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Transformations of flows

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§ 0. Preliminaries.

Given a topological space (X, σ) with the family of open sets σ, the corresponding topological measurable space is defined to be the pair $(X, \mathcal{B}_\sigma)$, where $\mathcal{B}_\sigma$ is the family of Borel sets. If in particular σ is defined by a metric with distance function d, then we write it $\mathcal{B}_d$ instead of $\mathcal{B}_\sigma$, Furthermore, if the topological measurable space is furnished with a measure ν, the completion of $\mathcal{B}_\sigma$ or $\mathcal{B}_d$ will be denoted by $\mathcal{L}_\sigma$ or $\mathcal{L}_d$ respectively.

In this paper, for basic measure space is taken Lebesgue space $(M, \mathcal{L}, \mu)$, where $\mathcal{L}$ is the underlying σ-algebra, complete under μ, and $\mu(M)=1$[1].

Let (Ω, d) be a complete metric separable space with distance function d, and probability measure P over $\mathcal{B}_d$. Then $(\Omega, \mathcal{L}_d, P)$ is a Lebesgue space. The family of Lebesgue spaces coincides with that of such probability spaces; any Lebesgue space is isomorphic with $[0, 1]$ endowed with an ordinary probability distribution.

A flow[2] on M is a set $\mathfrak{S}=\{M, T_t, \mu\}=\{M, T_t\}$, where $t \in T=(-\infty, \infty)$, T_t is a one-parameter group of automorphisms on M. In the theory of flows, the choice of Lebesgue space for underlying measure space eliminates measure-theoretic complexities. This depends on the separability properties specific to every Lebesgue space.

Now given a flow $\mathfrak{S}=\{M, T_t, \mu\}$, one can carry it over a metric space Ω, getting a flow $\mathfrak{S}'=\{\Omega, S_t, P\}$ which is isomorphic with $\mathfrak{S}$, with the path $\omega_t=T_t\omega$, $\omega \in \Omega$, continuous in (t, ω). Such a method of representing a given flow on a metric space goes back to Ambrose and Kakutani [**2**].

Among those representations a convenient one is that in which the underlying space Ω consists of measurable t-functions, and S_t is the one-parameter family of shifts acting on Ω. For such a flow (Ω, S_t, P), the trajectory $S_t\omega$, $-\infty<t<\infty$, is also continuous in (t, ω). The advantage of this method consists in the fact that the regularity of the paths and metric topology in

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1) For basic properties of the Lebesgue space, see Rohlin [**6**].

2) Rohlin [**7**] presents an excellent exposition of the theory of automorphisms and flows developed up to 1949.

Ω often simplify arguments.

In the analysis of flows, very often required is its measurability but not mere continuity. For such examples from recent studies we may quote the determination of the spectral type of a Kolmogorov flow (Sinai [**10**]) and computation of the entropy of a flow (Abramob [**1**]). However, the definitions of the spectral type or the entropy of a flow do not require its measurability. So that as Rohlin ([8], p. 10, pp. 21-22) points out, it would be useful to find a general principle which makes those problems free from the measurability. For this purpose, Theorem 3 will provide a general way. Results obtained are related to the theory of stationary processes.

In the beginning sections we will prove fundamental theorems on the isomorphism of flows or 1-parameter semi-groups of endomorphisms. Then they are applied, on the one hand, to give a direct proof of Sinai's theorem on the spectra of the Kolmogorov flows, and on the other hand, to construct a natural extension of a 1-parameter semi-group[3].

§ 1. Fundamental concepts.

When a proposition depending on points of M is true except on a set of μ-measure zero, we say that it is true (mod 0).

A measurable partition ζ on M is generated by an at most denumerable system of measurable sets $\{B_n\}_1^\infty$, a base of ζ. The corresponding factor space will be denoted by M/ζ. M is the sum of disjoint sets C of the form $C = A_1 A_2 \cdots$, $A_i = B_i$ or B_i^c. C is called a ζ-cell. A set represented as a sum of ζ-cells will be called a ζ-set. The system of measurable sets of the form $D = A_1 A_2 \cdots A_m$ also serves as a base of ζ (multiplicative base). By $\mathcal{B}(\zeta)$ and $\mathcal{L}(\zeta)$ will be denoted the σ-algebra generated by $\{B_n\}_1^\infty$ and its completion under μ. There are two extreme partitions ν and ε, ε the partition into individual points, whereas ν consists of the single cell M. Thus $\mathcal{L}(\varepsilon) = \mathcal{L}$.

In the following, two time domains $T = (-\infty, \infty)$ and $T_+ = [0, \infty)$ are used. We metrize the space $L_0(M)$ of all real-valued measurable functions on M by the distance

$$\delta(f, g) = \int_M \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} d\mu ,$$

$$f, g \in L_0(M) .$$

Next consider the space $L_0(T)$, $L_0(T_t)$ of real-valued measurable, with respect to the ordinary Lebesgue measure on T and T_+, functions and metrize them

3) Rohlin [**9**] discussed the same problem for the time-discrete semi-groups of endomorphisms.

respectively by

$$d(f, g)=\int_{-\infty}^{\infty}p(u, f(u)-g(u))du\ ,$$

$$d_+(f, g)=\int_0^{\infty}p(u, f(u)-g(u))du\ ,$$

where

$$p(u, x)=\frac{1}{1+u^2}\frac{|x|}{1+|x|}\ .$$

An endomorphism T over M is a measurable and measure-preserving point transformation from M onto itself. Further, if T is 1—1, then it is an automorphism. Besides such a (strict) endomorphism and automorphism, there are their variations. A measurable and measure-preserving point transformation T from M onto itself (mod 0) is called an endomorphism (mod 0). Two endomorphisms (mod 0) S and T are equal when

$$Sx=Tx \qquad (\text{mod}\ 0)\ .$$

Distinction between strict and (mod 0) measurable transformations (automorphisms and endomorphisms) are essential only in the discussion of a 1-parameter family of such transformations, because a single transformation (mod 0) can be made a strict one after dropping from M a set of measure zero.

Suppose now U is a mapping from a measure algebra $(\mathcal{L}, \mu)$ into itself which commutes with the operations of making union, intersection and complementation, and further suppose that it is measure preserving, then it is called an endomorphism on $(\mathcal{L}, \mu)$. Furthermore, if U is an onto mapping, then it is called an automorphism on $(\mathcal{L}, \mu)$.

To every endomorphism (automorphism) on $(\mathcal{L}, \mu)$ there corresponds a unique endomorphism (automorphism) (mod 0) on M such that

$$T^{-1}A=UA \qquad \text{for any } A\in\mathcal{L}\ .$$

Therefore we may identify endomorphisms and automorphisms over the measure algebra $(\mathcal{L}, \mu)$ with those (mod 0) over M. Let $\mathfrak{G}$, $\mathfrak{G}'$ be respectively the set of all automorphisms and endomorphisms (mod 0) over M. As is well-known $\mathfrak{G}$, $\mathfrak{G}'$ can be made complete separable metric spaces, and under the corresponding topologies they are respectively topological group and topological semi-group.

DEFINITION I. Let $\mathfrak{S}_+=\{M, T_t, \mu\}=\{M, T_t\}$, $t\in T_+$, satisfy the following conditions.

(i) For any $t\in T_+$, T_t is an endomorphism over M.

(ii) $T_sT_tx=T_{s+t}x$ for every $x\in M$, s, $t\in T_+$, $T_0=I$ (indentity). Then $\mathfrak{S}_+$

is called a 1-parameter semi-group of endomorphisms (simply 1-parameter semi-group) over M. Further, if for any $f(x)\in L_0(M), f(T_tx)$ is measurable in (t, x) under the obvious product measure, then $\mathfrak{S}_+$ is said to be measurable.

This is an obvious generalization of the definition of a one-parameter group of automorphisms (or flow) over M, and its measurability.

Correspondingly we may make the following definition.

DEFINITION II. Suppose that there is given a homomorphic mapping ψ from the topological semi-group T_+ (group T) into the topological semi-group $\mathfrak{G}'$ (group $\mathfrak{G}$) $: t\in T_+(T)\rightarrow\psi(t)=T_t\in\mathfrak{G}'(\mathfrak{G})$, then $\{M, T_t\}$, $t\in T_+(T)$, is called a continuous 1-parameter semi-group (group) of endomorphisms (automorphisms) (mod 0) over M. In this case the equality $T_tT_sx=T_{t+s}x$ is understood to be true (mod 0), and the exceptional null-sets may depend on s, t.

Corresponding to Definition I, II, we may introduce two kinds of isomorphisms among 1-parameter groups (semi-groups).

DEGINITION III. Given two continuous 1-parameter semi-groups (mod 0) $\mathfrak{S}_+=\{M, T_t, \mu\}$, $\mathfrak{S}'_+=\{M', T'_t, \mu'\}$. When there is an isomorphism (mod 0) φ from M to M' such that for $t\in T_+$,

$$(1) \qquad T_tx=\varphi^{-1}T'_t\varphi x \qquad (\mathrm{mod}\ 0),$$

$\mathfrak{S}_+$, $\mathfrak{S}'_+$ are said to be isomorphic (mod 0) each other, denoted by $\mathfrak{S}_+\sim\mathfrak{S}'_+$ (mod 0).

Similarly for the definition of the isomorphism (mod 0) for 1-parameter groups. The equality (1) is precisely understood as follows. For any $t\in T_+$, there exist $N_t\in\mathcal{L}$, $N'_t\in\mathcal{L}'$ of μ, μ'-measure zero, and an isomorphism (mod 0) φ from M to M' such that φ is an exact isomorphism from $M-N_t$ to $M'-N'_t$; T_t and T'_t are defined respectively on $M-N_t$, $M'-N'_t$, and is valid $T_tx=\varphi^{-1}T'_t\varphi x$ on $M-N_t$.

DEFINITION IV. Given two measurable 1-parameter semi-groups $\mathfrak{S}_+=\{M, T_t, \mu\}$, $\mathfrak{S}'_+=\{M', T'_t, \mu'\}$. Suppose there exist measurable M_0, M'_0 such that $\mu(M_0)=\mu'(M'_0)=1$, $T_tM_0=M_0$, $T'_tM'_0=M'_0$ for any $t\in T_+$, and an isomorphism (in the exact sense) φ from M_0 to M'_0 such that $T_tx=\varphi^{-1}T'_t\varphi x$ for any $x\in M_0$, then $\mathfrak{S}_+$ and $\mathfrak{S}'_+$ are said to be isomorphic each other, denoted by $\mathfrak{S}_+\sim\mathfrak{S}'_+$; the same definition is made for the exact isomorphism between measurable flows.

It should be noted that in the defining equality of the isomorphism between two measurable flows $\mathfrak{S}$, $\mathfrak{S}'$, the exceptional null sets depend only on $\mathfrak{S}$, $\mathfrak{S}'$ but not on t, and they are strictly invariant under T_t, T'_t. As is well-known, a measurable 1-parameter group (semi-group) is continuous.

When we deal with mappings of a Lebesgue space into another measure space, the following proposition and its corollary are useful.

