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Topological covering of $SL(2)$ over a local field

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The purpose of this paper is to show that, if F is a local field different from the field $\boldsymbol{C}$ of complex numbers, and if F contains the m-th roots of unity, then the topological group $SL(2, F)$ has an m-fold, non-trivial, topological covering group which is of a fairly number-theoretical nature. In the sequel, a local field will always mean a completion, by a finite or infinite place, of an algebraic number field of finite degree, and $SL(2, F)$ will mean the topological group of all 2×2 matrices with determinant 1 over a local field F.

We shall obtain a topological covering of $SL(2, F)$ by proving in a elementary way that an expression containing Hilbert's symbols is actually a factor set of $SL(2, F)$.

Hilbert's symbol[1] of degree m of a local field F containing the m-th roots of unity will be denoted by (α, β), where α, β are non-zero numbers of F. In addition to fundamental properties of Hilbert' symbol, the relation

$$(1) \qquad (\alpha, \beta)(-\alpha^{-1}\beta, \alpha+\beta)=1$$

is useful in our arguments. This formula is valid whenever $\alpha\neq 0$, $\beta\neq 0$, and $\alpha+\beta\neq 0$[2].

Now, our result is the following

THEOREM. *Let $F\neq \boldsymbol{C}$ be a local field containing the m-th roots of unity, let $G=SL(2, F)$, and, for an element $\sigma=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in G$, put $x(\sigma)=\gamma$ or δ according to $\gamma\neq 0$ or $=0$. Furthermore, for $\sigma, \tau\in G$, put*

$$(2) \qquad a(\sigma, \tau)=(x(\sigma), x(\tau))(-x(\sigma)^{-1}x(\tau), x(\sigma\tau)).$$

Then, $a(\sigma, \tau)$ satisfies the factor set relation

$$(3) \qquad a(\sigma, \tau)a(\sigma\tau, \rho)=a(\sigma, \tau\rho)a(\tau, \rho)$$

for any $\sigma, \tau, \rho\in G$, and determines an m-fold topological covering group of G.

The proof will be performed in § 2, and the non-triviality of the covering in the theorem will be treated in § 3.

From the form of the factor set (2), it is understood that our theorem has

1) As for the definition of Hilbert's symbol, see [**1**].
2) [**1**], p. 55, formula (15).

a global meaning as well as local one. Namely, let F temporarily denote an algebraic number field containing the m-th roots of unity, let $F_{\mathfrak{p}}$ be the completion of F by a place $\mathfrak{p}$ of F, and let $a(\sigma, \tau/\mathfrak{p})$ be the factor set (2) for $F_{\mathfrak{p}}$. Then, each $SL(2, F_{\mathfrak{p}})$ is given a covering group by $a(\sigma, \tau/\mathfrak{p})$, and moreover the product formula

$$\prod_{\mathfrak{p}} a(\sigma, \tau/\mathfrak{p}) = 1 \tag{4}$$

holds, which means that the set of all $a(\sigma, \tau/\mathfrak{p})$ determines a global covering group of the adele group of $SL(2, F)$.

A global covering group of this kind was constructed in [4] for the case of $m=2$. For the multiplicative group of a number field, a related investigation was done also in [2]. On the other hand, the results in the present paper have some overlap with the work of Calvin Moore[3]. But, he and the author were working independently.

§ 1. Lemmas.

As before, we denote by F a local field containing the m-th roots of unity, and we write $G=SL(2, F)$. Furthermore, we use the notation $\gamma(\sigma)=\gamma$ for $\sigma=\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in G$.

LEMMA 1. *If σ, τ are two elements of G with $\gamma(\sigma) \neq 0$, $\gamma(\tau)=0$, then*

$$a(\sigma, \tau) = a(\tau, \sigma) = (x(\sigma), x(\tau)) \tag{5}$$

holds for $a(\sigma, \tau)$ in (2). *If $\gamma(\sigma)=\gamma(\tau)=0$, then*

$$a(\sigma, \tau) = (x(\sigma), x(\tau))^{-1}\,. \tag{6}$$

PROOF. If $\gamma(\sigma) \neq 0$, $\gamma(\tau)=0$, then

$$x(\sigma\tau) = x(\sigma)x(\tau)^{-1}\,, \qquad x(\tau\sigma) = x(\tau)x(\sigma)\,. \tag{7}$$

