of the conformal vector fields, such as Z, derived from scalar fields satisfying (4.4) in the product of space forms.

§ 5. Conformal vector fields in the product of space forms.

Let M_1 and M_2 be hyperbolic and spherical spaces respectively and normalize the metric of $M=M_1\times M_2$ such as $k=-1$ by a homothety. Then the equations (4.4) turn to

$$\nabla_j\nabla_i\rho=\rho g_{ji}\,,\qquad \nabla_q\nabla_p\rho=-\rho g_{qp}\,. \tag{5.1}$$

Now we take coordinate systems (x^h) and (y^p) in M_1 and M_2, with respect to which the metric forms are expressed in

$$ds_1^2=\frac{\sum(dx^h)^2}{\left\{1-\frac{1}{4}\sum(x^h)^2\right\}^2}\,,\qquad ds_2^2=\frac{\sum(dy^p)^2}{\left\{1+\frac{1}{4}\sum(y^p)^2\right\}^2}$$

respectively. We put

$$R_1^2=\sum(x^h)^2\,,\qquad R_2^2=\sum(y^p)^2$$

and

$$S_1=1-\frac{1}{4}R_1^2\,,\qquad S_2=1+\frac{1}{4}R_2^2\,.$$

By use of the results in § 3, the general solution of (5.1) is written in

$$\rho=\frac{1}{S_1S_2}\Big[A\Big(1+\frac{1}{4}R_1^2\Big)\Big(1-\frac{1}{4}R_2^2\Big)+\Big(1-\frac{1}{4}R_2^2\Big)B_ix^i$$
$$+\Big(1+\frac{1}{4}R_1^2\Big)C_py^p+D_{ip}x^iy^p\Big]\,,$$

where A, B_i, C_p and D_{ip} are arbitrary constants. Hence the vector space of solutions of (5.1) is spanned by the four kinds of the following scalar fields:

$$\rho_{(o)}=\frac{1}{S_1S_2}\Big(1+\frac{1}{4}R_1^2\Big)\Big(1-\frac{1}{4}R_2^2\Big)\,,$$
$$\rho_{(i)}=\frac{1}{S_1S_2}\Big(1-\frac{1}{4}R_2^2\Big)x^i\,,$$

$$\rho_{(p)}=\frac{1}{S_1S_2}\left(1+\frac{1}{4}R_1^2\right)y^p,$$

$$\rho_{(ip)}=\frac{1}{S_1S_2}x^iy^p.$$

The corresponding vector fields defined by (4.13) are given by

$$(5.2)\quad \begin{aligned} &Z_o\left\{\begin{aligned} z^h_{(o)}&=\left(1-\frac{1}{4}R_2^2\right)x^h/S_2,\\ z^s_{(o)}&=\left(1+\frac{1}{4}R_1^2\right)y^s/S_1,\end{aligned}\right.\\ &Z_i\left\{\begin{aligned} z^h_{(i)}&=\left(1-\frac{1}{4}R_2^2\right)\left(S_1\delta_{ih}+\frac{1}{2}x^ix^h\right)/S_2,\\ z^s_{(i)}&=x^iy^s/S_1,\end{aligned}\right.\\ &Z_p\left\{\begin{aligned} z^h_{(p)}&=x^hy^p/S_2,\\ z^s_{(p)}&=-\left(1+\frac{1}{4}R_1^2\right)\left(S_2\delta_{ps}-\frac{1}{2}y^py^s\right)/S_1,\end{aligned}\right.\\ &Z_{ip}\left\{\begin{aligned} z^h_{(ip)}&=\left(S_1\delta_{ih}+\frac{1}{2}x^ix^h\right)y^p/S_2,\\ z^s_{(ip)}&=-\left(S_2\delta_{ps}-\frac{1}{2}y^py^s\right)x^i/S_1.\end{aligned}\right. \end{aligned}$$

Let L be the Lie algebra of conformal vector fields in $M=M_1\times M_2$, and L_0, L_1, L_2 and L_{12} the vector subspaces of L spanned by Z_0, Z_i, Z_p and Z_{ip} respectively.