PROPOSITION 1°. *Let M be a Lebesgue space, $(X, \mathcal{L}, \nu)$ a separable*[4] *measure space, and φ a homomorphic mapping from M into X.*

Then the image $\varphi(M)$ is a Lebesgue space, and therefore $\varphi(M) \in \mathcal{L}$.

COROLLARY. *Let (X, σ) be a Hausdorff space satisfying the second countability axiom, $\mathcal{B}_\sigma$ its topological Borel field, M a Lebesgue space, and φ a measurable mapping from M into (X, σ). Let $(X, \mathcal{L}_\sigma, \nu)$ be the measure space with ν, the induced measure*[5] *(denoted as $\nu = \varphi\mu$), and $\mathcal{L}_\sigma$ the completion of $\mathcal{B}_\sigma$ under ν. Then $\varphi(M)$ is a Lebesgue space and therefore $\varphi(M) \in \mathcal{L}_\sigma$.*

§ 2. Isomorphic mapping of a flow into a metric space.

Consider a product space $\Omega(\Omega_+)$ with a finite or denumerable number of components in $L_0(T)$ $(L_0(T_+))$ with metric $d(d_+)$. Define metrics over Ω and Ω_+:

$$d(\omega_1, \omega_2) = \sum_{k=1}^{\infty} \frac{1}{2^k} d(f_k, g_k), \tag{2.1}$$

$$\omega_1 = \{f_1(u), f_2(u), \cdots\}, \quad \omega_2 = \{g_1(u), g_2(u), \cdots\}$$

$$f_i, g_i \in L_0(T) \qquad (1 \leqq i < \infty);$$

$$d_+(\omega_1, \omega_2) = \sum_{k=1}^{\infty} \frac{1}{2^k} d_+(f_k, g_k), \tag{2.2}$$

$$\omega_1 = \{f_1(u), f_2(u), \cdots\}, f_1, f_2, \cdots \in L_0(T_+),$$

etc..

Regardless of any measure on $\Omega(\Omega_+)$ there is a flow (1-parameter semi-group) S_t, $t \in T(T_+)$ over $\Omega(\Omega_+)$;

$$\begin{aligned} \omega = \{f_1(t), f_2(t), \cdots\} &\rightarrow S_\tau \omega \\ &= \{f_1(t+\tau), f_2(t+\tau), \cdots\}, \ \tau \in T(T_+). \end{aligned} \tag{2.3}$$

Obviously (i) S_t, $t \in T(T_+)$ is a one-to-one (many-to-one) mapping from $\Omega(\Omega_+)$ onto itself, (ii) $(t, \omega) \in \Omega \times T(\Omega_+ \times T_+) \rightarrow S_t\omega \in \Omega(\Omega_+)$ is a continuous mapping.

Let $\mathcal{F} = \{f_k(x)\}$ be a family of a finite or denumerable number of $L_0(M)$-functions. Given a measurable flow (1-parameter semi-group) over M, define a mapping $\varphi = \varphi(\mathcal{F})$ from M into $\Omega(\Omega_+)$ as follows:

$$x \in M \rightarrow \omega = \varphi x = \{f_1(T_t x), f_2(T_t x), \cdots\}, \tag{2.4}$$

whenever $\varphi x \in \Omega(\Omega_+)$.

4) The separability is in the sense of Rohlin [6], and $\mathcal{L}$ is supposed to be completed under ν.

5) The induced measure is completely determined by assigning to $A \in \sigma$ the value $\nu(A) = \mu(\varphi^{-1}A)$. It is enough to make this assignment of the value $\nu(A)$ only for all sets of $\sigma' \subset \sigma$, where σ' is a countable base of σ.

PROPOSITION 2°. *In either the case, a flow or a 1-parameter semi-group, there is an* M_0, $\varphi(M-M_0)=0$, $T_tM_0=M_0$ *and such that if we put* $\Omega_0=\varphi M_0 \subset \Omega(\Omega_+)$, *then* $S_t\Omega_0=\Omega_0$ *and*

$$\varphi T_t x = S_t \varphi x \qquad \textit{for any } x \in M_0 .$$

PROOF. In the case of a flow, if we put $M_0=\{x:\varphi x\in\Omega\}$, then by Fubini's theorem M_0 is measurable, $\mu(M_0)=1$, and $T_tM_0=M_0$. On the other hand, in the case of a 1-parameter semi-group, if we put $M'=\{x:\varphi x\in\Omega_+\}$, we have $\mu(M-M')=0$, $T_tM'\subset T_sM'$ for $0\leqq s<t<\infty$. Denote now by $\tilde{T}_t$ the restriction of T_t to M', then $\tilde{T}_t$ is a homomorphic mapping (Rohlin [6]) from (M',μ) into itself, because for any measurable $A\in M'$, $\tilde{T}_t^{-1}A=(T_t^{-1}A)M'$, $\mu(\tilde{T}_t^{-1}A)=\mu(T_t^{-1}A)=\mu(A)$. Therefore T_tM' is measurable by Proposition 1° and $\mu(T_tM')=1$. Define now $M_0=\bigcap_{t\geqq 0}T_tM'$, then M_0 is measurable, $\mu(M_0)=1$, and $T_tM_0=M_0$ for any $t\in T_+$.

In either the case, a flow or 1-parameter semi-group, for $x\in M_0$

$$\begin{aligned}\varphi(T_tx)&=\{f_k(T_s(T_tx)),\ -\infty<s<\infty\}\\&=\{f_k(T_{s+t}x),\ -\infty<s<\infty\}=S_t\varphi x .\end{aligned}\tag{2.5}$$

Therefore $\Omega_0=\varphi M_0$ satisfies $S_t\Omega_0=\Omega_0$, and

$$\varphi T_t x = S_t\varphi x \qquad \text{on} \quad M_0 .\tag{2.6}$$

THEOREM 1. *There is given a flow* $\mathfrak{S}$ (1-*parameter semi-group* $\mathfrak{S}^+$) *over* M. *Suppose* $\mathcal{F}$ *satisfies either the following* (i) *or* (ii): (i) $\mathcal{F}$ *is dense in* $L_0(M)$, (ii) *the set* $\mathfrak{G}=\{g_k\}$ *of all linear combinations of the finite products of functions from* $\mathcal{F}$ *is dense in* $L_0(M)$.

Then $\varphi=\varphi(\mathcal{F})$ *is an isomorphic mapping from M into* $\Omega(\Omega_+)$; $\mathfrak{S}'=\{\Omega, S_t, \mu\}$ ($\mathfrak{S}'_+=\{\Omega_+, S_t, P\}$), *where over* $\Omega(\Omega_+)$ *are defined a* σ-*algebra* $\mathcal{L}_d$ ($\mathcal{L}_{d+}$) *and a probability* P, *is a flow* (1-*parameter semi-group*) *over* $\Omega(\Omega_+)$; *through* φ

$$\mathfrak{S}\sim\mathfrak{S}' \qquad (\textit{strict isomorphism}),\tag{2.7}$$

$$\mathfrak{S}_+\sim\mathfrak{S}'_+ \qquad (\text{mod } 0).\tag{2.8}$$

Suppose that $\mathfrak{S}_+$ *satisfies the additional condition* (C): *there exists a measurable* $M_0\subset M$ *such that* $\mu(M_0)=1$, $T_tM_0=M_0$, *and whenever* $T_tx=T_tx'$ *for all* $t>0$, x, $x'\in M_0$, *then* $x=x'$.

Then the isomorphism (2.8) *can be strengthened to the strict one.*

REMARK. $\mathfrak{S}'_+$ in the above satisfies (C).

PROOF. Proof of (2.7). φ is obviously a measurable mapping from M into (Ω, d), and the P-measure is completely determined by setting

$$P(\omega: d(\omega,\omega_0)<a)=\mu(x: d(\varphi x,\omega_0)<a),\ a>0,\ \omega_0\in\Omega .\tag{2.9}$$

As we have seen in the above, there is a (strictly) T_t-invariant set M_0, $\mu(M_0)=1$.

Let $\Sigma=\{D_k\}$ be a multiplicative base of M, and χ_{D_k} the indicator of D_k. Now if $\{h_n\}\subset\mathfrak{F}$ satisfies the relation $h_n\to\chi_D$ (in $L_0(M)$), $n\to\infty$, then

$$\int_{M_0}d\mu\int_{-\infty}^{\infty}P(t,\ h_n(T_tx)-\chi_D(T_tx))dt$$
$$=\pi\delta(h_n,\ \chi_D)\to 0,\ n\to\infty\,.$$

This implies that there is a subsequence $\{h_n'\}$ of $\{h_n\}$ such that $h_n'(T_tx)\to\chi_D(T_tx)$ (in $L_0(T)$), except on a null subset of M_0. More precisely, if we set

$$N_k=\{x:x\in M_0,\ \overline{\lim_{n\to\infty}}\, d(h_n'(T_tx),\ \chi_{D_k}(T_tx))\neq 0\}\,,$$

N_k is T_t-invariant[6], and $h_n'(T_tx)\to\chi_{D_k}(T_tx)$ (in $L_0(T)$) for all $x\in M_0-N_k$. Therefore if we put $N=\bigcup_k N_k$, N is a T_t-invariant subset of M_0, $\mu(N)=0$, and whenever $\varphi x=\varphi x'$, x, $x'\in M_0-N$, then $\chi_{D_k}(T_tx)=\chi_{D_k}(T_tx')$ (as elements of $L_0(T)$) for all k. This means $x=x'$.

Therefore we have the T_t-invariant $\bar{M}_0=M_0-N$, and S_t-invariant $\bar{\Omega}_0=\varphi(\bar{M}_0)$, $\mu(\bar{M}_0)=P(\bar{\Omega}_0)=1$ (cf. Corollary to Proposition 1°), φ is an isomorphic mapping from $\bar{M}_0$ onto $\bar{\Omega}_0$, and as in (2.6) $T_tx=\varphi^{-1}S_t\varphi x$ on $\bar{M}_0$. That S_t is P-measure preserving is obvious from the relation that for $a>0$, if we put

$$A=\{\omega:d(\omega,\ \omega_0)<a\},\ \omega_0\in\Omega\,,\tag{2.10}$$
$$S_\tau^{-1}A=\{\omega:d(S_\tau\omega,\ \omega_0)<a\}\,,$$

one has

$$P(\omega:d(S_\tau\omega,\ \omega_0)<a)$$
$$=\mu(x:d(S_\tau\varphi x,\ \omega_0)<a)=\mu(x:d(\varphi T_\tau x,\ \omega_0)<a)$$
$$=\mu(x:d(\varphi x,\ \omega_0)<a)=P(\omega:d(\omega,\ \omega_0)<a)$$

or

$$P(A)=P(S_\tau^{-1}A)\,.\tag{2.11}$$

Collecting the discussions in the above, we finally have $\mathfrak{S}\sim\mathfrak{S}'$.

