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The notion of restricted ideles with application to some extension fields II

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Let k be an algebraic number field of finite degree, K be a normal extension of k of degree n, and $\mathfrak{G}$ be its galois group. Denote by s resp. $\hat{s}$ the set of all primes of k resp. of K which has degree 1 in K/k. We defined in the preceeding paper [3], which will be referred to as RI, the restricted idele group J_s resp. $J_{\hat{s}}$ of k resp. of K. And we proved that there is a one to one correspondence between some ($\mathfrak{G}$-invariant $\hat{s}$-admissible) closed subgroups H of $J_{\hat{s}}$ and abelian extensions M of K normal over k.

In this paper we shall strengthen the above consequence and the condition of H to be $\hat{s}$-admissible in RI, by studying the norm residue mapping of $J_{\hat{s}}$ to the group of the maximal abelian extension (theorem 1 and 2). Moreover we shall determine the conductor of the field M corresponding to H (theorem 3). Since the $\hat{s}$-restricted idele group $J_{\hat{s}}$ of K is $\mathfrak{G}$-isomorphic to the direct product J_s^n of n-folds of the s-restricted idele group J_s of k, H is considered a subgroup of J_s^n. So it will be interest to characterize the condition of $\hat{s}$-admissibility by terms of the ground field k. We shall do it for a special case of K/k, by substantially using the theorem 2 (theorem 4).

§1. Norm residue symbols.

Let k be an any algebraic number field of finite degree and $J=J_k$ be the (ordinary) idele group of k. Let $S=S(k)$ be the set of all (finite or infinite) primes $\mathfrak{p}$ of k, s be a subset of S, and s' be its complement in S; $S-s$. We defined in RI the *s-restricted idele group* J_s by the restricted direct product of $\mathfrak{p}$-adic completions $k_{\mathfrak{p}}$ over $\mathfrak{p}$-adic unit groups $U_{\mathfrak{p}}$ of k, where $\mathfrak{p}$ runs over s.

Then we have

$$(1) \qquad J=J_s \times J_{s'} \quad \text{(direct).}$$

We shall fix this isomorphism and embed naturally J_s into J. Denote by π_s the projection of J to J_s. The *s-restriction* ρ_s is defined by any subset A of J_s by

$$(2) \qquad \rho_s(A)=\pi_s(A \cap J_s).$$

For any normal extension K/k, denote by $\mathfrak{G}(K/k)$ its galois group. Let A_k be the maximal abelian extension of k and $\mathfrak{G}_k$ be its galois group, which is the projective limit of $\mathfrak{G}(A/k)$ of abelian extensions A over k of finite degree.

For any $\mathfrak{a} \in J$ and any abelian extension A/k of finite degree, let $(\mathfrak{a}, A/k)$ be the norm residue symbol. Let further $(\mathfrak{a}, k)$ be the (generalized) norm residue symbol of k, which is defined as an element of $\mathfrak{G}_k$ whose $\mathfrak{G}(A/k)$ component is $(\mathfrak{a}, A/k)$. Then $(\mathfrak{a}, k)$ gives a homomorphism of J_k onto $\mathfrak{G}_k$. We denote this homomorphism by Φ and call the *reciprocity map*. Denoting by $\mathfrak{a}_\mathfrak{p}$ the $\mathfrak{p}$-component of $\mathfrak{a}$, we have

$$(3) \qquad (\mathfrak{a}, k) = \prod_{\mathfrak{p} \in S} (\mathfrak{a}_\mathfrak{p}, k_\mathfrak{p})$$

where $(\mathfrak{a}_\mathfrak{p}, k_\mathfrak{p})$ is the (generalized) local norm residue symbol. For any subset s of S denote by Φ_s the restriction of Φ to J_s. Then

$$(4) \qquad \Phi_s(\mathfrak{a}_s) = (\mathfrak{a}_s, k) = \prod_{\mathfrak{p} \in s} (\mathfrak{a}_\mathfrak{p}, k_\mathfrak{p})$$

for any $\mathfrak{a}_s \in J_s$. Moreover we have immediately from the definition

$$(5) \qquad \rho_s(\Phi^{-1}(\mathfrak{H})) = \Phi_s^{-1}(\mathfrak{H})$$

for any subgroup $\mathfrak{H}$ of $\mathfrak{G}_k$[1].

