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On the rate of growth of Hurwitz functions of a complex or p-adic variable

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§ 1. Introduction.

A Hurwitz function is a function of one variable z whose derivatives of all orders at $z=0$ are integral[1].

It is nearly obvious that any transcendental entire Hurwitz function must be at least of exponential order and type one. Various improvements on this fact have been found by Kakeya [1] and Pólya [2].

S. Kakeya proved that any entire Hurwitz function whose maximum modulus $M(r)=\max_{|z|=r}|f(z)|$ satisfies the relation

$$\lim_{r\to\infty}\frac{r^{1/2}M(r)}{e^r}=0 \tag{1.1}$$

must be a polynomial.

G. Pólya sharpened this theorem to show that a transcendental entire Hurwitz function must satisfy

$$\limsup_{r\to\infty} M(r)e^{-r}\sqrt{r}\geqq\frac{1}{\sqrt{2\pi}} \tag{1.2}$$

while the function

$$\varphi(z)=\sum_{\nu=0}^{\infty}\frac{z^{(2^\nu)}}{(2^\nu)!} \tag{1.3}$$

satisfies

$$M(r)e^{-r}\sqrt{r}<\frac{1}{\sqrt{2\pi}}+\varepsilon \tag{1.4}$$

for $r>r_0(\varepsilon)$ and $\varepsilon>0$.

All these conditions are in terms of certain upper limits involving the maximum function $M(r)$. It may therefore be of interest to establish a precise dividing line for the growth of $M(r)$ below which one finds only polynomials. We do this in the present paper and discuss certain generalizations on the concept of entire Hurwitz functions.

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1) All results discussed here are equally valid if we consider Gaussian integers or integers in any imaginary quadratic number field instead of rational integers.

§ 2. Transcendental entire Hurwitz function of slow growth.

DEFINITION.

$$\varphi(r)=\max_n \frac{r^n}{\Gamma(n+1)} \qquad (r \geqq 0).$$

THEOREM 1. *Let $\psi(r)$ be any increasing function such that for every N, there exists an r_N so that $\psi(r)>r^N$ for all $r>r_N$, then there exists a non-denumerable set of Hurwitz functions satisfying $M(r)<\varphi(r)+\psi(r)$ for all $r \geqq R$, where R is a suitable positive number depending only on ψ.*

PROOF. Let $0 \leqq n_1 < n_2 < \cdots < n_i < \cdots$ be a sequence of integers such that $n_{i+1} > 4n_i$ and

$$\psi(r) > 2r^{n_i-1} \tag{2.1}$$

for any $r > n_i/4$. Then the function

$$f(z)=\sum_{i=1}^{\infty} \frac{z^{n_i}}{n_i!} \tag{2.2}$$

has the described property. For $\frac{n_k}{4} < r \leqq \frac{n_{k+1}}{4}$ we have

$$M(r)=\sum_{i=1}^{\infty} \frac{r^{n_i}}{n_i!}=\sum_{i<k} \frac{r^{n_i}}{n_i!}+\frac{r^{n_k}}{n_k!}+\sum_{i>k} \frac{r^{n_i}}{n_i!}=S_1+S_2+S_3. \tag{2.2'}$$

We now estimate S_1, S_2 and S_3. By (2.1) we have

$$S_1=\sum \frac{r^{n_i}}{n_i!} \leqq n_{k-1} \cdot \frac{r^{n_{k-1}}}{n_{k-1}!} \leqq r^{n_k-1} < \frac{1}{2}\psi(r). \tag{2.3}$$

By the definition of φ, we have

$$S_2=\frac{r^{n_k}}{n_k!} \leqq \varphi(r). \tag{2.4}$$

Now, since $n! > (n/e)^n$, we have

$$\begin{aligned} S_3=\sum_{i>k} \frac{r^{n_i}}{n_i!} &\leqq \frac{r^{n_{k+1}}}{n_{k+1}!} \cdot \sum_{l=0}^{\infty}\left(\frac{1}{4}\right)^l \\ &< 2\cdot\left(\frac{e}{4}\right)^{n_{k+1}} < 1 < \frac{1}{2}\psi(r) \end{aligned} \tag{2.5}$$

for r sufficiently large, say $r > R$.