We shall prove (2.8) and the succeeding additional proposition. As in the above, there exists a measurable $M'\subset M$, $\mu(M')=1$, $T_tM'\subset M'$, $t\in T_+$. Therefore, by the same device as in the proof of Proposition 2°, there is defined a measurable $M_0\subset M$ such that $\mu(M_0)=1$, $T_tM_0=M_0$, and for x, $x'\in M_0$, $\varphi x=\varphi x'$ implies

$$\chi_{D_k}(T_tx)=\chi_{D_k}(T_tx')\tag{2.12}$$

for all $t>0$, and $1\leqq k<\infty$. But this does not necessarily imply that the partition ζ generated by the system $\{T_{1/n}^{-1}D_k,\ 1\leqq n<\infty,\ 1\leqq k<\infty\}$ separate

6) Hereafter T_t-invariance is understood in the strict sense.

x, x'. Now ζ can be written as $\zeta = \bigvee_{n \geqq 1} T_{1/n}^{-1}\varepsilon$, and the continuity of T_t implies

$$\mathcal{L} = \mathcal{L}(\varepsilon) = \bigvee_{n \geqq 1} \mathcal{L}(T_{1/n}^{-1}\varepsilon) = \mathcal{L}(\bigvee_{n \geqq 1} T_{1/n}^{-1}\varepsilon),$$

i. e. $\zeta = \varepsilon$ (mod 0). So that there is an N, $\mu(N) = 0$ such that whenever $\varphi x = \varphi x'$ for $x, x' \in M_0 - N$, then $x = x'$, i. e. φ is an isomorphism from $M_0 - N$ into Ω_0. Since $\varphi(T_t x) = S_t \varphi x$ for $x \in M_0$, $T_t x = \varphi^{-1} S_t \varphi x$ on the set $\{x : x \in M_0, T_t x \in M_0 - N\}$, i. e. for every $x \in M_0 \cap T_t^{-1}(M_0 - N) = M_0 - N_t$, where N_t is a suitable null set.

That S_t is measure preserving is proved as in (2.11). Collecting results in the above one has

$$\mathfrak{S}_+ \sim \mathfrak{S}'_+ \ (\mathrm{mod}\ 0).$$

The fact that under (C), the last isomorphism is strengthened to the strict one follows immediately from (2.12).

In connection with the flow (1-parameter semi-group) generated by shifts on $\Omega(\Omega_+)$ we can state the

PROPOSITION 3°. *If there is given a S_t-invariant probability measure over $\Omega(\Omega_+)$, then $\mathfrak{S} = (\Omega, S_t, P)$ ($\mathfrak{S}_+ = (\Omega_+, S_t, P)$) becomes a measurable flow (1-parameter semi-group satisfying (C))*

PROOF. Since the proof is the same in either the case, a flow or 1-parameter semi-group, we will state it for a flow. Let $\mathcal{F}_T$ be the family of Borel sets on T, $\overline{\mathcal{B}_d \times \mathcal{F}_T}$ the completion of $\mathcal{B}_d \times \mathcal{F}_T$ under $dP \times dt$, and $f(\omega)$ a d-continuous function, then $f(S_t(\omega))$ is continuous in (t, ω). Therefore, $f(S_t\omega)$ is $\overline{\mathcal{B}_d \times \mathcal{F}_T}$-measurable, if $f(\omega)$ is a Baire function. Further, if f is $\mathcal{L}_d$-measurable, there are two Baire functions $f_i(\omega)$ $(i = 1, 2)$ such that

$$f_1(\omega) \leqq f(\omega) \leqq f_2(\omega), \int (f_2(\omega) - f_1(\omega)) dP = 0.$$

Then

$$f_1(S_t\omega) \leqq f(S_t\omega) \leqq f_2(S_t\omega),$$

$$\int_0^{t_0} \int_\Omega (f_2(S_t\omega) - f_1(S_t\omega))\, dt dP = t_0 \int_\Omega (f_2(\omega) - f_1(\omega)) dP = 0,$$

for any $t_0 > 0$, which implies that $f(S_t\omega)$ is $\overline{\mathcal{B}_d \times \mathcal{F}_T}$-measurable, as was to be proved.

We shall show further that as an underlying space Ω a more restricted function space serves as well.

PROPOSITION 4°. *Given a flow $\mathfrak{S} = \{M, T_t\}$, there exists a flow $\mathfrak{S}'$ isomorphic with $\mathfrak{S}$*

$$\mathfrak{S}' = \{\Omega, S_t\},$$

where Ω is a compact metric space.

PROOF. Take a multiplicative base $\{D_k\}$ and put

$$C_{n,k}(t, x) = \int_t^{t+1/n} \chi_{D_k}(T_t x) dt, \ 1 \leqq n, k < \infty .$$

Without loss of generality we may assume that for every $x \in M$, all $\chi_{Dk}(T_t x) \in L_0(T)$. Let $C(T)$ be the F-space consisting of all continuous functions $f(t)$, $-\infty < t < \infty$, satisfying $|f(t) - f(s)| \leqq 2|s-t|$, $\sup_{-\infty<s<\infty} |f(s)| \leqq 1$, with the relevant metric

$$|f| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f\|_n}{1+\|f\|_n}, \ \|f\|_n = \max_{|t| \leqq n} |f(t)| .$$

Ascoli-Arzela's theorem implies that $C(T)$ is compact. Define now Ω to be the countable product of $C(T)$, and consider the mapping

$$x \in M \to \omega = \varphi x = \{C_{n,k}(t)\} \in \Omega .$$

If $\varphi x = \varphi x'$, $x, x' \in M$, then since $nC_{n,k}(t, y) \to \chi_{D_k}(T_t y)$, as $n \to \infty$, for almost all t and every $y \in M$, there holds the equality $\{\chi_{Dk}(T_t x)\} = \{\chi_{Dk}(T_t x')\}$ $(L^0(T))$ for all k. Therefore, from the proof of Theorem 1, $x = x'$. The remaining part in the proof is similar to that of Theorem 1.

REMARK TO THEOREM 1. Theorem 1 provides us with a principle of realizing isomorphisms among flows or semi-groups. Sometimes, it is useful to construct a homomorphic mapping of a flow into the function space or its factor flow, using a smaller class $\mathcal{F}$ than that in Theorem 1. Let $\mathcal{F} = \{f_k(x)\}$ be a family of $L_0(M)$-functions, $\mathcal{B}^0$ the smallest σ-algebra, with respect to which any $f_k(x)$ is measurable, and put

$$\mathcal{B} = \bigvee_{-\infty<t<\infty} T_t \mathcal{B}^0$$

To $\mathcal{B}$ there exists a unique (mod 0) partition ζ such that $\mathcal{B} = \mathcal{B}(\zeta)$, ζ is T_t-invariant. Then by the same argument as in Theorem 1, one can prove the following statement which generalizes Theorem 1. Let T_t^ζ be the factor flow induced by T_t on the factor space M/ζ, then we have an isomorphism

$$T_t^\zeta \sim S_t$$

Especially if $\bigvee_{-\infty<t<\infty} T_t \mathcal{B}^0 = \mathcal{L}$, then

$$T_t \sim S_t .$$

§ 3. Representation of stationary processes by means of flows and 1-parameter semi-groups.

In this section by a stationary process $\xi(t, \alpha) = \{\xi_n(t, \alpha), t \in T\}$ or $\eta_+(t, \alpha) = \{\eta_n(t, \alpha), t \in T_+\}$ is meant a measurable stationary process with real-valued component processes $\xi_1(t, \alpha), \xi_2(t, \alpha), \cdots, \eta_1(t, \alpha), \cdots$ on an abstract probability space $(A, \mathcal{L}, Q)$. If for any set of finite number of points $t_1, t_2, \cdots, t_n \in T$, two stationary processes ξ, η have the same joint distribution, then they are said to be equivalent in probability law, written as $\xi \sim \eta$ (in law), and the same definition with ξ_+, η_+.

It is usual to represent a given stationary process on the space of sample functions with a measure invariant under the shifts. Unfortunately, the measure space thus obtained is not necessarily Lebesgue space. A more convenient way is to make use of the function spaces Ω, Ω_+.

Suppose that $\xi(\xi_+)$ is a (measurable) stationary process over $T(T_+)$, then by dropping a set of Q-measure zero, $\xi_n(t, \alpha), t \in T(T_+)$ are supposed to be elements of $L_0(T)(L_0(T_+))$ for all $\alpha \in A$, and the correspondence

$$\alpha \in A \to \omega = \varphi\alpha = \{\xi_1(t, \alpha), \xi_2(t, \alpha), \cdots\} \in \Omega(\Omega_+)$$

is a mapping from A into $\Omega(\Omega_+)$.

PROPOSITION 5°. *φ is a measurable mapping from $(A, \mathcal{L})$ into $(\Omega, \mathcal{B}_d)$ $((\Omega_+, \mathcal{B}_{d+}))$, and under $P = \varphi Q$, $\mathfrak{S} = (\Omega, S_t, P)$ $(\mathfrak{S}_+ = (\Omega_+, S_t, P))$ is a measurable flow (1-parameter semi-group satisfying (C)).*

PROOF. Since for any $\omega_0 = (\omega_1^0(t), \omega_2^0(t), \cdots) \in \Omega$, $a > 0$, $\tau \in T$,

$$d(\varphi\alpha, \omega_0) = \sum_{k=1}^{\infty} \frac{1}{2^k} \int_{-\infty}^{\infty} p(t, \omega_k^0(t) - \xi_k(t, \alpha))dt$$

is measurable in α, and

$$Q(\alpha : \sum_{k=1}^{\infty} \frac{1}{2^k} \int_{-\infty}^{\infty} p(t, \omega_k^0(t) - \xi_k(t+\tau, \alpha))dt < a)$$
$$= Q(\alpha : \sum_{k=1}^{\infty} \frac{1}{2^k} \int_{-\infty}^{\infty} p(t, \omega_k^0(t) - \xi_k(t, \alpha))dt < a),$$

φ is a measurable mapping as designated in the above, and

$$P(\omega : d(\omega, \omega_0) < a) = Q(\alpha : d(\varphi\alpha, \omega_0) < a)$$
$$= Q(\alpha : d(S_\tau\varphi\alpha, \omega_0) < a) = P(\omega : d(S_\tau\omega, \omega_0) < a),$$

i.e. P is S_t-invariant; similarity for $\mathfrak{S}_+$.

DEFINITION. $\mathfrak{S}(\mathfrak{S}_+)$ in Proposition 5° is called a flow (1-parameter semi-group) determined by the stationary process $\xi(\xi_+)$.

By this definition, if the two stationary processes are equivalent, $\xi \sim \eta$ $(\xi_+ \sim \eta_+)$ (in law), then the flows (1-parameter semi-groups) determined by the two processes coincide.

THEOREM 2. *Suppose there is given a stationary process $\xi(\xi_+)$, and consider the flow (1-parameter semi-group) $\mathfrak{S}(\mathfrak{S}_+)$ (determined by them), then there exists a (measurable) stationary process $x(t, \omega)$ $(x_+(t, \omega))$ satisfying the conditions*

$$x(t, \omega) = x(0, S_t\omega) \qquad \textit{for all } t, \omega,$$

$$x(t, \omega) \sim \xi(t, \alpha) \quad (in\ law);$$

the same is true with ξ_+, x_+.

PROOF. First we notice that an $\omega \in \Omega$ is an equivalent class of measurable functions. We will select from every ω a representative in a unique way.