Now let K be a normal extension field of k of finite degree and denote by $S(K/k)$ the set of all primes of k which are of degree 1 in K/k. Moreover denote by $\hat{S}$ the set of all primes of K and by $\hat{S}(K/k)$ the set of primes of K whose norms belong to $S(K/k)$. Put $S(K/k) = s$, $\hat{S}(K/k) = \hat{s}$.

Let A_K be as before the maximal abelian extension of K, and $\mathfrak{G}_K$ its galois group. Let further $M_1, M_2, \cdots$ be a sequence of abelian extensions of K such that $k \subset M_1 \subset M_2, \cdots$, every M_i is normal over k, and the union of all M_i is equal to A_K. Then $\mathfrak{G}_K$ is equal to the projective limit of $\mathfrak{G}(M_i/K)$. So we denote an element σ of $\mathfrak{G}_K$ by $\sigma = \{\sigma_i\}$ where $\sigma_i \in \mathfrak{G}(M_i/K)$. Then $\{\sigma_i\}$ belongs to $\mathfrak{G}_K$ if and only if the restriction of σ_i to M_j is equal to σ_j when $i \geqq j$. Denote by D_K the complete inverse image of the connected component of the unity by the canonical homomorphism of the ordinary idele group J_K to the ordinary idele class group C_K. Then we have

THEOREM 1. *The image of the norm residue mapping $\Phi_{\hat{s}}$ of $J_{\hat{s}}$ is equal to $\mathfrak{G}_K$, and the kernel of $\Phi_{\hat{s}}$ is equal to $\rho_{\hat{s}}(D_K)$. Hence we have $J_{\hat{s}}/\rho_{\hat{s}}(D) = \mathfrak{G}_K$.*

PROOF. Notations being as above, $\{\sigma_i\}$ be an any element of $\mathfrak{G}_K$ where $\sigma_i \in \mathfrak{G}(M_i/K)$. Let $\mathfrak{a}_i$ be an element of $J_{\hat{s}}$ such that $\sigma_i = (\mathfrak{a}_i, M_i/K)$, whose existence follows from theorem 1 in *RI*. Let further $H_{\hat{s}}^{(i)}$ be the subgroup of $J_{\hat{s}}$ corresponding to M_i by theorem 2 in *RI*. Then $\mathfrak{a}_i H_{\hat{s}}^{(i)} \supset \mathfrak{a}_j H_{\hat{s}}^{(j)}$ when

1) By Φ^{-1} we mean always the complete converse image.

$j \geqq i$. Let $\bigcap_i \mathfrak{a}_i H_{\hat{s}}^{(i)} = \mathfrak{a}_{\hat{s}}$, whose existence in $J_{\hat{s}}$ follows from that $H_{\hat{s}}^{(i)}$ is open and $J_{\hat{s}}$ is locally compact. Then $(\mathfrak{a}_{\hat{s}}, M_i/K) = (\mathfrak{a}_i, M_i/K) = \sigma_i$ for every i. Hence we have $\Phi_{\hat{s}}(\mathfrak{a}_{\hat{s}}) = \sigma_i$ which proves the first assersion of the theorem. Since the kernel of Φ is D_K, the other assertions of the theorem follows immediately from the definition of $\Phi_{\hat{s}}$.

We called in *RI* a subgroup $H_{\hat{s}}$ of $J_{\hat{s}}$ is $\hat{s}$-admissible if $H_{\hat{s}} = \rho_{\hat{s}}(\overline{H_{\hat{s}} D_K})$, where the bar stand for the closure in J_K. Now we have

THEOREM 2. *Let $H_{\hat{s}}$ be a closed subgroup of $J_{\hat{s}}$ of finite index. Then $H_{\hat{s}}$ is $\hat{s}$-admissible if and only if $H_{\hat{s}}$ contains $\rho_{\hat{s}}(D_K)$. If $H_{\hat{s}}$ is $\hat{s}$-admissible, then there exists uniquely the admissible*[2] *subgroup H of J of finite index such that $\rho_{\hat{s}}(H) = H_{\hat{s}}$. When that is so we have moreover $\Phi(H) = \Phi_{\hat{s}}(H_{\hat{s}})$.*