So, by definition, we have

$$\begin{aligned} a(\sigma, \tau) &= (x(\sigma), x(\tau))(-x(\sigma)^{-1}x(\tau), x(\sigma)^{-1}x(\tau))^{-1} \\ &= (x(\sigma), x(\tau))\,, \\ a(\tau, \sigma) &= (x(\tau), x(\sigma))(-x(\tau)^{-1}x(\sigma), x(\tau))(-x(\tau)^{-1}x(\sigma), x(\sigma)) \\ &= (x(\sigma), x(\tau))\,, \end{aligned}$$

which proves (5). If $\gamma(\sigma)=\gamma(\tau)=0$, then $x(\sigma\tau)=x(\sigma)x(\tau)$. Therefore, (6) is shown in the same way as the above equality for $a(\tau, \sigma)$.

LEMMA 2. *Fix an element $\sigma \in G$. Then,*

3) Unpublished at present.

$$a(\varepsilon, \sigma)=a(\sigma, \varepsilon)=\begin{cases} 1, & \gamma(\sigma)\neq 0, \\ (x(\varepsilon), x(\sigma)), & \gamma(\sigma)=0, \end{cases}$$

for $\varepsilon \in G$ *which is sufficiently near to* 1.

PROOF. We have

$$a(\varepsilon, \sigma)=(x(\varepsilon), x(\sigma))(-x(\varepsilon)^{-1}x(\sigma), x(\varepsilon\sigma))$$

by definition. If $\gamma(\varepsilon)=0$ or $\gamma(\sigma)\neq 0$, then $x(\varepsilon\sigma)\to x(\sigma)$ as $\varepsilon\to 1$, and $a(\varepsilon, \sigma) \to(-x(\sigma), x(\sigma))=1$ because of the continuity of Hilbert's symbol. So, we may assume $\gamma(\varepsilon)\neq 0$ and $\gamma(\sigma)=0$. This case reduces to Lemma 1.

LEMMA 3. *Let* $\sigma, \tau \in G$ *be given. Then, two special cases*

$$a(\varepsilon, \sigma)a(\varepsilon\sigma, \tau)=a(\varepsilon, \sigma\tau)a(\sigma, \tau) \tag{8}$$

and

$$a(\sigma, \tau)a(\sigma\tau, \varepsilon)=a(\sigma, \tau\varepsilon)a(\tau, \varepsilon) \tag{9}$$

of (3) *hold for* $\varepsilon=\begin{pmatrix}1 & \\ \mu & 1\end{pmatrix}=\begin{pmatrix}1 & 0\\ \mu & 1\end{pmatrix}\in G$, *provided that* μ *is sufficiently near to* 0.

PROOF. To prove (8), it is enough to prove

$$a(\varepsilon, \sigma)a(\varepsilon\sigma, \sigma^{-1}\tau)=a(\varepsilon, \tau)a(\sigma, \sigma^{-1}\tau). \tag{10}$$

Consider

$$\begin{aligned} &a(\varepsilon\sigma, \sigma^{-1}\tau)a(\sigma, \sigma^{-1}\tau)^{-1} \\ &\quad=(x(\varepsilon\sigma), x(\sigma^{-1}\tau))(x(\sigma), x(\sigma^{-1}\tau))^{-1}(-x(\varepsilon\sigma)^{-1}x(\sigma^{-1}\tau), x(\varepsilon\tau)) \\ &\qquad\cdot(-x(\sigma)^{-1}x(\sigma^{-1}\tau), x(\tau))^{-1}, \end{aligned} \tag{11}$$

and suppose $\gamma(\sigma)\neq 0$, $\gamma(\tau)\neq 0$. Then, $x(\varepsilon\sigma)\to x(\sigma)$, $x(\varepsilon\tau)\to x(\tau)$, and therefore (11) tends to 1, as $\mu\to 0$. Suppose next $\gamma(\sigma)\neq 0$, $\gamma(\tau)=0$. Then, it follows from (7) and $x(\sigma^{-1})=-x(\sigma)$ that (11) is equal to

$$(-x(\sigma)^{-1}x(\sigma^{-1}\tau), x(\varepsilon\tau)x(\tau)^{-1})=(x(\tau)^{-1}, \mu x(\tau)^{-2})=(\mu, x(\tau))$$

for sufficiently small μ. If $\gamma(\sigma)=0$, $\gamma(\tau)\neq 0$, it is similarly seen that (11) tends to

$$(\mu x(\sigma)^{-2}, x(\sigma)^{-1}x(\tau))(\mu^{-1}x(\sigma)^{2}, x(\tau))=(\mu, x(\sigma))^{-1}.$$