Take a natural number λ, put

$$g_\lambda(y) = y, \qquad |y| \leqq \lambda,$$
$$= \lambda, \ |y| > \lambda,$$
$$\omega = \{\omega_1(t), \omega_2(t), \cdots\},$$

and define

$$x_k(t, \omega) = \lim_{\lambda\to\infty} \overline{\lim_{n\to\infty}} G_\lambda^n(t, \omega_k),$$

where

$$G_\lambda^n(t, \omega_k) = n\int_\lambda^{t+1/n} g_\lambda(\omega_k(s))ds.$$

Then $x = \{x_k(t, \omega), t \in T\}$ belongs to ω and is the required representative, and since $G_\lambda^n(t, \omega_k) = G_\lambda^n(0, (S_t\omega)_k)$, $x(t, \omega)$ is defined for all $t \in T$, $\omega \in \Omega$, and

$$x(t, \omega) = x(0, S_t\omega).$$

Since $x(0, \omega)$ is $\mathcal{L}_d$-measurable, $x(t, \omega)$ is a measurable process by Proposition 3°.

Since for every ω, there exists

$$\tilde{x}(t, \omega) = \{\lim_{\lambda\to\infty} \lim_{n\to\infty} G_\lambda^n(t, \omega_k)\} \tag{3.1}$$

for almost all t, by Fubini's theorem, this limit exists for almost all ω, for almost all fixed t. However, since P is S_t-invariant, the limit actually exists for almost all ω, for every fixed t, and

$$x(t, \omega) = \tilde{x}(t, \omega)$$

for almost all $(t, \omega) \in \Omega \times T$.

If we use the stationarity of ξ, the above argument applies also to $\varphi\alpha$, instead of ω. For all α, t define now

$$\tilde{x}(t, \varphi\alpha) = \{\lim_{\lambda\to\infty} \overline{\lim_{n\to\infty}} G_\lambda^n(t, \xi_k(t, \alpha))\}$$

which exists for almost all α, for every fixed t. Again by the stationarity, $\tilde{x}(t, \varphi\alpha) = \xi(t, \alpha)$ for almost all α, for every fixed t. By the definition of the P-measure $\tilde{x}(t, \varphi\alpha) \sim x(t, \omega)$ (in law). Combination of these gives $\xi \sim x$ (in law).

The same argument applies to ξ_+.

§ 4. Measurable representation of continuons flows.

In this section we shall prove that if there is given a continuous flow (1-parameter semi-group), there exists a measurable one isomorphic (mod 0) to the given one. Thus obtained flow (semi-group) is called the measurable representation.

THEOREM 3. (i) *Let T_t be a continuous flow (1-parameter semi-group)* (mod 0) *over M (in the sense of Definition II), then there is a measurable flow (semi-group satisfying (C)) which isomorphic* (mod 0) *to T_t.*

(ii) *The representation given in* (i) *is unique,* i. e. *if we have two representations $\mathfrak{S}=(M, S_t)$, $\mathfrak{S}'=(M', S_t')$, then $\mathfrak{S}\sim\mathfrak{S}'$ (strict isomorphism); the same is true with the semi-group case.*

We will state the proof for a flow. For the proof we require several lemmas.

LEMMA 1. *If there is given an automorphism (endomorphism) T* (mod 0) *over M, then there exists an $M_0\subset M$, $\mu(M-M_0)=0$, such that the restriction of T to M_0 is a (strict) automorphism (endomorphism) over M_0.*

PROOF. The following argument applies to either the case, an automorphism, or an endomorphism.

Let $M'\subset M$ be the domain of T, and write $M''=\bigcap_{n=-\infty}^{\infty} T^n M'$, then $TM''\subset M''$. Therefore if we define $M_0=\bigcap_{n\geqq 0} T^n M''$, then $TM_0=M_0$, $\mu(M_0)=1$.

LEMMA 2. *Let (Y, σ) be a Hausdorff space satisfying the second countability axiom, and λ, μ respectively equi-measurable mappings from Lebesgue spaces (M, μ), (M', μ') into Y, i. e. $\mu(\pi^{-1}A)=\mu'(\lambda^{-1}A)$, $A\in\mathcal{L}_\nu$, $\nu=\pi\mu=\lambda\mu'$. Let ζ_π be the partition of M generated by the cells of the form $C=\pi^{-1}y, y\in Y$, and the same with the partition ζ_λ. Then there exists an isomorphism* (mod 0) *U from M/ζ_π to M'/ζ_λ such that, for any $y\in Y_0=\pi(M)\cap\lambda(M')$, $\pi^{-1}y\in M/\zeta_\pi\to C'=UC=\lambda^{-1}y\in M'/\zeta_\lambda$.*

LEMMA 3. *Let φ be a measurable mapping from (M, μ) into a Hausforff space (Y, σ) satisfying the second countability axiom, and suppose that there is given an automorphism S over Y associated with $(\mathcal{L}_\nu, \nu)$, $\nu=\varphi\mu$. Then*

(i) *There exists a $Y_0\subset\varphi(M)$, $Y_0\in\mathcal{L}_\nu$, $\nu(Y_0)=1$, such that $V=\varphi^{-1}S\varphi$ is a strict automorphism over M_0/ζ_φ, $M_0=\varphi^{-1}(Y_0)$*; (ii) *if f is a measurable mapping from $(Y, \mathcal{L}_\nu, \nu)$ into a Hausdorff topological measurable space Z, and write $c=\zeta_\varphi(C)$, $c'=\zeta_\varphi(C')$, $C=\{x:f(S\varphi x)=z, x\in M_0\}$, $C'=\{x:f(\varphi x)=z, x\in M_0\}$, $z\in Z$, then $c'=Vc$, where $\zeta_\varphi(\cdot)$ implies the mapping of a ζ_φ-cell to the corresponding point on M_0/ζ_φ.*

There are corresponding lemmas concerning endomorphisms. In either case, the proof is easily derived from fundamental properties of measurable partitions and factor spaces.

PROOF OF THEOREM 3. We shall prove the theorem for a flow. Take a multiplicative base of the space M and put

$$\xi_0(t, x) = \{\chi_{D_k}(T_t x)\}\,, \tag{4.1}$$

then by the continuity of T_t, $\xi_0(t, x)$ is a stationary process continuous in probability, and it has a measurable modification $\xi(t, x)$; $\xi(t, x) = \xi_0(t, x)$ for almost all x, for every fixed t. Let $x(t, \omega)$, $\omega = \varphi x$, be the measurable representation of $\xi(t, x)$, obtained in § 3, and apply Lemma 1 to $S = S_t$, then we have a measurable $Y_0 \subset \varphi(M)$, $P(Y_0) = 1$, over which S is a strict automorphism. If we write $f(\omega) = x(0, \omega)$, Theorem 2 implies $x(t, \omega) = x(0, S_t\omega)$, for all t, ω, and

$$f(S\varphi x) = x(t, \varphi x) = \xi(t, x) = \xi_0(t, x) \tag{4.2}$$

$$f(\varphi x) = x(0, \varphi x) = \xi(0, x) = \xi_0(0, x)\,, \tag{4.3}$$

for almost all x.

Now $\pi(x) = \xi_0(t, x)$, $\lambda(x) = \xi_0(0, x)$ are equi-measurable mapping, and their dependence on the base as defined in (4.1) implies

$$\zeta_\pi = \zeta_\lambda = \varepsilon \,(\mathrm{mod}\, 0). \tag{4.4}$$

So that if we denote by $\zeta_{f\circ\varphi}$ the partition generated by the mapping $x \to f\circ\varphi(x)$, the obvious relation $\zeta_{f\circ\varphi} \leqq \zeta_\varphi$ together with (4.2), (4.3) and (4.4) gives

$$\zeta_\varphi = \varepsilon \,(\mathrm{mod}\, 0) \tag{4.5}$$

An application of Lemma 2 guarantees the existence of an automorphism (mod 0) U from $M/\zeta_\pi = M\,(\mathrm{mod}\, 0)$ to $M/\zeta_\lambda = M\,(\mathrm{mod}\, 0)$, whereas in view of (4.1) we must have $U = T_t\,(\mathrm{mod}\, 0)$. Also from Lemma 3 there is the automorphism $V = \varphi^{-1}S\varphi = \varphi^{-1}S_t\varphi$ over $M_0/\zeta_\varphi = M_0$ (mod 0), and from Lemma 3, (ii), with $f(S\varphi x) = \pi(x)$, $f(\varphi x) = \lambda(x)$ (mod 0), one has $V = U\,(\mathrm{mod}\, 0)$, therefore $T_t = \varphi^{-1}S_t\varphi$ (mod 0).

To show the uniqueness of the representation, notice that $S_t \sim S_t'\,(\mathrm{mod}\ 0)$ through an isomorphism ψ from M into M'. Now let $\{D_k\}$ be a base of M and put $D_k' = \psi D_k$, then $\{D_k'\}$ becomes a base of M'. Clearly mappings φ, φ', such that $x \to \varphi x = \{D_k(S_t x)\}$, $x' \to \varphi' x' = \{D_k'(S_t' x')\}$ from M, M' into Ω are equi-measurable; dropping apropriate null sets we may assume $\varphi(M)$, $\varphi(M')$ $\subset \Omega$. If we denote by $\bar{S}_t$ the shifting flow on Ω, $Y_0 = \varphi(M) \cap \varphi(M')$ is $\bar{S}_t$-invariant, with $P(Y_0) = 1$, and $M_0 = \varphi^{-1}Y_0$, $M_0' = \varphi'^{-1}Y_0$ are respectively S_t, S_t'-invariant, having $(\bar{S}_t)_{Y_0} \sim (S_t')_{M_0'}$, $(\bar{S}_t)_{Y_0} \sim (S_t)_{M_0}$ (either strictly), where $(\ )_A$ means the reduction of a flow to an invariant set A. So that $S_t \sim S_t'$ in the strict sense.

As an application of Theorem 3 we prove the

COROLLARY. *Let x_t, y_t, $-\infty < t < \infty$, be real Gaussian stationary processes,* *with mean zero, spectral measures $dF(\lambda)$, $dG(\lambda)$. If they are mutually absolutly continuous, $dF(\lambda)\sim dG(\lambda)$, then the flows $\mathfrak{S}_x, \mathfrak{S}_y$ generated by x_t, y_t respectively are isomorphic. If in particular, $dF(\lambda)\sim d\lambda$, $\mathfrak{S}_x$ is isomorphic with the flow generated by the stationary random destribution $\frac{dB(t)}{dt}$, where $B(t)$ is the one-dimensional Brownian motion.*

PROOF. By Theorem 3, we suppose that x_t, y_t are the measurable representations over Ω, and write them as usual in the forms

$$x(t, \omega)=\int_{-\infty}^{\infty} e^{i\lambda t}d\xi(\lambda, \omega), \quad y(t, \omega)=\int_{-\infty}^{\infty} e^{i\lambda t}d\eta(\lambda, \omega).$$

Let us denote by E_x, E_y the expectations under the probability measures P_x, P_y generated respetively by x_t, y_t. One has $E_x(d\xi)=E_y(d\eta)=0$, $E_x(|d\xi(\lambda)|^2)=dF(\lambda)$, $E_y(|d\eta(\lambda)|^2)=dG(\lambda)$, and may write $q(\lambda)dF(\lambda)=dG(\lambda)$.