PROOF. We first note that $\mathfrak{G}_K$ is compact, J resp. $J_{\hat{s}}$ is locally compact, and Φ resp. $\Phi_{\hat{s}}$ maps J resp. $J_{\hat{s}}$ onto $\mathfrak{G}_K$. Hence both Φ and $\Phi_{\hat{s}}$ are open[3]. Suppose that $H_{\hat{s}}$ contains $\rho_{\hat{s}}(D_K)$, which is the kernel of $\Phi_{\hat{s}}$. Put $\Phi_{\hat{s}}(H_{\hat{s}}) = \mathfrak{H}$. Then since $\Phi_{\hat{s}}$ is an open and onto mapping, $\mathfrak{H}$ is a closed subgroup of $\mathfrak{G}_K$ of finite index. Put $H = \Phi^{-1}(\mathfrak{H})$. Then H is an admissible subgroup of J of finite index, and $\rho_{\hat{s}}(H) = \rho_{\hat{s}}(\Phi^{-1}(\mathfrak{H})) = \Phi_{\hat{s}}^{-1}(\mathfrak{H}) = \Phi_{\hat{s}}^{-1}(\Phi_{\hat{s}}(H_{\hat{s}})) = H_{\hat{s}}$ by (5).

Suppose that H' be also an admissible subgroup of J of finite index such that $\rho_{\hat{s}}(H') = H_{\hat{s}}$. Put $\Phi(H') = \mathfrak{H}'$. Then by using (5), $\Phi_{\hat{s}}^{-1}(\mathfrak{H}') = \rho_{\hat{s}}(\Phi^{-1}(\mathfrak{H}')) = \rho_{\hat{s}}(H') = H_{\hat{s}}$. Hence $\mathfrak{H}' = \Phi_{\hat{s}}(H_{\hat{s}}) = \mathfrak{H}$. Then since both H and H' are admissible and closed in J, we have $H = H'$. Thus the last two assertions of the theorem are proved. The assertion about the $\hat{s}$-admissibility is now an immediate consequence of the definition.

§ 2. Conductor.

Let K/k be as before a normal extension of finite degree, and put $s = S(K/k)$, $\hat{s} = \hat{S}(K/k)$. Let further $H_{\hat{s}}$ be an $\hat{s}$-admissible subgroup of $J_{\hat{s}}$ of finite degree. Then by theorem 2 there exists an abelian extension M of K which corresponds to the admissible subgroup H by means of the class field theory, where $\rho_{\hat{s}}(H) = H_{\hat{s}}$. We shall call such an M the abelian extension of K *corresponding to $H_{\hat{s}}$*. In this section we shall study the conductor of M/K.

Let $\mathfrak{P}$ be a prime of K and $\nu_{\mathfrak{P}}$ be a non negative integer. If $\mathfrak{P}$ is archimedean, $\nu_{\mathfrak{P}} = 0$ or 1. For $\mathfrak{a}_{\mathfrak{P}} \in K_{\mathfrak{P}}$ we define[4] the congruence $\mathfrak{a}_{\mathfrak{P}} \equiv 1$ (mod. $\mathfrak{P}^{\nu_{\mathfrak{P}}}$) to mean the usual congruence if $\mathfrak{P}$ finite and $\nu_{\mathfrak{P}} \geqq 1$; $\mathfrak{a}_{\mathfrak{P}}$ is a $\mathfrak{P}$-unit if $\mathfrak{P}$ finite and $\nu_{\mathfrak{P}} = 0$; $\mathfrak{a}_{\mathfrak{P}} > 0$ if $\mathfrak{P}$ real and $\nu_{\mathfrak{P}} = 1$; and if $\mathfrak{P}$ is complex, or if $\mathfrak{P}$ is real but $\nu_{\mathfrak{P}} = 0$, then we put no restriction on $\mathfrak{a}_{\mathfrak{P}}$. Denote by $\gamma_{\mathfrak{P}}(\mathfrak{P}^{\nu_{\mathfrak{P}}})$ the

2) This means that H is closed and contains D_K.
3) See Pontrjagin [4], Ch. 3, Theorem 13.
4) See Artin-Tate [2], Ch. 8, 2.