If finally $\gamma(\sigma)=\gamma(\tau)=0$, then (11) becomes

$$(\mu x(\sigma)^{-2}, x(\sigma)^{-1}x(\tau))(-\mu^{-1}x(\tau), \mu x(\tau)^{-1})(-x(\sigma)^{-2}x(\tau), x(\tau))^{-1}=(\mu, x(\sigma)^{-1}x(\tau))$$

when μ is sufficiently small. Since $\mu=\gamma(\varepsilon)$, (8) follows from Lemma 2 in every case.

To prove (9), it is enough to prove

$$a(\sigma\tau^{-1}, \tau)a(\sigma, \varepsilon)=a(\sigma\tau^{-1}, \tau\varepsilon)a(\tau, \varepsilon). \tag{12}$$

The value of

$$(13)\qquad a(\sigma\tau^{-1}, \tau\varepsilon)a(\sigma\tau^{-1}, \tau)$$
$$=(x(\sigma\tau^{-1}), x(\tau\varepsilon))(x(\sigma\tau^{-1}), x(\tau))^{-1}(-x(\sigma\tau^{-1})^{-1}x(\tau\varepsilon), x(\sigma\varepsilon))$$
$$\cdot(-x(\sigma\tau^{-1})^{-1}x(\tau), x(\sigma))^{-1}$$

as $\mu\to 0$ is 1 for $\gamma(\sigma)\neq 0$, $\gamma(\tau)\neq 0$,

$$(x(\sigma)x(\tau), \mu)(\mu, x(\sigma))=(\mu, x(\tau))^{-1}$$

for $\gamma(\sigma)\neq 0$, $\gamma(\tau)=0$,

$$(-x(\sigma)^{-1}x(\tau^{-1})^{-1}x(\tau), \mu)=(\mu, x(\sigma))$$

for $\gamma(\sigma)=0$, $\gamma(\tau)\neq 0$, and

$$(x(\sigma)x(\tau)^{-1}, \mu)(-\mu x(\sigma)^{-1}x(\tau)^2, \mu x(\sigma))(-x(\sigma)^{-1}x(\tau)^2, x(\sigma))^{-1}$$
$$=(x(\sigma)x(\tau)^{-1}, \mu)(x(\sigma)^{-1}x(\tau)^2, \mu)(\mu, x(\sigma))$$
$$=(\mu, x(\sigma))(\mu, x(\tau))^{-1}$$

for $\gamma(\sigma)=0$, $\gamma(\tau)=0$. Hence, Lemma 2 implies (9).

Thus, the lemma is completely proved.

LEMMA 4. *Set*

$$b(\sigma, \tau, \rho)=a(\sigma, \tau)a(\sigma\tau, \rho)a(\sigma, \tau\rho)^{-1}a(\tau, \rho)^{-1},$$

$(\sigma, \tau, \rho\in G)$. Then, for any fixed $\sigma, \tau, \rho\in G$, we have

$$b(\varepsilon\sigma, \tau, \rho)=b(\sigma, \tau, \rho\varepsilon)=b(\sigma, \tau, \rho),$$

where $\varepsilon=\begin{pmatrix}1 & \\ \mu & 1\end{pmatrix}$, *and μ is sufficiently near to 0.*

PROOF. If ε is close to 1, then (8) of Lemma 3 yields

$$b(\varepsilon\sigma, \tau, \rho)=a(\varepsilon\sigma, \tau)a(\varepsilon\sigma\tau, \rho)a(\varepsilon\sigma, \tau\rho)^{-1}a(\tau, \rho)^{-1}$$
$$=a(\varepsilon, \sigma)^{-1}a(\varepsilon, \sigma\tau)a(\sigma, \tau)a(\varepsilon, \sigma\tau)^{-1}a(\varepsilon, \sigma\tau\rho)a(\sigma\tau, \rho)$$
$$\cdot a(\varepsilon, \sigma)a(\varepsilon, \sigma\tau\rho)^{-1}a(\sigma, \tau\rho)^{-1}a(\tau, \rho)^{-1}$$
$$=a(\sigma, \tau)a(\sigma\tau, \rho)a(\sigma, \tau\rho)^{-1}a(\tau, \rho)^{-1}=b(\sigma, \tau, \rho).$$