Now for any $\omega_0\in\Omega$, the distance $d(\omega, \omega_0)$ is a bounded function of ω and can be approximated, in the $L^2(P_x)$ and $L^2(P_y)$-norms, by L^2-functions of the form

$$F(\xi(t_1, \omega)-\xi(s_1, \omega), \cdots, \xi(t_n, \omega)-\xi(s_n, \omega)),$$

$$G(\eta(t_1, \omega)-\eta(s_1, \omega), \cdots, \eta(t_n, \omega)-\eta(s_n, \omega)),$$

where s_1, t_1 etc. are all rational numbers. This implies that, if $\{\chi_n(\lambda)\}$ is the set of indicator functions of finite intervals with rational end points, then the family of all linear combinations of finite products of functions from $\left\{\int_{-\infty}^{\infty}\chi_n(\lambda)d\xi(\lambda, \omega)\right\}$ and the family similarly defined from $\left\{g_n(\omega)=\int_{-\infty}^{\infty}\chi_n(\lambda)d\eta(\lambda, \omega)\right\}$ are dense respectively in $L^2(P_x)$, $L^2(P_y)$, so that they are dense respectively in $L_0(P_x)$, $L_0(P_y)$, where L_0 means the set of measurable functions regarded as an F-space under the designated measure.

On the other hand

$$f_n(\omega)=\int_{-\infty}^{\infty}\chi_n(\lambda)\sqrt{q(\lambda)}\,d\xi(\lambda, \omega) \qquad 1\leqq n<\infty$$

is dense in $L_0(P_x)$. For, if we take any even real-valued function $f(\lambda)\in L^2(dF)$, then

$$E_x\Big(\int_{-\infty}^{\infty}f(\lambda)d\xi(\lambda)\cdot\int_{-\infty}^{\infty}\chi_n(\lambda)\sqrt{q(\lambda)}\,d\xi(\lambda)\Big)=0$$

implies

$$\int_{-\infty}^{\infty}f(\lambda)\chi_n(\lambda)\sqrt{q(\lambda)}\,dF(\lambda)=0,$$

or $f(\lambda)=0$ almost everywhere in the dF-measure. Consider now stationary processes $\{f_n(S_t\omega), 1\leqq n<\infty\}$ over (Ω, P_x) and $\{g_n(S_t\omega), 1\leqq n<\infty\}$ over $(\Omega, P_y\}$, and notice that

$$f_n(S_t\omega)=\int_{-\infty}^{\infty}e^{i\lambda t}\chi_n(\lambda)\sqrt{q(\lambda)}\,d\xi(\lambda,\omega)\,,$$

$$g_n(S_t\omega)=\int_{-\infty}^{\infty}e^{i\lambda t}\chi_n(\lambda)d\eta(\lambda,\omega)\,.$$

Then the above processes are equivalent in probability law, so that

$$\mathfrak{S}_x\sim\mathfrak{S}_y\,.$$

§5. Natural extension of a continuous 1-parameter group of endomorphisms.

To every endomorphism over M, there corresponds an isometry from $L^2(\mu)$ into itself, so that the natural extension of a 1-parameter semi-group of endomorphisms gives rise to an extension of a 1-parameter semi-group of isometries in Hilbert space.

The time-discrete case was discussed by Rohlin [9]. After Rohlin we make the

DEFINITION. Let $\mathfrak{S}=\{M, T_t\}$ be a measurable flow, ζ a partition subject to the following conditions (i) $T_t\zeta\geqq\zeta$ for $t>0$, (ii) $\bigvee_{t\geqq 0}T_t\zeta=\varepsilon$, T_t^ζ the 1-parameter semi-group induced over $(M/\zeta, \mu^\zeta)$ from $\mathfrak{S}$[7], and $\mathfrak{S}_0=(\bar{M}, U_t)$ be any 1-parameter semi-group of endomorphisms, satisfying

$$(5.1)\qquad (M/\zeta,\mu^\zeta,T_t^\zeta)\sim\mathfrak{S}_0\qquad(\mathrm{mod}\,0)^{8)}\,.$$

Then $\mathfrak{S}$ is called a natural extension of $\mathfrak{S}_0$.

THEOREM 4. *Given a continuous 1-parameter semi-group of endomorphisms* (mod 0) *over M, its natural extension exists and is unique up to strict isomorphism.*

PROOF. Let $\mathfrak{S}_0=(\bar{M}, U_t)$ be the given 1-parameter semi-group. Take a dense set $\{f_k(x)\}$ in $L_0(M)$, and write $\xi_+(t, x)=\{\xi_k(t, x)\}$, $t\geqq 0$, $\xi_k(t, x)=f_k(U_tx)$. There is then a stationary process $\eta(t, \alpha)$, $t\in T$, such that

$$(5.2)\qquad \{\eta(t,\alpha),t\geqq 0\}\sim\{\xi(t,x),t\geqq 0\}\qquad(\text{in law})\,.$$

Let $\mathfrak{S}=\{\Omega, S_t\}$, $\mathfrak{S}_0'=\{\Omega_+, V_t\}$ be the flow and the 1-parameter semi-group determined respectively by η and ξ_+, then $\mathfrak{S}_0\sim\mathfrak{S}_0'$ (mod 0).

Define now a mapping ϕ from Ω onto Ω_+:

$$(5.3)\qquad \omega=\{f_1(t),f_2(t),\cdots\}\in\Omega\to\phi\omega=\{f_1(t),f_2(t)\}_{t\geqq 0}\in\Omega_+\,.$$

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7) For a ζ-cell C, $T_t^\zeta C(t\geqq 0)$ is defined to be the ζ-cell C' such that $T_tC\subset C'$. Obviously such a C' is unique.

8) μ^ζ is the measure induced by μ on the space M/ζ.

Since for an $\omega_0=(f_1^0(u), f_2^0(u), \cdots)\in\Omega_+$, $d_+(\psi\omega, \omega_0)$ is d-continuous in ω, ψ is a measurable mapping from $(\Omega, \mathcal{B}_d)$ into $(\Omega_+, \mathcal{B}_{d+})$. Let ζ be the partition determined by ψ, i.e. the partition whose cell C is given by

$$C=\{\omega : f_i(t)=f_i^0(t),\ t\geqq 0,\ \omega\in\Omega\}\,,$$

then obviously

$$S_s\zeta\leqq S_t\zeta\ (-\infty<s<t<\infty),\quad \bigvee_{t=0}^{\infty} S_t\zeta=\varepsilon\,. \tag{5.4}$$

If we put $C=S_t^\zeta C_0$, by the definition of S_t^ζ, we have $V_t\psi C_0=\psi C$, and $\psi S_t^\zeta C_0=V_t\psi C_0$ for all $C_0\in\zeta$, or $S_t^\zeta=\psi^{-1}V_t\psi$. Further, if we write

$$A=\psi^{-1}\{\omega\in\Omega_+ : d_+(\omega, \omega_0)<a\},\quad \omega_0\in\Omega_+,\ a>0\,,$$

we have $A\in\Omega/\zeta$ and[9]

$$\begin{aligned}P^\zeta(A)&=P(\omega\in\Omega : d_+(\psi\omega, \omega_0)<a)\\ &=P\Big(\omega : \sum_{k=1}^{\infty}\frac{1}{2^k}\int_0^\infty p(u, f_k(u)-f_k^0(u))du<a\Big)\\ &=P\Big(\alpha : \sum_{k=1}^{\infty}\frac{1}{2^k}\int_0^\infty p(u, \eta_k(u, \alpha)-f_k^0(u))du<a\Big)\\ &=\mu\Big(\sum_{k=1}^{\infty}\frac{1}{2^k}\int_0^\infty p(u, \xi_k(u, x)-f_k^0(u))du<a\Big)\\ &=Q(\omega\in\Omega_+ : d_+(\omega, \omega_0)<a)\,.\end{aligned}$$

Therefore

$$(\Omega/\zeta, P^\zeta, T_t^\zeta)_{t\geqq 0}\sim\mathfrak{S}_0' \qquad \text{(strict isomorphism)},$$

which together with $\mathfrak{S}_0\sim\mathfrak{S}_0'$ (mod 0) implies that $\mathfrak{S}$ is a natural extension of $\mathfrak{S}_0$.

Uniqueness of the extension. Suppose that there are two such extensions $\mathfrak{S}=(M, T_t)$, $\hat{\mathfrak{S}}=(\hat{M}, \hat{T}_t)$ with respective underlying partitions ζ, $\hat{\zeta}$, then there is an isomorphism λ:

$$(M/\zeta, T_t^\zeta)_{t\geqq 0}\sim(\hat{M}/\hat{\zeta}, \hat{T}_t^{\hat{\zeta}})_{t\geqq 0} \qquad (\text{mod } 0)\,.$$

Now we want to show that

$$\mathfrak{S}\sim\mathfrak{S}' \qquad \text{(strict isomorphism)}. \tag{5.5}$$

For this purpose take the respective bases $\{B_n\}$, $\{\hat{B}_n\}$ of M/ζ, $\hat{M}/\hat{\zeta}$, such that $\hat{B}_n=\lambda B_n$. Put $T=T_{t_0}(t_0>0)$ and define

$$D=T^{j_1}A_{\alpha_1}T^{j_2}A_{\alpha_2}\cdots T^{j_k}A_{\alpha_k}\quad \hat{D}=\hat{T}^{j_1}\hat{A}_{\alpha_1}\hat{T}^{j_2}\hat{A}_{\alpha_2}\cdots\hat{T}^{j_k}\hat{A}_{\alpha_k}\,,$$

9) P^ζ is the measure induced by P on the space Ω/ζ.

where α's and j's are positive integers, $1 \leqq j_1 \leqq j_2 \leqq \cdots \leqq j_k$, $A_n = B_n$ or B_n^c, $\hat{A}_n = \hat{B}_n$ or $\hat{B}_n^c$. Then the set of all $D(\hat{D})$ becomes a base of $M(\hat{M})$, and the correspondence $D \leftrightarrow \hat{D}$ gives rise to an isomorphism φ between M and $\hat{M}$, whose reduction to the factor spaces is λ. The isomorphism $(\hat{M}/\hat{\zeta}, \hat{T}^{\zeta}) \sim (M/\zeta, T^{\zeta})$ implies $T^{-j}D \leftrightarrow \hat{T}^{-j}\hat{D}$, $\mu(D) = \hat{\mu}(\hat{D})$, and

$$(M, T_{t_0}) \sim (\hat{M}, \hat{T}_{t_0}) \qquad (\text{mod } 0).$$

φ being independent of t, this implies $\mathfrak{S} \sim \mathfrak{S}'$ (mod 0). By Theorem 1, we have then the stronger result $\mathfrak{S} \sim \mathfrak{S}'$ (strictly). Q. E. D.