Combining (2.3), (2.4) and (2.5), we get

$$M(r) < \varphi(r)+\psi(r) \qquad \text{for} \quad r > R.$$

It is clear that any infinite subsequence of our sequence $\{n_i\}$ gives rise to a transcendental entire Hurwitz function whose rate of growth is not greater than that of $f(z)$ and there are $2^{\aleph_0}$ such subsequences.

F. Gross [15] has generalized this result to functions of several complex variables.

§ 3. Lower bounds for the growth of transcendental entire Hurwitz functions.

THEOREM 2. *If $M(r) < \varphi(r) + r^n$ for some n and all $r > r_0$, then $f(z)$ is a polynomial.*

PROOF. For

$$f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} z^k$$

we have

$$[M(r)]^2 \geqq \frac{1}{2\pi r} \oint_{|z|=r} |f(z)|^2 \cdot |dz| = \sum_{k=0}^{\infty} \frac{|a_k|^2}{(k!)^2} r^{2k}. \tag{3.1}$$

Now pick $r_1 \geqq r_0$ so that $\varphi(r) > r^n$ for all $r > r_1$, then for any $k > r_1$ we have $|a_k| \leqq 1$, since otherwise (i.e. $|a_k| \geqq 2$) we should have

$$[M(r)]^2 \geqq 4[\varphi(r)]^2 > (\varphi(r) + r^n)^2 \tag{3.2}$$

when $r^k/k! = \varphi(r)$ or $r \approx k + \frac{1}{2} > r_1$, contrary to hypothesis.

Assume now $|a_k| = 1$ for some sufficiently large k and that $a_l \neq 0$ for $l < k$ then, if we pick r so that $\varphi(r) = r^k/k!$ we have, from (3.1)

$$[\varphi(r)]^2 + \frac{r^{2l}}{(l!)^2} < [\varphi(r)]^2 + 2r^n\varphi(r) + r^{2n} \tag{3.3}$$

or cancelling $[\varphi(r)]^2$, taking logarithm, considering $r \approx k + \frac{1}{2}$ and $\log l = O(\log r)$ we have

$$2l \log r - 2l \log l + 2l < r + O(\log r). \tag{3.4}$$

Dividing by r and setting $\lambda = r/l$ (> 1), we get

$$\frac{2(\log \lambda + 1)}{\lambda} < 1 + O\left(\frac{\log r}{r}\right) = 1 + o(1). \tag{3.5}$$

The function $y = 2(\log x + 1)/x$ satisfies $y > 1$ for $1 \leqq x \leqq 5$.

Since $\lambda > 1$ this inequality leads to $\lambda > 4$ for sufficiently large r and hence we get $l < k/4$ for all large k. In other words $a_m = 0$ for all $k/4 \leqq m < k$ whenever $a_k \neq 0$, $k > k_0$.

If $f(z)$ is transcendental we can pick $k_0 < k_1 < k_2$ so that $a_{k_1} \neq 0$, $a_{k_2} \neq 0$ and $a_m = 0$ for $k_1 < m < k_2$. If we pick r so that $r^{k_2}/k_2! = \varphi(r)$ and $\arg(z)$ so that $\arg(a_{k_1} z^{k_1}) = \arg(a_{k_2} z^{k_2})$, then $M(r) \geqq |f(z)|$ for this particular z. Thus we have

$$M(r) \geqq \varphi(r) + \frac{r^{k_1}}{k_1!} - \sum_{l < k_1/4} \frac{|a_l|}{l!} \cdot r^l - \sum_{m > k_2} \frac{r^m}{m!}. \tag{3.6}$$

Now we estimate these terms.

Since $r>k_1/4$, we have

$$\sum_{l<k_1/4}\frac{|a_l|}{l!}\cdot r^l\leqq A\cdot\sum_{l<k_1/4}\frac{r^l}{l!} \tag{3.7}$$
$$\leqq A\cdot\frac{r^{k_1/4}}{\Gamma\left(\frac{k_1}{4}+1\right)}\cdot\sum_{l<k_1/4}1=A\cdot\frac{k_1}{4}\frac{r^{k_1/4}}{\Gamma\left(\frac{k_1}{4}+1\right)}$$
$$=A\cdot\frac{r^{k_1/4}}{\Gamma\left(\frac{k_1}{4}\right)}<\frac{1}{2}\cdot\frac{r^{k_1}}{k_1!}$$

where $A=\max\limits_{l=1,2,3,\cdots,(k_1/4)}\{|a_l|\}$ is bounded, since $|a_l|\leqq 1$ for sufficiently large l.

$$\sum_{m>4k_2}\frac{r^m}{m!}<\frac{r^{4r}}{\Gamma(4r+1)}\sum_{s=0}^{\infty}\left(\frac{1}{4}\right)^s<2\cdot\left(\frac{e}{4}\right)^{4r}<1 \tag{3.8}$$

for sufficiently large r.