On the other hand, it is known that an isometric vector fields in a product Riemannian manifold is decomposed into the sum of isometric ones in the parts, [2]. Hence, from the results in § 3, the Lie algebra K of isometric vector fields in M is spanned by four kinds of the following vector fields:

$$(5.3)\quad \left\{\begin{aligned} &W_i\ :\ w^h_{(i)}=\left(1+\frac{1}{4}R_1^2\right)\delta_{ih}-\frac{1}{2}x^ix^h, && w^s_{(i)}=0,\\ &W_{ij}:\ w^h_{(ij)}=\delta_{ih}x^j-\delta_{jh}x^i, && w^s_{(ij)}=0,\\ &W_p\ :\ w^h_{(p)}=0, && w^s_{(p)}=\left(1-\frac{1}{4}R_2^2\right)\delta_{ps}+\frac{1}{2}y^py^s,\\ &W_{pq}:\ w^h_{(pq)}=0, && w^s_{(pq)}=\delta_{ps}y^q-\delta_{qs}y^p.\end{aligned}\right.$$

Let K_{11}, K_{12}, K_{21} and K_{22} be the subspaces of K spanned by W_i, W_{ij}, W_p and W_{pq} respectively.

By straightforward computations, the bracket products of the vector fields

W's with Z's are listed in

$$[W_i,\ Z_0]=Z_i\,,$$
$$[W_i,\ Z_j]=\delta_{ij}Z_0\,,$$
$$[W_i,\ Z_p]=Z_{ip}\,,$$
$$[W_i,\ Z_{jp}]=\delta_{ij}Z_p\,,$$
$$[W_{ij},\ Z_0]=0\,,$$
$$[W_{ij},\ Z_k]=\delta_{ik}Z_j-\delta_{jk}Z_i\,,$$
$$[W_{ij},\ Z_p]=0\,,$$
$$[W_{ij},Z_{kp}]=\delta_{ik}Z_{jp}-\delta_{jk}Z_{ip}\,,$$
$$[W_p,\ Z_0]=-Z_p\,,$$
$$[W_p,\ Z_i]=-Z_{ip}\,,$$
$$[W_p,\ Z_q]=\delta_{pq}Z_0\,,$$
$$[W_p,\ Z_{iq}]=\delta_{pq}Z_i\,,$$
$$[W_{pq},\ Z_0]=0\,,$$
$$[W_{pq},\ Z_i]=0\,,$$
$$[W_{pq},\ Z_r]=\delta_{pr}Z_q-\delta_{qr}Z_p\,,$$
$$[W_{pq},Z_{ir}]=\delta_{pr}Z_{iq}-\delta_{qr}Z_{ip}\,.$$

The relation of inclusion of the above bracket products in L's is shown in the following table:

	L_0	L_1	L_2	L_{12}
K_{11}	L_1	L_0	L_{12}	L_2
K_{12}	0	L_1	0	L_{12}
K_{21}	L_2	L_{12}	L_0	L_1
K_{22}	0	0	L_2	L_{12}

From this table, we see that a number of times of bracket products of any conformal vector field with suitable isometric vector fields is contained in L_0, for example,

$$[K_{21},[K_{11},[K_{22},[K_{12},\ L]]]]\subset L_0\,.$$

Now we are going to give the

PROOF OF MAIN THEOREM. In virtue of de Rham's theorem [**1**], the universal covering space of a complete reducible Riemannian manifold M is a product space. The completeness of M and of vector fields in M is carried into the covering space. We may therefore assume that M is a complete product Riemannian manifold. Let V be a nonisometric conformal vector field with associated scalar field ρ in M. Since an isometric vector field is complete in a complete Riemannian manifold [3] and the group of conformal transformations is a Lie group, the decomposition (4.14) implies that, if V would be complete in M, then so would be the conformal vector field Z defined by (4.13). In order to prove the theorem, it is sufficient to show that Z is not complete in the product of a hyperbolic space M_1 with a spherical space M_2. Since the product manifold $M_1 \times M_2$ admits the isometric vector fields W's in (5.3) and a number of times of bracket products of Z with W's becomes Z_0 multiplied with a non-zero constant factor, we suffice to show that the conformal vector field Z_0 is not complete in $M_1 \times M_2$.

Let O be the point with coordinates $x^h = 0$, $y^p = 0$. The vector field Z_0 has components

$$z^h_{(0)} = x^h, \qquad z^p_{(0)} = 0$$

on the part $M_1(O)$ through the point O. Hence the part $M_1(O)$ is invariant under transformations of Z_0 and the trajectories of points of $M_1(O)$ satisfy the equations

$$\frac{dx^h}{dt} = x^h \tag{5.4}$$

with respect to the canonical parameter t of Z_0. The trajectory issuing from a point with coordinate x^h_0 in $M_1(O)$ is given by

$$x^h = x^h_0 \exp t$$

and overruns the boundary of the ball $R_1 < 2$ within a finite value of t. Thus the vector field Z_0 is not complete. Q. E. D.

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