When U_t in the above satisfies the condition

$$\bigwedge_{t \geqq 0} T_t^{-1}\zeta = \nu \tag{5.6}$$

$\mathfrak{S}_0$ is said to be exact (Rohlin [**9**]). In this connection, Kolmogorov's flow is such that there exists a partition ξ satisfying the condition

$$\begin{gathered} \xi \geqq T_t^{-1}\xi \ (t \geqq 0), \\ \bigvee_{t \geqq 0} T_t\xi = \varepsilon, \quad \bigwedge_{t \geqq 0} T_t^{-1}\xi = \nu \qquad (\text{mod } 0). \end{gathered} \tag{5.7}$$

It is easy to show that the natural extension of an exact $\mathfrak{S}_0$ is a Kolmogorov's flow.

To U_t there corresponds a semi-group of isometries $\tilde{U}_t$ from $L^2(\bar{M})$ into itself. The group of unitary operators $\tilde{T}_t$ on $L^2(M)$, corresponding to T_t, is an extension of $\tilde{U}_t$.

§ 6. Spectral type of the Kolmogorov flow.

In this section we present a proof of Sinai's theorem, not using the Ambrose representation.

THEOREM 5. (Sinai [**10**]) *The spectral type of the Kolmogorov flow consists of Lebesgue spectrum with countable multiplicity.*

A few remarks and notations before going to the proof.

According to the proof of Theorem 4, the natural extension of $(M/\zeta, T_t^{\xi})_{t \geqq 0}$ on the function space Ω enables us to assume that T_t is measurable and ξ is such that the defining properies (5.7) are valid in the strict sense.

Let us write $\xi^t = T_t\xi$. Take a multiplicative base $\{D_k\}$ of ξ, then $\{D_{k,l}\}$, $1 \leqq k, l < \infty$, $D_{k,l} = T_l D_k$, is a base of M. Consider the usual metric space Ω of all the element ω of the form

$$\omega = \{\omega_{k,l}(t), t \in T, 1 \leqq k, l < \infty\}, \ \omega_{k,l}(t) \in L_0(T),$$

with metric

(6.1) $$d(\omega, \omega')=\sum_{k,l=1}^{\infty}\frac{1}{2^{k+l}}\int_{-\infty}^{\infty}p(u, \omega_{k,l}(u)-\omega'_{k,l}(u))du\,,$$

$$\omega, \omega' \in \Omega\,,$$

the mapping φ, $x\in M\to\omega=\varphi x=\{\chi_{D_{kl}}(T_t x)\}$, and define auxiliary function

(6.2) $$\tilde{d}(\omega, \omega')=\sum_{l=1}^{l_0}\sum_{k=1}^{\infty}\frac{1}{2^{k+l}}\int_{u_0}^{\infty}p(u, \omega_{k,l}(u)-\omega'_{k,l}(u))du\,,$$

which depends on l_0 and u_0.

From (6.1), (6.2) we see easily that

(6.3) $$0\leqq d(\omega, \omega')-\tilde{d}(\omega, \omega')\leqq \rho(l_0, u_0)\,,$$

where

$$\rho(l_0, u_0)=\frac{\pi}{2^{l_0}}+\left(\frac{\pi}{2}+\arctan u_0\right),$$

and that there is an integer-valued $N(\varepsilon)$, $N(\varepsilon)\uparrow\infty$ as $\varepsilon\to+0$, such that $\rho(l_0, u_0)<\varepsilon$ for $l_0, -u_0\geqq N(\varepsilon)$.

We are now considering the isomorphic image (Ω, S_t) of (M, T_t). Define η^t to be the partition induced by ξ^t through φ over $\varphi M\subset\Omega$, take a dense set $\{\omega^n\}_1^\infty$ in Ω, and put

(6.4) $$U_\varepsilon(\omega^n)=\{\omega : d(\omega, \omega^n)<\varepsilon, \omega\in\varphi M\}\,,$$

(6.5) $$\tilde{U}_\varepsilon(\omega^n)=\{\omega : \tilde{d}(\omega, \omega^n)<\varepsilon, \omega\in\varphi M\}\,.$$

Then the following relations are deduced immediately from (6.3):

(6.6) $$d(\omega, \omega')<\tilde{d}(\omega, \omega')+\varepsilon \qquad \text{for } l_0, -u_0>N(\varepsilon)\,,$$

(6.7) $$U_\varepsilon(\omega^n)\subset\tilde{U}_\varepsilon(\omega^n)\,.$$

Now for the proof of the theorem we require several lemmas.

LEMMA 1.

(6.8.a) $$\bigcup_{n\geqq1}\tilde{U}_\varepsilon(\omega^n)=\varphi M\,;$$

(6.8.b) $$\text{dia}\,\tilde{U}_\varepsilon(\omega^n)\leqq4\varepsilon \qquad \textit{for } l_0, -u_0>N(\varepsilon)\,;$$

(6.8.c) $$\text{dia}\,C\leqq4\varepsilon \qquad \textit{for any } \eta^{2N(\varepsilon)}\textit{-cell } C.$$

PROOF. (6.8.a) follows from (6.7) and the fact that

$$\bigcup_{n\geqq1}U_\varepsilon(\omega^n)=\varphi M\,;$$

(6.8.b) follows from (6.6).

The right-hand side expression in (6.2) implies that

$$\varphi^{-1}\tilde{U}_\varepsilon(\omega^n)\in\mathcal{B}(\xi^{l_0-u_0})\,,$$

$$\tilde{U}_\varepsilon(\omega^n) \in \mathscr{B}(\eta^{l_0-u_0})$$
$$= \mathscr{B}(\eta^{2N(\varepsilon)}), \qquad \text{for } l_0 = -u_0 = N(\varepsilon).$$

This together with (6.8.a), (6.8.b) implies (6.8.c).

LEMMA 2. *Let $(\Omega, \mathcal{L}, P)$, $P = \varphi\mu$, be the measure space considered in the above, and take a $K \in \mathcal{L}$, with $P(K) > 0$. Then, for any $n \geqq 1$, there exists distinct n points, ω_k $(1 \leqq k \leqq n)$, such that $P(U_\varepsilon(\omega_k) \cap K) > 0$ for any $\varepsilon > 0$.*

PROOF. Since $\mathfrak{S}$ is a Kolmogorov flow, P is a continuous measure, i.e. there is no atomic point on Ω. Split K into n disjoint measurable sets, K_i with $P(K_i) > 0$, $1 \leqq i \leqq n$. Take a compact $\bar{K}_i \subset K_i$, with $P(\bar{K}_i) > 0$. Then there exists an $\omega_i \in K_i$, such that $P(U_\varepsilon(\omega_i) \cap \bar{K}_i) > 0$ for any $\varepsilon > 0$.

Consider the usual metric δ in $\mathcal{L}_d$, i.e. $\delta(A, B) = P((A-B) \cup (B-A))$ for $A, B \in \mathcal{L}_d$.

LEMMA 3. *There exists an $A \in \mathscr{B}(\eta^0)$ such that $\delta(A, \mathscr{B}(\eta^t)) > 0$ for any $t<0$.*

PROOF. Take a $B \in \mathscr{B}(\eta^0)$ such that $\delta(t) = \delta(B, \mathscr{B}(\eta^t)) > 0$ for some $t < 0$. $\delta(t)$ is continuous, because $|\delta(s) - \delta(t)| \leqq \delta(S_{|s-t|}B, B)$. Let $t_0 = \inf(t : \delta(t) = 0)$, then $-\infty < t_0 \leqq 0$, and there exists a sequence $t_n \uparrow t_0$, and $A_n \in \mathscr{B}(\eta^{t_n})$, such that $\delta(A_n, B) = 0$. This means that $B \in \mathscr{B}(\eta^{t_0})$, $\notin \mathscr{B}(\eta^t)$ for any $t < t_0$. Put $A = S_{-t_0}B$, then A satisfies the required conditions.

For a partition ξ, the set of $\mathscr{B}(\xi)$-measurable square-integrable functions will be denoted by $L^2(\xi)$.

LEMMA 4. *In a Lebesgue space M, there are given partitions ξ, η with $\xi < \eta$ (strict refinement). Let $\{\mu_c, C = \xi\text{-cell}\}$ be the canonical system of measure for ξ.*

Take an $A \in \mathscr{B}(\eta)$, $\notin \mathscr{B}(\xi)$, and define

$$\tilde{A} = \cup \{C : 0 < \mu_c(A) < 1, C = \xi\text{-ell}\},$$

then $\mu(A\tilde{A}) > 0$. Take $B \in \mathcal{L}$, put $E = AB\tilde{A}$ and assume that $\mu(E) > 0$, and define

$$\tilde{E} = \cup \{C : 0 < \mu_c(E), C = \xi\text{-cell}\}.$$

Finally define

$$F = \tilde{E}A,$$

$$f(x) = \chi_F(x) - \mu_c(F) \qquad \text{for } x \in C,$$

then $f(x)$ is a non-null element of $L^2(\eta)$, orthogonal with $L^2(\xi)$.

PROOF. Obviously $\tilde{A}$, $\tilde{E}$ are ξ-sets, and $\mu_c(E) = \mu_c(AB)\chi_{\tilde{A}}$; $\mu_c(E) > 0$ implies that $C \in \tilde{A}$, therefore $\tilde{E} \subset \tilde{A}$. On the other hand $\mu(E) > 0$ implies $\mu(\tilde{E}) > 0$; since $\mu_c(A) > 0$ for $C \in \tilde{A}$, and therefore for $C \in \tilde{E}$, one has

$$\mu(F) = \int_{\tilde{E}} \mu_c(A) d\mu^\xi > 0, \ 0 < \mu_c(F) < 1 \qquad \text{for } C \in \tilde{E}.$$

Combination of the results in the last sentence implies that $F \in \mathscr{B}(\eta)$, $\notin \mathscr{B}(\xi)$, and f is a non-null $\mathscr{B}(\eta)$-measurable function, orthogonal with $L^2(\xi)$.

LEMMA 5. *Let Ω be the space introduced in Lemma 2, and $\tau(\omega)$ the hitting time of the trajectory $S_t\omega$, $t \geqq 0$, for closed F, i.e.*

$$\tau(\omega) = \inf\{t : S_t\omega \in F, t \geqq 0\}\ .$$

Then $\tau(\omega)$ is lower semi-continuous in ω.

PROOF. $\tau(\omega)$ is clearly lower semi-continuous at an $\omega_0 \in F$, because $\tau(\omega_0) = 0$. When $\omega_0 \notin F$, the lower semi-continuity at ω_0 follows from the fact that ω_t is d-continuous in (t, ω).