group of all elements $\mathfrak{a}_{\mathfrak{P}}$ of $K_{\mathfrak{P}}$ such that $\mathfrak{a}_{\mathfrak{P}}\equiv 1$ (mod. $\mathfrak{P}^{\nu_{\mathfrak{P}}}$). Furthermore for an idele $\mathfrak{a}$ and an integral divisor $\mathfrak{m}=\prod_{\mathfrak{P}}\mathfrak{P}^{\nu_{\mathfrak{P}}}$ define $\mathfrak{a}\equiv 1$ (mod. $\mathfrak{m}$) to mean $\mathfrak{a}_{\mathfrak{P}}\equiv 1$ (mod. $\mathfrak{P}^{\nu_{\mathfrak{P}}}$) for every $\mathfrak{P}$, and denote by $\gamma(\mathfrak{m})$ the group of all such ideles. For an integral divisor $\mathfrak{m}$ we denote by $\mathfrak{m}_{\hat{s}}$ resp. $\mathfrak{m}_{\hat{s}'}$ its $\hat{s}$ resp. $\hat{s}'$-part, and put $\gamma_{\hat{s}}(\mathfrak{m}_{\hat{s}})=\rho_{\hat{s}}(\gamma(\mathfrak{m}_{\hat{s}}))$, $\gamma_{\hat{s}'}(\mathfrak{m}_{\hat{s}})=\rho_{\hat{s}'}(\gamma(\mathfrak{m}_{\hat{s}'}))$.

Now let H be an admissible subgroup of J_K of finite index and M be the abelian extension of K corresponding to H by means of the class field theory. Then[5] it is well known that the conductor of M/K is equal to an integral divisor $\mathfrak{f}=\prod_{\mathfrak{P}}\mathfrak{f}_{\mathfrak{P}}$ where $\mathfrak{f}_{\mathfrak{P}}=\mathfrak{P}^{\nu_{\mathfrak{P}}}$, $\nu_{\mathfrak{P}}$ is the smallest non-negative integer such that $H\supset\gamma_{\mathfrak{P}}(\mathfrak{f}_{\mathfrak{P}})$ for every prime $\mathfrak{P}$.

LEMMA 1. *Let $A_{\mathfrak{P}}$ be any subgroup of $K_{\mathfrak{P}}$. Then* $\Phi_{\hat{s}}^{-1}(\Phi(A_{\mathfrak{P}}))=A_{\mathfrak{P}}\cdot\rho_{\hat{s}}(D_K)$ *or* $=\pi_{\hat{s}}(D_K\cap(J_{\hat{s}}\times A_{\mathfrak{P}}))$ *according to* $\mathfrak{P}\in\hat{s}$ *or* $\in\hat{s}'$.

PROOF. If $\mathfrak{P}\in\hat{s}$, then $\Phi_{\hat{s}}^{-1}(\Phi(A_{\mathfrak{P}}))=\Phi_{\hat{s}}^{-1}(\Phi_{\hat{s}}(A_{\mathfrak{P}}))=A_{\mathfrak{P}}\cdot\Phi_{\hat{s}}^{-1}(1)=A_{\mathfrak{P}}\cdot\rho_{\hat{s}}(D_K)$ by theorem 1. If $\mathfrak{P}\in\hat{s}'$, then $\Phi_{\hat{s}}^{-1}(\Phi(A_{\mathfrak{P}}))$ is of all $\mathfrak{a}\in J_{\hat{s}}$ such that $\Phi_{\hat{s}}(\mathfrak{a})=\Phi(\mathfrak{b}_{\mathfrak{P}})$ for some $\mathfrak{a}_{\mathfrak{P}}\in A_{\mathfrak{P}}$.

This is equivalent to $\mathfrak{a}\mathfrak{b}_{\mathfrak{P}}^{-1}\in D_K$, since the kernel of Φ is D_K. Hence $\Phi_{\hat{s}}^{-1}(\Phi(A_{\mathfrak{P}}))=\pi_{\hat{s}}(D_K\cap(J_{\hat{s}}\times A_{\mathfrak{P}}))$. Thus the lemma is proved.

THEOREM 3. *Let $H_{\hat{s}}$ be an $\hat{s}$-admissible subgroup of $J_{\hat{s}}$ and M be the abelian extension of K corresponding to $H_{\hat{s}}$. Then the conductor of M/K is equal to an integral divisor $\mathfrak{f}=\prod_{\mathfrak{P}}f_{\mathfrak{P}}$ where $\mathfrak{f}_{\mathfrak{P}}=\mathfrak{P}^{\nu_{\mathfrak{P}}}$, $\nu_{\mathfrak{P}}$ is the smallest non negative integer such that $H_{\hat{s}}\supset\gamma_{\mathfrak{P}}(\mathfrak{f}_{\mathfrak{P}})$ or $\supset\pi_{\hat{s}}(D_K\cap(J_{\hat{s}}\times\gamma(\mathfrak{f}_{\mathfrak{P}})))$ according to $\mathfrak{P}\in s$ or $\in s'$.*