Now we must show that $r^{k_1}/k_1!$ is greater than r^N for sufficiently large r.

Here we have two possibilities. If $\sqrt{r}\leqq k_1<r/4$, then

$$\frac{r^{k_1}}{k_1!}>\left(\frac{r}{k_1}\right)^{k_1}>4^{k_1}>4^{\sqrt{r}}>2r^N \tag{3.9}$$

for r sufficiently large. If $k_1<\sqrt{r}$, then

$$\frac{r^{k_1}}{k_1!}>\left(\frac{r}{\sqrt{r}}\right)^{k_1}=r^{k_1/2}>2r^N \tag{3.10}$$

for r, and hence k_1, sufficiently large.

Thus for sufficiently large choices of k_1, k_2 and suitable choices of r, we get $M(r)>\varphi(r)+r^N$ contrary to hypothesis.

Using Stirling's formula for $\Gamma(n)$ we can obtain an asymptotic expansion for $\log\varphi(r)$ of the form

$$\log\varphi(r)=r-\frac{1}{2}\log r-\frac{1}{2}\log 2\pi+\frac{c_1}{r}+\frac{c_2}{r^2}+\cdots \tag{3.11}$$
$$+\frac{c_n}{r^n}+O\left(\frac{1}{r^{n+1}}\right)$$

where the c_ν are rational and can be computed successively. In particular, we have $c_1=1/24$ and $c_2=0$ which leads to the following improvement on the results of Kakeya and Pólya.

COROLLARY. *For every $\varepsilon>0$ there exists a transcendental entire Hurwitz function with*

$$\limsup_{r\to\infty}\frac{\sqrt{2\pi r}\cdot e^{-r}M(r)}{\left(1+\frac{1+\varepsilon}{24r}\right)}<1 \tag{3.12}$$

while every entire Hurwitz function for which

$$\limsup_{r\to\infty} \frac{\sqrt{2\pi r}\cdot e^{-r}M(r)}{\left(1+\frac{1-\varepsilon}{24r}\right)} \leqq 1 \tag{3.13}$$

is a polynomial.

§ 4. Two-point Hurwitz functions.

Both authors [3], [4] have considered functions all of whose derivatives are integral valued at several integral points. While the results are satisfactory in so far as the minimal order of such transcendental functions is concerned, we have so far been able to determine the minimal type only in the case of two points. In the particular case of functions which are entire Hurwitz functions at two consecutive integers, say 0 and 1, much of the analysis of § 2 and § 3 can be carried out to give sharper results on their rate of growth. In this and later sections we use $[x]$ to denote the greatest integer, not exceeding x.

LEMMA. *If $f(z)$ is an entire Hurwitz function with $M(r)=\psi(r)$, then $g(z)=f(z(z-1))$ is an entire Hurwitz function at 0 and 1 with $M(r)\leqq\psi(r(r+1))$.*

PROOF. Obvious.

As a consequence of Theorem 1 we have, therefore,

THEOREM 3. *Let $\varphi(r)$ and $\psi(r)$ be as in Theorem 1. Then there exists a non-denumerable set of entire Hurwitz functions at 0 and 1 with $M(r)<\varphi(r(r+1))+\psi(r)$ for $r>R$ where R depends only on ψ.*

On the other hand, we can prove

THEOREM 4. *If $g(z)$ is an entire Hurwitz function at 0 and 1 and $M(r)\leqq\varphi(r(r-1))$ for all $r>r_0$, then $g(z)$ is a polynominal.*

PROOF. We can expand $g(z)$ in a series

$$g(z)=\sum_{n=0}^{\infty} a_n z^{\left[\frac{n+1}{2}\right]}\cdot(z-1)^{[n/2]} \tag{4.1}$$

where

$$a_n=\frac{1}{2\pi i}\oint_{|\zeta|=r}\frac{g(\zeta)}{\zeta^{\left[\frac{n+2}{2}\right]}(\zeta-1)^{\left[\frac{n+1}{2}\right]}}\,d\zeta. \tag{4.2}$$