PROOF OF THEOREM 5. We will prove that the multiplicity of the spectrum is exactly countable. Take a compact $K \subset \Omega$ with $P(K) > 0$, and let $\{\omega_k\}_1^n$ be the set designated in Lemma 2. Put

$$p = \min_{i \neq j} d(\omega_i, \omega_j)\ ,$$

choose ε, $\delta_1 > 0$ such that

$$p > 2(2\delta_1 + 4\varepsilon) \tag{6.9}$$

and define

$$K_i^{\delta_1} = U_{\delta_1}(\omega_i) \cap K\ ,$$

where $U_r(\omega)$ is the closed sphere with radius r, center ω. Take $\delta_2 > 0$ sufficiently small, then since the tube $\mathfrak{T} = \bigcup_{|t| \leqq \delta_2} S_t K_i^{\delta_1}$ with the compact base $K_i^{\delta_1}$ is compact, one has

$$\mathfrak{T} \subset U_{2\delta_1}(\omega_i)\ . \tag{6.10}$$

Choose a $\delta_0 < \delta_2$, and according to Lemma 3, take an $A^* \in \eta^0$, $\notin \eta^{-\delta_0}$. Define $\tilde{A}^*$ from A^*, as $\tilde{A}$ was from A in Lemma 4; write $A = S_\tau A^*$, $\tilde{A} = S_\tau \tilde{A}^*$, $B = K_i^{\delta_1}$ and apply Lemma 4 with ξ, η replaced respectively by $\eta^{\tau-\delta_0}$, η^τ. We must have then

$$E = S_\tau A^* \cap S_\tau \tilde{A}^* \cap K_i^\delta\ ,$$

and from the mixing property (of order 1) of the Kolmogorov flow, there exists $t_0 > 0$ such that

$$P(E) \geqq \frac{1}{2} P(A^* \tilde{A}^*) P(K_i^{\delta_1}) > 0 \qquad \text{for all } \tau \geqq t_0\ .$$

We will further fix t_0 so large that

$$t_0 - \delta_0 - \delta_2 > 2N(\varepsilon)\ . \tag{6.11}$$

Now by the definition in Lemma 4,

$$\tilde{E} = \bigcup \{C : 0 < \mu_c(E),\ C = \eta^{\tau-\delta_0}\text{-cell}\}\ ,$$

$$F=\tilde{E}A\,,$$

and

$$f_i(\omega)=\chi_F(\omega)-\mu_c(F)$$

is a non-null element in $L^2(\eta^\tau)$, orthogonal with $L^2(\eta^{\tau-\delta_0})$. From (6.8.c) it follows that

(6.12) $$\operatorname{dia} C \leqq 4\varepsilon \qquad \text{for all } \eta^{\tau-\delta_0}\text{-cell } C,\ \tau \geqq t_0\,.$$

Since $\operatorname{Car} f_i \subset \tilde{E}$[10], we have also

(6.13) $$\operatorname{Car} f_i \subset U_{\delta_1+4\varepsilon}(\omega_i)\,.$$

From (6.10), (6.11), and (6.12), one has

(6.14) $$\bigcup_{|t|\leqq\delta_2} \operatorname{car} S_t f_i \subset U_{2\delta_1+4\varepsilon}(\omega_i)\,.$$

Suppose $i\neq j$, $1\leqq i, j\leqq n$, $|t-s|\leqq\delta_2$, then to evaluate $\rho_{ij}(t-s)=(S_s f_i, S_t f_j)$, we may assume $-\delta_2\leqq s, t\leqq\delta_2$. Then by (6.14)

$$\rho_{ij}(t-s)=0\,.$$

Suppose next $1\leqq i, j\leqq n$, $t-s>\delta_2$, then $\rho_{ij}(t-s)=(f_i, S_{t-s}f_j)=0$, since $S_{s-t}f_j\in L^2(\eta^{\tau-\delta_0})$; in the same way $\rho_{ij}(t-s)=0$ for $s-t>\delta_2$. Therefore, the trajectories $T_s f_i$, $-\infty<s<\infty$, $1\leqq i\leqq n$, are orthogonal each other, and

(6.15) $$\rho_{ii}(t-s)=0 \qquad \text{for } |s-t|\geqq\delta\,.$$

By the Paley-Wiener theorem, (6.15) implies

$$\rho_{jj}(t-s)=\int_{-\infty}^{\infty} e^{i\lambda(t-s)}g(\lambda)d\lambda$$

with $g\in L(-\infty,\infty)$, $g(\lambda)>0$ almost everywhere.

Since only spectrum for the Kolmogorov flow is of uniform Lebesgue, and n in the above arguments is arbitrary, its multiplicity must be uniformly countable.

Concluding this section, we observe an operator-theoretical implication of Sinai's theorem. Let $\mathcal{H}_0=L^2(\xi)$ and $\mathcal{H}$ be the space of square-integrable functions on M. Then $\tilde{T}_t$ defined by

$$f(x)\in\mathcal{H}_0\rightarrow\tilde{T}_t f(x)=f(T_t x)$$

is a semi-group of isometries. P. Masani [5] proved a decomposition theorem connected with a semi-group of isometries on a Hilbert space. Let iH be the infinitesimal generator of $(\tilde{T}_t, t\geqq 0, \mathcal{H}_0)$, V the corresponding Cayley transformation

$$V=(H-i)(H+i)^{-1}\,.$$

10) $\operatorname{Car} f$ means the support of f, i.e. the set of all points at which f is non-null.

Then H is maximal symmetric with deficiency index $(0, n)$, $n=\dim(K)$, $K=\mathcal{H}_0\ominus V\mathcal{H}_0$. Let $(\xi_\lambda, \lambda\in\Lambda)$ be a CON base of K, and define

$$W_{st}=\frac{1}{\sqrt{2}}(\tilde{T}_t-\tilde{T}_s-\int_s^t\tilde{T}_u du) \qquad \text{(cf. [5])},$$

$$\xi_\lambda(I)=W_{st}(\xi_\lambda),\ I=[s, t],$$

then $\{\xi_\lambda(dt), \lambda\in\Lambda\}$ is an orthogonal set of orthogonal random measures, $E(\xi_\lambda(I)\xi_\mu(J))=\delta_{\lambda\mu}|I\cap J|$, which enjoys the relation

$$\tilde{T}_\tau(\xi_\alpha(I))=\xi_\alpha(\theta_\tau I),$$

where θ_τ, $\tau\geqq 0$ is the shift

$$I=[s, t]\to\theta_\tau I=[s+\tau, t+\tau].$$

Now denote by $L^2(T_+)$, $L^2(T)$, the spaces of functions which are square-integrable in the Lebesgue measure, on the respective spaces T_+, T. Every $x\in\mathcal{H}_0$ is represented as

$$x=\sum_{\lambda\in\Lambda}\int_0^\infty f_\lambda(t)\xi_\lambda(dt),\ f_\lambda\in L^2(T_+),$$

with

$$\|x\|^2=\sum_{\lambda\in\Lambda}\|f_\lambda\|^2<\infty.$$

This gives a decomposition of $\mathcal{H}_0$ into a direct sum of $\tilde{T}_t$-invariant spaces and T_t is isomorphic with that operator which transform $\{f_\lambda(t), \lambda\in\Lambda\}$ to $\{\tilde{\theta}_\tau f_\lambda(t), \lambda\in\Lambda\}$, where

$$\tilde{\theta}_\tau f(t)=f(t-\tau) \qquad \text{for } t\geqq\tau,$$

$$=0 \quad \text{for } 0\leqq t<\tau,\ f\in L^2(T_+).$$

$T_\tau\xi(dt)$, $-\infty<\tau<0$, enables us to get a random measure, $\xi_\lambda(dt)(-\infty<t<\infty)$, which is an extension of the original ξ_λ. With this extension, every $x\in\mathcal{H}$ is now represented as $x=\sum_{\lambda\in\Lambda}\int_{-\infty}^\infty f_\lambda(t)\xi_\lambda(dt)$, with $f_\lambda\in L^2(T)$, $\|x\|^2=\sum_{\lambda\in\Lambda}\|f_\lambda\|^2$. Now $\tilde{T}_t$ over $\mathcal{H}$ is isomorphic with the direct sum of the shift operators

$$\tilde{\theta}_\tau: f_\lambda(t)\to\tilde{\theta}_\tau f_\lambda(t)=f_\lambda(t-\tau).$$

Sinai's theorem signifies that for the Kolmogorov flow Λ is always countable.

§7. Transformations of a flow by means of an additive functional.

The additive functional, which has been worked out in recent few years, acquired a central position among tools for studying Markov processes [3]. A certain class of Markov processes are derived from a given one by random changes of time defined through appropriate additive functionals. However, it is worthy of remembering that the same device was used earlier by E. Hopf in ergodic theory ([4], p. 43).

Suppose that T_t, $-\infty < t < \infty$, is a measurable flow without wandering sets on the Lebesgue space (Ω, m). The case $m(\Omega) = \infty$ is not excluded, and in this case Ω is a countable union of disjoint Lebesgue spaces with finite total measures.

E. Hopf defines the functional φ:

$$\varphi_t(\omega) = \int_0^t g(\omega_s) ds, \ -\infty < t < \infty,$$

where $g(\omega)$ is a positive measurable function. φ satisfies the additivity condition

$$\varphi_{s+t}(\omega) = \varphi_s(\omega) + \varphi_t(\omega_s).$$

Precisely speaking, there exists a T_t-invariant set Ω_0, $m(\Omega - \Omega_0) = 0$ such that for any $\omega \in \Omega_0$, the mapping $t \in [0, \infty) \to \varphi_t(\omega) \in [0, \infty]$ is well defined and satisfies the above additivity condition.

Now for simplicity, we further assume that $g(\omega)$ is integrable, $m(\omega : g(\omega) = 0) = 0$, then as is well-known (cf. [4], § 13), $\Omega_0 = (\omega : \varphi_\infty = -\varphi_{-\infty} = \infty$, $\varphi_t(\omega)$ is strictly increasing in t) is a T_t-invariant set with $m(\Omega - \Omega_0) = 0$. The mapping $\omega \in \Omega_0 \to S_t\omega = T_{\tau_t(\omega)}$, $\tau_t(\omega)$ the inverse to the function $\varphi_t(\omega)$, is a $1-1$ and onto mapping. Further it is shown that $S_t\omega$ is a measurable flow with invariant measure $d\mu = g(\omega)dm$ (cf. Theorem 6). S_t is what we may call the transformed flow of T_t by the time change through φ. As its application, we will observe that the "ratio-limit theorem" (cf. [4], p. 53), can be derived as a corollary of Birkhoff-Khintchine's ergodic theorem.

Let $f(\omega)$ be an integrable with respect to m function and put $\tilde{f}(\omega) = f(\omega)/g(\omega)$, then since $\tilde{f} \in L(d\mu)$, for almost (in μ) every ω, there exists

$$\bar{f}(\omega) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \tilde{f}(\omega_s) ds. \tag{1}$$

Recalling that m and μ are mutually absolutely continuous and using the relations

$$T = \int_0^{\tau_T} g(\omega_t) dt,$$

$$\int_0^T \tilde{f}(\omega_{\tau_s(\omega)}) ds = \int_0^{\tau_T} \tilde{f}(\omega_t) g(\omega_t) dt = \int_0^{\tau_T} f(\omega_t) dt,$$

$$\mu(\lim_{t\to\infty} \tau_t(\omega) < \infty) = 0,$$

we deduce from (1) that for almost (in m) every ω

$$\bar{f}(\omega) = \lim_{T\to\infty} \left(\int_0^T f(\omega_t) dt \Big/ \int_0^T g(\omega_t) dt \right),$$

and that $(\bar{f}, g)_m = (\bar{f}, 1)_\mu = (\tilde{f}, 1)_\mu = (f, 1)_m$ [11].

Important classes of time changes depend on φ built on $g(\omega)$ which vanishes on a certain set of positive measure. For instance, if g is the characteristic function of a measurable set A, then roughly speaking, through the corresponding time change, one obtains a flow over A derived from T_t. Including such a case, a general study of transformations with a reasonably wide class of additive functionals needs a considerably careful analysis. Although we aim at considering a transformation of a flow with as wide a class of additive functionals as in Markov processes, here we confine ourselves to dealing with an intermediate class[12].