PROOF. We have $\Phi^{-1}(\Phi_{\hat{s}}(H_{\hat{s}}))=H$ by theorem 2. Hence $H\supset\gamma_{\mathfrak{P}}(\mathfrak{f}_{\mathfrak{P}})$ if and only if $\Phi_{\hat{s}}(H_{\hat{s}})\supset\Phi(\gamma_{\mathfrak{P}}(\mathfrak{f}_{\mathfrak{P}}))$. This is equivalent that $H_{\hat{s}}\supset\Phi_{\hat{s}}^{-1}(\Phi(\gamma_{\mathfrak{P}}(\mathfrak{f}_{\mathfrak{P}})))$, since $H_{\hat{s}}$ is $\hat{s}$-admissible. Then the theorem implies from lemma 1 immediately.

We note that the proposition 5 in *RI* implies that the condition $H_{\hat{s}}\supset\pi_{\hat{s}}(D_K\cap(J_{\hat{s}}\times\gamma(\mathfrak{f}_{\mathfrak{P}})))$ can be replaced by $H_{\hat{s}}\supset\pi_{\hat{s}}(K^{\times}\cap(J_{\hat{s}}\times\gamma(\mathfrak{f}_{\mathfrak{P}})))$.

§3. Condition of the admissibility in the ground field (special case).

Let K/k be a normal extension of finite degree, and put $s=S(K/k)$, $\hat{s}=\hat{S}(K/k)$. It is easily proved that the number of independent units of K is equal to that of k if and only if k is totally real and K is totally imaginary and quadratic over k. In this case we shall characterize in terms of the ground field k the condition of a subgroup of $J_{\hat{s}}$ to be $\hat{s}$-admissible.

We have proved in theorem 2 that a subgroup $H_{\hat{s}}$ of $J_{\hat{s}}$ is $\hat{s}$-admissible if and only if $H_{\hat{s}}$ contains $\rho_{\hat{s}}(D_K)$. Therefore our purpose in this section is to study on $\rho_{\hat{s}}(D_K)$. The structure of D_K is known by Artin [1] as follows[6]:

5) See for instance Artin-Tate [2], Ch. 8, 2.
6) Cf. Artin and Tate [2], Ch. 9.

Let U be the group of unit ideles of K, and $U_{\mathfrak{P}}$ be the group of $\mathfrak{P}$-adic units of $K_{\mathfrak{P}}$. Then we have

$$U = \bar{U}\tilde{U} \tag{6}$$

where $\bar{U} = \prod_{\mathfrak{P} \neq \mathfrak{P}_\infty} U_{\mathfrak{P}}$ and $\tilde{U} = \prod_{\mathfrak{P}_\infty} U_{\mathfrak{P}_\infty}$. We split each unit idele $\mathfrak{a}$ as a product

$$\mathfrak{a} = \bar{\mathfrak{a}}\tilde{\mathfrak{a}}, \tag{7}$$

where $\bar{\mathfrak{a}} \in \bar{U}$, $\tilde{\mathfrak{a}} \in \tilde{U}$ and embedded ordinarily in U. Denote by $\bar{Z}$ the completion of the group Z of rational integers under the topology whose fundamental system of neighborhoods of 0 consists of all ideals of Z. Put $V = \bar{Z} + R$ (direct), where R is the group of real numbers, and denote any element $\lambda \in V$ as $\lambda = (x, h)$, where $x \in \bar{Z}$ and $h \in R$. For any element $\mathfrak{a} = U$, the power $\mathfrak{a}^\lambda$ is defined by

$$\mathfrak{a}^\lambda = \bar{\mathfrak{a}}^x \tilde{\mathfrak{a}}^h, \tag{8}$$

where $\bar{\mathfrak{a}}^x$ is the generalization of the ordinary power with regard to the above topology. Let $\phi_j(t)$ the idele which has the component $e^{2\pi i t}$ at j-th complex prime and 1 at all other primes. Denote by T the group generated by all such $\phi_j(t)$, $j = 1, \cdots, r_2$. Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r$ be a system of independent totally positive units of K, and denote by E_K the group of all elements $\varepsilon_1^{\lambda_1} \cdots \varepsilon_r^{\lambda_r}$ where $\lambda_i = (x_i, h_i) \in V$ $(i = 1, \cdots, r)$. Furthermore denote by L the group of ideles which has a real number as the component at the infinite prime fixed once for all, and 1 at all other primes. Then we have by Artin [1]

$$D_K = E_K \cdot T \cdot L \cdot K^*, \tag{9}$$

where K^* is the multiplicative group of non zero elements of K which is embedded ordinarily in J_K.