Now, $[n/2]!\cdot a_n$ is an integer, because

$$(g^{([n/2])}(\zeta))_{\zeta=0,1}=\sum_{k=[n/2]}^{n-1} c_k a_k \pm [n/2]!\cdot a_n=\text{integer} \tag{4.3}$$

and c_k is divisible by $[n/2]!$, so by induction hypothesis

$$\sum_{k=[n/2]}^{n-1} c_k a_k=\text{integer} \tag{4.3'}$$

and $[n/2]!\cdot a_n$ is an nteger. Thus $g(z)$ is an entire Hurwitz function at 0 and 1 if and only if

(4.4) $$g(z)=\sum_{n=0}^{\infty}\frac{b_n}{[n/2]!}z^{[\frac{n+1}{2}]}(z-1)^{[n/2]}$$

with $b_n=$ integer for $n=0, 1, 2, \cdots$. From (4.2) we get

(4.5) $$|b_n|<\frac{[n/2]!\,M(r)}{r^{[n/2]}(r-1)^{[\frac{n+1}{2}]}}\leqq\frac{M(r)}{\varphi(r(r-1))},$$

if we choose r so that

(4.6) $$\varphi(r(r-1))=\frac{(r(r-1))^{[n/2]}}{[n/2]!}.$$

Thus, if $M(r)\leqq\varphi(r(r-1))$ for $r>r_0$ then $|b_n|<1$, and hence $b_n=0$, for $n>n_0$. This means that $g(z)$ is a polynomial.

REMARK. Comparing Theorems 3 and 4 we see that there exist transcendental entire Hurwitz functions at 0 and 1 with $M(r)<\exp(r^2+r-\log r+O(1))$ while there do not exist such functions with $M(r)<C\exp(r^2-r-\log r)$ for some positive constant C. Therefore, there remains the question of the best constant $-1\leqq\sigma_1\leqq 1$ so that all transcendental entire Hurwitz functions at 0 and 1 must satisfy $M(r)<\exp(r^2+\sigma_1 r+o(r))$ for arbitrarily large r.

We can recapture the precision attained in the one point case if we modify the region in which we maximize $|f(z)|$.

DEFINITION.

(4.7) $$M^*(r)=\max_{|z(z-1)|\leqq r^2}|f(z)|.$$

THEOREM 5. *If $f(z)$ is an entire function at* 0 *and* 1 *and $M^*(r)<\varphi(r^2)+r^n$ for some n and all $r>R$, then $f(z)$ is a polynomial. On the other hand, if $\psi(r)$ is as in Theorem* 1, *then there exists a non-denumerable set of entire Hurwitz functions at* 0 *and* 1 *with $M^*(r)<\varphi(r^2)+\psi(r)$.*

PROOF. Equation (4.4) states that $f(z)$ is an entire Hurwitz function at 0 and 1 if and only if it can be expressed in the form $f(z)=F_1(w)+zF_2(w)$ where $w=z(z-1)$ and F_1, F_2 are entire Hurwitz functions. Since M^* for F_1, F_2 is the ordinary maximum function expressed in terms of $|w|^{1/2}$ if we consider them as functions of w, the second part of the theorem follows immediately from Theorem 1 if we set $F_2=0$ and pick F_1 as one of the functions constructed there with $\psi(r)$ replaced by $\psi(r^{1/2})$.

If F_2 is a polynomial, then the first part of the theorem follows immediately from Theorem 2 applied to F_1. If F_2 is transcendental, let w_0 be a point on $|w|=r^2$ for which $|F_2(w)|$ is maximal.

To this w_0 there correspond the two values $z_0=\frac{1}{2}\pm\sqrt{w_0+\frac{1}{4}}$ whose arguments differ by an amount which approaches π as $r\to\infty$. All we need is that the angle is greater than 2θ for some $\theta>0$ for r sufficiently large. By the cosine theorem we have

$$(4.8)\qquad |F_1(w_0)+z_0F_2(w_0)|^2 \geqq |F_1(w_0)|^2+|z_0|^2|F_2(w_0)|^2-2|z_0|\,|F_1(w_0)|\,|F_2(w_0)|\cos\theta$$
$$=(|F_1(w_0)|-|z_0|\,|F_2(w_0)|\cos\theta)^2+|z_0|^2\,|F_2(w_0)|^2\sin^2\theta$$

for at least one of the choices of z_0, and hence $M^*(r)<\varphi(r^2)+r^n$ would imply

$$(4.9)\qquad \max_{|w|=r^2}|F_2(w)|<\frac{c}{r^2}(\varphi(r^2)+r^n)\,,\qquad c=\text{constant}$$

which would imply, by Theorem 2, that F_2 is a polynomial, contrary to hypothesis.