On the other hand, (9) of Lemma 3 yields

$$b(\sigma, \tau, \rho\varepsilon)=a(\sigma, \tau)a(\sigma\tau, \rho\varepsilon)a(\sigma, \tau\rho\varepsilon)^{-1}a(\tau, \rho\varepsilon)^{-1}$$
$$=a(\sigma, \tau)a(\sigma\tau, \rho)a(\sigma\tau\rho, \varepsilon)a(\rho, \varepsilon)^{-1}$$
$$\cdot a(\sigma, \tau\rho)^{-1}a(\sigma\tau\rho, \varepsilon)^{-1}a(\tau\rho, \varepsilon)a(\tau, \rho)^{-1}a(\tau\rho, \varepsilon)^{-1}a(\rho, \varepsilon)$$
$$=a(\sigma, \tau)a(\sigma\tau, \rho)a(\sigma, \tau\rho)^{-1}a(\tau, \rho)^{-1}=b(\sigma, \tau, \rho),$$

which completes the proof.

§ 2. Proof of the theorem.

Using the notation of Lemma 4, the assertion of our theorem is that $b(\sigma, \tau, \rho)=1$ for every $\sigma, \tau, \rho \in G$. While $b(\sigma, \tau, \rho)=1$ means that G has an abstract, m-fold covering group, Lemma 2 entails $a(\sigma, \tau)=1$ whenever σ, τ are near to 1; therefore the covering group posesses a topology locally isomorphic to G, and is a topological covering group.

Let now $\sigma=\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in G$ and $\varepsilon=\begin{pmatrix}1 & \\ \mu & 1\end{pmatrix} \in G$. Then, there is at most one μ with $\gamma(\varepsilon\sigma)=0$, because such μ is a solution of $\alpha\mu+\gamma=0$. Similarly, there is at most one μ with $\gamma(\sigma\varepsilon)=0$, such μ being a solution of $\delta\mu+\gamma=0$. From this and from Lemma 4, it follows that there exists $\varepsilon \in G$ with $b(\varepsilon\sigma, \tau, \rho)=b(\sigma, \tau, \rho)$, $\gamma(\varepsilon\sigma) \neq 0$, $\gamma(\varepsilon\sigma\tau) \neq 0$. Analogously, one can find $\varepsilon' \in G$ such that $b(\varepsilon\sigma, \tau, \rho\varepsilon')=b(\varepsilon\sigma, \tau, \rho)$, $\gamma(\rho\varepsilon') \neq 0$, $\gamma(\tau\rho\varepsilon') \neq 0$, $\gamma(\varepsilon\sigma\tau\rho\varepsilon') \neq 0$. Hence, our theorem may be proved under the assumptions $\gamma(\sigma) \neq 0$, $\gamma(\rho) \neq 0$, $\gamma(\sigma\tau) \neq 0$, $\gamma(\tau\rho) \neq 0$, $\gamma(\sigma\tau\rho) \neq 0$.

Denote by N the subgroup of G consisting of all elements of the form $\begin{pmatrix}1 & \beta \\ & 1\end{pmatrix}$. Then, since $\gamma(\nu\sigma\nu')=\gamma(\sigma)$, $(\nu, \nu' \in N, \sigma \in G)$, the definition of $a(\sigma, \tau)$ implies

$$a(\nu\sigma, \tau)=a(\sigma, \tau\nu)=a(\sigma, \tau), \qquad a(\sigma\nu, \tau)=a(\sigma, \nu\tau) \tag{14}$$

for every $\sigma, \tau \in G$, and $\nu \in N$. Consequently, we have

$$b(\sigma, \tau, \rho)=b(\nu_1\sigma\nu_2, \nu_2^{-1}\tau\nu_3, \nu_3^{-1}\rho\nu_4) \tag{15}$$