To preserve for the transformed path ω_{τ_t} necessary regularities, prerequisite must be a nice behaviour of the original path ω_t. The representation of a flow on a compact metric space (σ-finite as a measure space) is so smooth that it guarantees the regularity in the transformed path, and as is clear from § 2, there is no loss of generality in assuming that the given flow is already such one.

Suppose that there is given a flow T_t over a compact metric measure space $(\Omega, \mathcal{M}, m)$, $m(\Omega) \leqq \infty$ (when $m(\Omega) = \infty$, m is assumed to be σ-finite), whose path $T_t\omega = \omega_t$ is continuous in (ω, t), and an additive functional $\varphi_t(\omega) = \varphi(t, \omega)$, $-\infty < t < \infty$, which satisfies the following properties:

(F.1) (a) For any t, $\varphi_t(\omega)$ is measurable in ω. For any $\omega \in \Omega$, $\varphi_t(\omega)$ is a real-valued function of t with the additivity

$$\varphi_{s+t}(\omega) = \varphi_s(\omega) + \varphi_t(\omega_s), \quad -\infty < s, t < \infty .$$

(b) For almost all ω, $\varphi_t(\omega)$ is strictly increasing and continuous in t, and $\varphi_\infty(\omega) = -\varphi_{-\infty}(\omega) = \infty$.

(F.2) $E_m(\varphi_1(\omega)) < \infty$[13].

THEOREM 6. *Define*

$$\tau_t = \inf (s : \varphi_s(\omega) > t) ,$$

then

$$S_t\omega = \omega_{\tau_t}(\omega), \quad -\infty < t < \infty ,$$

is a measurable flow over $(\Omega, \mathcal{M})$ *with an invariant measure* μ *such that*

$$\mu(A) = E(\chi_A(\omega)\varphi_1(\omega))$$

for and $A \in \mathcal{M}$.

PROOF. Since from the additivity, the set of ω satisfying (b) is T_t-invariant, we may assume that (b) is satisfied for all $\omega \in \Omega$.

11) Subscripts in the brackets signify the respective measures used.

12) Generalizations of such a transformation and an interesting study of related problems will be published in the forthcoming paper by H. Totoki.

13) E_m means the expectation under the measure m.

Let $C(\Omega)$ be the space of continuous functions on Ω, and define the functional

$$\lambda(f, t)=E_m\Big(\int_0^t f(\omega_u)d\varphi_u\Big), f\in C(\Omega),$$

then by the additivity of φ

$$\begin{aligned}\lambda(f, s+t)&=E_m\Big(\int_0^{s+t} f(\omega_u)d\varphi_u\Big)\\&=E_m\Big(\int_0^t f(\omega_u)d\varphi_u\Big)+E_m\Big(\int_t^{s+t} f(\omega_u)d\varphi_u\Big)\\&=\lambda(f, t)+E_m\Big(\int_0^s f((\omega_t)_u)d\varphi_u(\omega_t)\Big)\\&=\lambda(f, t)+\lambda(f, s).\end{aligned}$$

Therefore there exists a bounded linear functional $\lambda(f)$ over $C(\Omega)$ and the corresponding measure μ over Ω such that

$$\lambda(f, t)=t\lambda(f)=t\int f(\omega)d\mu\text{ [14]}. \tag{2}$$

The above equality being extended to satisfy for any bounded measurable f, one has

$$\mu(\Omega)=E_m(\varphi_1(\omega)),\ \mu(A)=E_m\Big(\int_0^1 \chi_A(\omega_u)d\varphi_u(\omega)\Big),$$

and taking the Laplace transform of (2)

$$\int f(\omega)d\mu=\alpha E_m\Big(\int_0^\infty e^{-\alpha t}f(\omega_t)d\varphi_t\Big),\ \alpha>0. \tag{3}$$

Now we will prove that S_t is a 1—1 mapping from Ω onto itself. Since

$$\begin{aligned}s+t&=\varphi(\tau_{s+t}(\omega), \omega),\ s=\varphi(\tau_s(\omega), \omega)\\t&=\varphi(\tau_{s+t}(\omega), \omega)-\varphi(\tau_s(\omega), \omega)\\&=\varphi(\tau_{s+t}(\omega)-\tau_s(\omega), \omega_{\tau_s}(\omega)),\end{aligned}$$

there holds

$$\tau_{s+t}(\omega)=t_1+t_2,\quad\text{with}\quad t_1=\tau_s(\omega),\ t_2=\tau_t(\omega_{t_1}),$$

therefore

$$\begin{aligned}S_{s+t}\omega&=(\omega_{t_1})_{t_2}=(\omega_{t_1})_{\tau_t(\omega_{t_1})}\\&=S_t\omega_{t_1}=S_t(S_s\omega),\ -\infty<s, t<\infty,\end{aligned}$$

and obviously $S_0\omega=\omega$.

14) In the last integral sign, as well as in the following, the suppressed domain is always the whole space.

This implies that S_t, $-\infty < t < \infty$, is a $1-1$ onto mapping over Ω.

S_t is a measurable flow. Define $\tau_t^n(\omega) = \tau_{t_i}(\omega) \equiv \sigma_i(\omega)$, $t_{i-1} < t \leqq t_i$, $t_i = \frac{i}{n}$, $-\infty < i < \infty$, then $\tau_t^n(\omega) \downarrow \tau_t(\omega)$, $n \to \infty$, and

$$S_t\omega = \lim \omega_{\tau_t^n}.$$

On the other hand

$$\omega_{\tau_t^n} = \omega_{\sigma_i(\omega)} \qquad \text{for } (\omega, t) \in \Omega \times (t_{i-1}, t_i],$$

and since $(\omega, t) \to \omega_t$ is a continuous mapping, $\omega_{\sigma_i(\omega)}$ is measurable in ω. This in turn means that $\omega_{\tau_t^n(\omega)}$ and therefore $\omega_{\tau_t(\omega)}$ is measurable in (t, ω).

Finally we shall prove that μ is an invariant measure. For this purpose we are going to verify the equivalent equality

$$\alpha E_\mu\Big(\int_0^\infty e^{-\alpha u} f(\omega_{\tau_u}) du\Big) = \int f(\omega) d\mu$$

for any $f \in C(\Omega)$ and $\alpha > 0$, or

$$\alpha E_\mu\Big(\int_0^\infty e^{-\alpha\varphi_t(\omega)} f(\omega_t) d\varphi_t(\omega) = \int f(\omega) d\mu . \tag{4}$$

Now by (3), (4) is in turn equivalent to the equality

$$\alpha^2 E_m\Big(\int_0^\infty\int_0^\infty e^{-\alpha u - \alpha\varphi_t(\omega_u)} f(\omega_{t+u}) d\varphi_u(\omega) d\varphi_t(\omega_u)\Big) = \int f(\omega) d\mu . \tag{5}$$

To prove this we make equidistant divisions of the intervals $0 \leqq u < \infty$, $0 \leqq t < \infty$, setting $u_i = t_i = i/n$, $1 \leqq n < \infty$, $0 \leqq i < \infty$. Then the left-hand side of (5) is equal to

$$\begin{aligned} &\alpha^2 \lim_{n\to\infty} E_m\{\sum_{0\leqq i,j<\infty} \exp(-\alpha u_{j+1} - \alpha\varphi(t_{i+1}, \omega_{u_j})) \\ &\qquad \times f(\omega_{u+u_i})(\varphi(u_{j+1}, \omega) - \varphi(u_j, \omega))(\varphi(t_{i+1}; \omega_{u_j}) - \varphi(t_i, \omega_{u_j}))\} \\ &= \alpha^2 \lim_{n\to\infty} E_m\{\sum_{0\leqq i,j<\infty} \exp(-\alpha u_{j+1} - \alpha\varphi(t_{i+1}, \omega_{-t_i})) f(\omega) \\ &\qquad \times (\varphi(u_{j+1}, \omega_{-u_j-t_i}) - \varphi(u_j, \omega_{-u_j-t_i}))(\varphi(t_{i+1}, \omega_{-t_i}) - \varphi(t_i, \omega_{-t_i}))\} . \end{aligned} \tag{6}$$

On the other hand, the relation $\varphi(\omega_s) = \varphi_{s+t}(\omega) - \varphi_s(\omega)$ implies

$$\varphi(u_{j+1}, \omega_{-u_j-t_i}) - \varphi(u_j, \omega_{-u_j-t_i}) = \varphi(-t_i + 1/n, \omega) - \varphi(-t_i, \omega),$$

$$\varphi(t_{i+1}, \omega_{-t_i}) - \varphi(t_i, \omega_{-t_i}) = \varphi(1/n, \omega),$$

$$\varphi(t_{i+1}, \omega_{-t_i}) = \varphi(1/n, \omega) - \varphi(-t_i, \omega).$$

Inserting those into the right-hand member of (6), (6) becomes

$$\alpha^2 \lim_{n\to\infty} E_m\{\sum_{0\leqq i,j<\infty} \exp(-\alpha u_{j+1}-\alpha\varphi(1/n,\omega)) \times\exp(\alpha\varphi(-t_i,\omega))f(\omega) \times(\varphi(-t_i+1/n,\omega)-\varphi(-t_i,\omega))\varphi(1/n,\omega)\} \tag{7}$$

$$=\alpha^2 \lim_{n\to\infty} \exp(-\alpha/n)(1-\exp(-\alpha/n))^{-1} \times E_m\{f(\omega)\varphi(1/n,\omega)\exp(-\alpha\varphi(1/n,\omega))(S_1+S_2)\},$$

where

$$S_1=\sum_{i=1}^{\infty}\exp(\alpha\xi_i)(\xi_{i-1}-\xi_i),\ \xi_i=\varphi(-t_i,\omega),$$

$$S_2=\exp(-\alpha\varphi(1/n,\omega))\varphi(1/n,\omega).$$

We observe firstly

$$0\leqq S_1\leqq 1/\alpha,\ S_1\to 1/\alpha \quad \text{as} \quad n\to\infty, \qquad 0\leqq S_1\leqq 1/\alpha e,\ S_2\to 0 \quad \text{as} \quad n\to\infty. \tag{8}$$

Secondly, since Ω is compact $f(\omega_t)\to f(\omega)$ uniformly in ω, as $t\to 0$, we have

$$\int_0^{1/n} f(\omega_t)d\varphi_t(\omega)=f(\omega)\varphi(1/n,\omega)+\varepsilon_n(\omega)\varphi(1/n,\omega), \tag{9}$$

$\varepsilon_n(\omega)\to 0$ uniformly in ω, as $n\to\infty$.

Insert (8), (9) into the right-hand member of (7), then one can easily conclude that the right-hand member of (7) reduces to the expression

$$nE_m\Big(\int_0^{1/n} f(\omega_t)d\varphi_t\Big)+nO(E_m(\varepsilon_n(\omega)\varphi(1/n,\omega)))$$

$$=\int f(\omega)d\mu+0(1), \qquad \text{as} \quad n\to\infty.$$

This completes the proof.

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