§5. On the p-adic behavior of Hurwitz functions.

One of the main interests of Hurwitz functions is the fact that they represent functions which are analytic, or at least formal power series, in every completion of the rationals, or, more generally, the completion of the field generated by the coefficients.

The classical results, like the Pólya-Carlson theorem, have been successfully combined with p-adic analysis by B. Dwork in order to prove the rationality of certain zeta-functions. Similarly, recent work by C. Pisot has used p-adic analysis to generalize results by Salem and himself on closed sets of algebraic numbers (P-V numbers), [**11**], [**12**].

In this section, we consider some simple consequences of the p-adic behavior of Hurwitz functions which generalize the results of the preceding sections.

As usual, we define the p-adic valuation of a rational number $a/b=p^{\alpha}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ by $|a/b|_p=p^{-\alpha}$.

The completion of the rational field Q under this valuation is the p-adic field Q_p which can be considered analogous to the field of real numbers which is the completion of Q under the ordinary absolute value.

The analogue of the complex numbers is obtained by taking the algebraic closure of Q_p and then completing this field under the extension of the p-adic valuation. We denote this field by Ω_p.

It is now possible to consider analytic functions over Ω_p. In particular, entire functions are power series with coefficients in Ω_p which converge (p-adically) for all values of the variable.

In the following, we use only the fact that Hadamard's formula for the radius of convergence of a power series remains valid in Ω_p, [**11**], [**12**].

Since a Hurwitz function has rational Taylor series coefficients $a_n/n!$ where a_n's are integers, and $|a_n/n!|_p\leqq|1/n!|_p<p^{n/(p-1)}$, it represents a function

analytic for p-adic numbers z with $|z|_p < p^{-1/(p-1)}$.

If the p-adic radius of convergence R_p exceeds $p^{-1/(p-1)}$, then the a_n will have to be divisible by increasing powers of p, thereby increasing the lower bound on the rate of growth of transcendental Hurwitz functions. More precisely we have the following.

THEOREM 6. *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n/n!$ *represents a transcendental Hurwitz function with p-adic radii of convergence* R_p, *then* $f(z)$ *is at least of exponential order and type*

$$\sigma = \prod_p R_p p^{\frac{1}{p-1}} . \tag{5.1}$$

Conversely, assume $\sigma > 1$, *then given any function* $\chi(r)$ *so that* $\liminf_{r\to\infty} \chi(r)/r \geqq \sigma$ *there exists a non-denumerable set of such Hurwitz functions with* $M(r) < e^{\chi(r)}$ *for* $r > R$ *where* R *depends only on* χ.

PROOF. By Hadamard's formula we have

$$R_p = \liminf_{n\to\infty} \sqrt[n]{|n!/a_n|_p} = p^{-\frac{1}{p-1}} \liminf_{n\to\infty} \sqrt[n]{|1/a_n|_p} . \tag{5.2}$$

In other words

$$|a_n|_p < (R_p \cdot p^{\frac{1}{p-1}} - \varepsilon)^{-n} \tag{5.2'}$$

for any $\varepsilon > 0$ and all $n > n_{\varepsilon,p}$. If $R_p = \infty$, this must be interpreted to mean that $|a_n|_p < \varepsilon^n$ for any $\varepsilon > 0$ and $n > n_{\varepsilon,p}$.

In case $R_p = \infty$ for some p, then for every $a_n \neq 0$ with $n > n_{\varepsilon,p}$ we have $|a_n| \geqq |a_n|_p^{-1} > \varepsilon^{-n}$ so that the type of $f(z)$ is

$$\limsup_{n\to\infty} |a_n|^{1/n} > 1/\varepsilon , \tag{5.3}$$

that is, $f(z)$ is of at least maximal type of exponential order.