for $\sigma, \tau, \rho \in G$, $\nu_1, \nu_2, \nu_3, \nu_4 \in N$. On the other hand, $\sigma \in G$ being an arbitrary element, there are $\nu, \nu' \in N$ such that $\nu\sigma\nu'$ is in a canonical form $\begin{pmatrix}\alpha & \\ & \alpha^{-1}\end{pmatrix}$ or $\begin{pmatrix} & -\gamma^{-1} \\ \gamma & \end{pmatrix}$. Furthermore, since $\gamma(\sigma) \neq 0$, and $\gamma(\rho) \neq 0$, $\nu_1\sigma\nu_2$ resp. $\nu_3^{-1}\rho\nu_4$ can be brought into a triangular form $\begin{pmatrix} & \beta \\ \gamma & \delta\end{pmatrix}$ resp. $\begin{pmatrix}\alpha & \beta \\ \gamma & \end{pmatrix}$. Owing to these facts, it is enough to prove the theorem for

$$\sigma=\begin{pmatrix} & -a^{-1} \\ a & d\end{pmatrix}, \qquad \tau=\begin{pmatrix} & -b^{-1} \\ b & \end{pmatrix}, \qquad \rho=\begin{pmatrix}e & -c^{-1} \\ c & \end{pmatrix}$$

when $\gamma(\tau) \neq 0$, and for

$$\sigma=\begin{pmatrix} & -a^{-1} \\ a & d\end{pmatrix}, \qquad \tau=\begin{pmatrix}b^{-1} & \\ & b\end{pmatrix}, \qquad \rho=\begin{pmatrix}e & -c^{-1} \\ c & \end{pmatrix}$$

when $\gamma(\tau)=0$. In both cases, $x(\sigma)=a$, $x(\tau)=b$, $x(\rho)=c$.

In the case of $\gamma(\tau) \neq 0$, we have $x(\sigma\tau)=bd$, $x(\tau\rho)=be$, $x(\sigma\tau\rho)=bde-ab^{-1}c$ by the assumption $\gamma(\sigma\tau) \neq 0$, $\gamma(\tau\rho) \neq 0$, $\gamma(\sigma\tau\rho) \neq 0$. So,

$$b(\sigma, \tau, \rho)=(a, b)(-a^{-1}b, bd)(bd, c)(-b^{-1}d^{-1}c, bde-ab^{-1}c)$$
$$\cdot(a, be)^{-1}(-a^{-1}be, bde-ab^{-1}c)^{-1}(b, c)^{-1}(-b^{-1}c, be)^{-1}$$
$$=(a, e)^{-1}(d, c)(a, b)^{-1}(-a^{-1}b, d)(c, b)^{-1}(-b^{-1}c, e)^{-1}$$
$$\cdot((bde)^{-1}ab^{-1}c, bde-ab^{-1}c).$$

Since here

$$((bde)^{-1}ab^{-1}c, bde-ab^{-1}c)=(-ab^{-1}c, bde)$$
$$=(ac, b)(-ab^{-1}, d)(c, d)(a, e)(-b^{-1}c, e)$$

follows from (1), we obtain

$$b(\sigma, \tau, \rho)=(a, b)^{-1}(c, b)^{-1}(ac, b)=1.$$

In the case of $\gamma(\tau)=0$, we have $x(\sigma\tau)=ab^{-1}$, $x(\tau\rho)=bc$, $x(\sigma\tau\rho)=ab^{-1}e+bcd$. So, from Lemma 1 follows

$$b(\sigma, \tau, \rho)=(a, b)(ab^{-1}, c)(-a^{-1}bc, ab^{-1}e+bcd)$$
$$\cdot(a, bc)^{-1}(-a^{-1}bc, ab^{-1}e+bcd)^{-1}(c, b)^{-1}=1.$$

This completes the proof.

§3. Non-triviality of the covering.

The aim of this § is to show that the covering given in the theorem is always non-trivial; we propose to show that there is no mapping s of G into the set of the m-th roots of unity such that

$$a(\sigma, \tau)=s(\sigma)s(\tau)s(\sigma\tau)^{-1}. \tag{16}$$

In the case of $m>2$, the impossibility of (16) is immediately to see, if we put $\sigma=\begin{pmatrix}\alpha & \\ & \alpha^{-1}\end{pmatrix}$, $\tau=\begin{pmatrix}\beta & \\ & \beta^{-1}\end{pmatrix}$, and use (6) of Lemma 1 together with the anti-symmetry of Hilbert's symbol. Since, however, this method is not effective for $m=2$, we state here from a different point of view a proof which is valid for arbitrary m[4]. Assume (16), and set $\nu=\begin{pmatrix}1 & \beta\\ & 1\end{pmatrix}$. Then, (14) yields