Now let k be a totally real number field of finite degree, and K be a totally imaginary and quadratic over k. Then we can take in k the above system $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r$ of independent units of K, and we have

$$\begin{aligned} D_k &= E \cdot L \cdot k^*, \\ D_K &= E \cdot T \cdot L \cdot K^*, \end{aligned} \tag{10}$$

where[7)] $E = E_k = E_K$.

LEMMA 2. *Let K/k be as above. Then we have*

$$\rho_{\mathfrak{s}}(D_K) = \rho_s(D_k).$$

PROOF. Let r be the number of independent units of k, which is equal to that of K. Generally denote by α, $\mathfrak{e}$, ϕ and ν elements of K, E, T and L respectively. Then $\rho_{\mathfrak{s}}(D_K)$ is of all $\pi_{\mathfrak{s}}(\mathfrak{e}\phi\nu\alpha)$ such that $\pi_{\mathfrak{s}'}(\mathfrak{e}\phi\nu\alpha) = 1$ by (10).

7) We always embed J_k into J_K by ordinal way.

This is equal to the set of all $\pi_{\hat{s}}(\mathfrak{e}\alpha^{-1})$ such that $\pi_{\hat{s}'}(\mathfrak{e})=\pi_{\hat{s}'}(\phi\nu\alpha)$. By the assumption of K/k, all infinite primes of K is contained in $\hat{s}'$. Hence $\pi_{\hat{s}'}(\mathfrak{e})=\pi_{\hat{s}'}(\phi\nu\alpha)$ is equivalent to $\pi_{\hat{s}'}(\bar{\mathfrak{e}})=\pi_{\hat{s}'}(\alpha)$ and $\pi_{\hat{s}'}(\tilde{\mathfrak{e}})=\pi_{\hat{s}'}(\phi\nu\tilde{\alpha})$. But the last condition is unnecessary. Because for any $\alpha \in K$ the equality $\tilde{\mathfrak{e}}=$ always a solution with respect to $\tilde{\mathfrak{e}}$, ϕ, ν. Now for any $\sigma \in \mathfrak{G}(K/k)$ and $x \in \bar{Z}$ we have easily $(\bar{\varepsilon}^{x})^{\sigma}=(\bar{\varepsilon}^{\sigma})^{x}$ by the definition of the generalized power. Then since $\bar{e}=\bar{\varepsilon}_1^{x_1}\cdots\bar{\varepsilon}_r^{x_r}$ where $\varepsilon_i \in k$, $\pi_{\hat{s}}(\bar{\mathfrak{e}})=\pi_{\hat{s}}(\alpha)$ implies $\alpha \in k$. Hence $\rho_{\hat{s}}(D_K)$ consists of all $\pi_{\hat{s}}(\mathfrak{e}\alpha^{-1})$ such that $\pi_{\hat{s}'}(\bar{\mathfrak{e}})=\pi_{\hat{s}'}(\alpha)$ where $\alpha \in k$.

By the same way as above we see $\rho_s(D_k)$ consists of all $\pi_s(\mathfrak{e}\alpha^{-1})$ such that $\pi_{s'}(\bar{\mathfrak{e}})=\pi_{s'}(\alpha)$ where $\alpha \in k$. Hence we have the lemma.

Now by theorem 2 and lemma 2 we have

THEOREM 4. *Let k be a totally real algebraic number field of finite degree and K be its quadratic extension which is totally imaginary. Put $s=S(K/k)$. We embed k^* and $\rho_s(D_k)$ diagonaly into the direct product $J_s \times J_s$ of s-restricted idele groups of k. Then there is a following one to one correspondence between the set of all closed subgroups H of $J_s \times J_s$ of finite index which contains $\rho_s(D_k)$ and the set of all abelian extensions M of K of finite degree: When M corresponds to H, a prime $\mathfrak{p}$ of k splits completely in M if and only if $\mathfrak{p} \in s$ and $k_{\mathfrak{p}} \times k_{\mathfrak{p}} \subset H$.*

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