If $R_p < \infty$ for all p, we pick positive N and δ. Define

$$\sigma_N = \prod_{p\leqq N} R_p \cdot p^{\frac{1}{p-1}} \tag{5.4}$$

(so that $\sigma_N \to \sigma$ as $N \to \infty$) and pick $\varepsilon > 0$ so small that $\prod_{p\leqq N} (R_p p^{\frac{1}{p-1}} - \varepsilon) > \sigma_N - \delta$.

Finally let $n_0 = \max_{p\leqq N} \{n_{\varepsilon,p}\}$. Then for every $a_n \neq 0$ with $n > n_0$ we have

$$|a_n| \geqq \prod_{p\leqq N} |a_n|_p^{-1} > (\sigma_N - \delta)^n \tag{5.5}$$

so that the type of $f(z)$ satisfies

$$\sigma = \limsup_{n\to\infty} |a_n|^{1/n} > \sigma_n - \delta . \tag{5.5'}$$

If we let $N \to \infty$ and $\delta \to 0$, then we get the desired result.

To construct functions with $M(r) < e^{\chi(r)}$, $r > r_0$, we set $f(z) = \sum_k a_{n_k} z^{n_k}/n_k!$ where $0 < n_1 < n_2 < \cdots$ so that $n_k^4 < n_{k+1}$ and the a_{n_k} are positive integers so that $a_{n_k} \leqq n_k^{n_k/4}$. Then for $\sqrt{n_k} < r \leqq \sqrt{n_{k+1}}$ we have

$$M(r) = \sum_{i=1}^{\infty} \frac{a_{n_i}}{n_i!} r^{n_i} = \sum_{i<k} \frac{a_{n_i}}{n_i!} \cdot r^{n_i} + \frac{a_{n_k}}{n_k!} \cdot r^{n_k} + \sum_{i>k} \frac{a_{n_i}}{n_i!} r^{n_i}. \tag{5.6}$$

Here

$$\sum_{i<k} \frac{a_{n_i}}{n_i!} r^{n_i} < n_{k-1} r^{n_{k-1}} < r^{\sqrt{r}+1} < \frac{1}{3} e^r < \frac{1}{3} e^{\chi(r)} \tag{5.7}$$

for $r > R$, and

$$\begin{aligned}\sum_{i>k} \frac{a_{n_i} r^{n_i}}{n_i!} &< \frac{n_{k+1}^{n_{k+1}/4} r^{n_{k+1}}}{n_{k+1}!} \sum_{s=0}^{\infty} \left(\frac{e}{\sqrt{r}}\right)^s < 2\left(\frac{e}{\sqrt{r}}\right)^{n_{k+1}} \\ &< 1 < \frac{1}{3} e^{\chi(r)}\end{aligned} \tag{5.8}$$

for $r > R$. If we impose the additional condition

$$(3a_{n_k})^{1/n_k} < \inf_{\sqrt{n_k} < r \leqq \sqrt{n_{k+1}}} \chi(r)/r = b_k \qquad (k > k_0)$$

we find that all three terms in (5.6) are less than $\frac{1}{3} e^{\chi(r)}$ and hence $M(r) < e^{\chi(r)}$ for $r > R$.

If $\sigma = \infty$, we set

$$c_k = \frac{1}{3} \min \{b_k, n_k^{n_k/4}\} \tag{5.9}$$

and choose

$$a_{n_k} = \prod_{p:\ \mathrm{prime}} p^{\left[\frac{\log c_k}{p^2 \log p}\right] n_k}. \tag{5.10}$$

This satisfies $a_{n_k}^{1/n_k} < c_k$ since

$$\sum_{p:\ \mathrm{prime}} \log p \cdot \left[\frac{\log c_k}{p^2 \log p}\right] \leqq \log c_k \sum_{p:\ \mathrm{prime}} \frac{1}{p^2} < \log c_k \tag{5.11}$$

so that indeed $M(r) < e^{\chi(r)}$. On the other hand we have

$$|a_{n_k}|_p^{1/n_k} = p^{-\left[\frac{\log c_k}{p^2 \log p}\right]} \to 0 \tag{5.12}$$

as $k \to \infty$ so that $R_p = \infty$ for all p.