$$s(\nu\sigma)s(\tau)s(\nu\sigma\tau)^{-1}=s(\sigma)s(\tau)s(\sigma\tau)^{-1}$$

for every $\sigma, \tau\in G$. Putting further $\tau=\sigma^{-1}$, we have $s(\nu\sigma)=s(\sigma)^{-1}=s(\nu)s(1)^{-1}$, while the mapping s becomes a homomorphism on the group N of the elements of the form $\begin{pmatrix}1 & \beta\\ & 1\end{pmatrix}$, and consequently is trivial. Therefore, $s(\nu\sigma)=s(\sigma)$ must hold. By a similar reason, $s(\sigma\nu)=s(\sigma)$ is also the case. Set $\sigma=\begin{pmatrix} & -1\\ 1 & \end{pmatrix}$, $\tau=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$, $(\gamma\neq 0,\ \alpha\neq 0)$, $\sigma\tau=\begin{pmatrix}-\gamma & -\delta\\ \alpha & \beta\end{pmatrix}$, and denote by $s_1(\gamma)$ the value of

4) The non-triviality of the covering for the case of $m=2$ is contained in [4], too.

s at $\begin{pmatrix} & -\gamma^{-1} \\ \gamma & \end{pmatrix}$. Then, from the definition, from (16), and from the N-invariance of s proved above, the equality

$$a(\sigma, \tau)=(-\gamma, \alpha)=s_1(1)s_1(\gamma)s_1(\alpha)^{-1}$$

is derived. This relation with $\alpha=1$ implies $s_1(\gamma)=1$ for any non-zero $\gamma \in F$. Accordingly, we have $(-\gamma, \alpha)=1$ identically, which is clearly a contradiction.

Let $\tilde{G}$ be the covering group of G determined by the theorem. Then, $\tilde{G}$ contains the group $\mathfrak{z}(m)$ of the m-th roots of unity, and the covering map $\tilde{G}\to G$ gives an isomorphism of G and $\tilde{G}/\mathfrak{z}(m)$. Let now $d \neq m$ be a natural number deviding m. Then, $\tilde{G}/\mathfrak{z}(d)$ is also a covering group of G, and the corresponding factor set is given by the natural image of $a(\sigma, \tau)$ into $\mathfrak{z}(m)/\mathfrak{z}(d)$. Since $\mathfrak{z}(m)/\mathfrak{z}(d) \cong \mathfrak{z}(m/d)$, we may regard $a(\sigma, \tau)^d$ as the factor set. The d-th power of Hilbert's symbol of degree m is Hilbert's symbol of degree m/d, which is not trivial in the present case. So, $a(\sigma, \tau)^d$ is also not trivial. Thus we have proved that all coverings $\tilde{G}/\mathfrak{z}(d)\to G$ are non-trivial as well as $\tilde{G}\to G$. In this sense, $\tilde{G}\to G$ is a proper covering.

§ 4. Further remarks.

Hilbert's symbol is a bi-multiplicative function, and is by itself a factor set of the multiplicative group of a field. Therefore, our factor set (2) can be regarded as a generalization of Hilbert's symbol to a matric algebra. In fact, the product formula (4) holds for global cases, and $a(\sigma, \tau)$ coincides with Hilbert's symbol when σ, τ are diagonal matrices. Moreover, it is possible for any other property of Hilbert's symbol to find the corresponding property of $a(\sigma, \tau)$. On the other hand, using an order of a matric algebra, one can consider through $a(\sigma, \tau)$ power residue symbols of a matric algebra. The multiplicativity of symbols for commutative cases should, however, always be replaced by the factor set relation.

Hilbert's symbol as a factor set of the multiplicative group of a field was partly investigated in [2] for the quadratic case. The formula (17) of [2] means that the Gauss sum gives rise to a multiplicative function on the 2-fold covering group, given by Hilbert's symbol, of the idele class group.

Something about a factor set is also mentioned at the very end of [3]. What the factor set determines is an ordinary, 2-fold topological covering group of $G=SL(2, R)$, which appears in the theorem of this paper for the special case of $F=\boldsymbol{R}$, $m=2$.

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