If $\sigma < \infty$, we set $a_{n_k} = 1$ for all k with $b_k > n_k^{n_k/4}$ and

$$a_{n_k} = \prod_{p:\ \mathrm{prime}} p^{\left[\left(\frac{\log R_p}{\log p} + \frac{1}{p-1}\right) \frac{n_k \log b_k - 3}{\log \sigma}\right]} \tag{5.13}$$

then clearly $a_{n_k} < \frac{1}{3} b_k^{n_k}$ and

$$(5.14)\qquad |a_{n_k}|_p^{1/n_k}=(R_p p^{\frac{1}{p-1}})^{-\frac{\log b_k}{\log\sigma}+o\left(\frac{1}{n_k}\right)}.$$

Thus

$$(5.15)\qquad \limsup_{n_k\to\infty}|a_{n_k}|_p^{1/n_k}=(R_p p^{\frac{1}{p-1}})^{-1}$$

and the p-adic radius of convergence is R_p.

It may be worth emphasising a special case of our theorem.

COROLLARY. *Every transcendental Hurwitz function which is entire in every p-adic field is at least of exponential order, maximal type. There exist such Hurwitz functions satisfying $M(r)<e^{\chi(r)}$, $r>R$, for any $\chi(r)$ with $\chi(r)/r\to+\infty$.*

The p-adic comments of this section can be extended to the two-point case of § 4 without difficulty.

THEOREM 7. *Let $f(z)$ be a transcendental Hurwitz function at 0 and 1 with p-adic radii of convergence R_p, then $f(z)$ is at least of order 2, and in case it is of order 2, then its type is at least σ, where*

$$(5.16)\qquad \sigma=\prod_{R_p\leqq 1}(R_p p^{1/(p-1)})\cdot\prod_{R_p>1}(R_p^2 p^{1/(p-1)}).$$

Conversely, assume that $\sigma>1$, then given any function $\chi^*(r)$ so that $\liminf_{r\to\infty}\chi^*(r)/r^2\geqq\sigma$, then there exists a non-denumerable set of such Hurwitz functions with $M(r)<e^{\chi^*(r)}$ for $r>R$, where R depends only on $\chi^*(r)$.

PROOF. As in the proof of Theorem 5, let $f(z)=F_1(w)+zF_2(w)$ where $w=z(z-1)$ and $F_i(w)$, $(i=1, 2)$ are Hurwitz functions, at least one of which is transcendental. Then for $|z|_p<1$ we have $|w|_p=|z|_p\cdot|z-1|_p=|z|_p$ and for $|z|_p>1$, we have $|w|_p=|z|_p\cdot|z-1|_p=|z|_p^2$. Therefore, the p-adic radius of convergence $R_{i,p}(w)$ of $F_i(w)$ is given by

$$(5.17)\qquad R_{i,p}(w)=\begin{cases} R_{i,p}(z) & \text{if } R_{i,p}(z)\leqq 1\\ (R_{i,p}(z))^2 & \text{if } R_{i,p}(z)>1\end{cases}\qquad (i=1, 2)$$

where $R_{i,p}(z)$ is the p-adic radius of convergence of the function $G_i(z)=F_i(z(z-1))$.

Since $F_i(w)$, $(i=1, 2)$ are Hurwitz functions with respect to w, it is the direct consequence of Theorem 6 that if $F_i(w)$ is transcendental, it is at least of exponential order and type σ_i where

$$(5.18)\qquad \sigma_i=\prod_p R_{i,p}(w)p^{1/(p-1)}$$
$$=\prod_{R_{i,p}(z)\leqq 1}R_{i,p}(z)p^{1/(p-1)}\cdot\prod_{R_{i,p}(z)>1}(R_{i,p}(z))^2p^{1/(p-1)}.$$

Since $R_p=\min\{R_{1,p}(z), R_{2,p}(z)\}$, we have

$$\sigma_i \geqq \prod_{R_p \leqq 1} R_p p^{1/(p-1)} \cdot \prod_{R_p > 1} R_p^2 p^{1/(p-1)} = \sigma . \tag{5.19}$$

This means that our function $f(z)$ is at least of order 2, and in case the order $=2$, then at least of type $\max(\sigma_1, \sigma_2) \geqq \sigma$, we get the desired result.

To construct functions with $M(r) < e^{\chi^*(r)}$, $r > R$, we use again the result of Theorem 6. Let $w = z(z-1)$, then, since $\sigma > 1$, we have $|w| = |z(z-1)| \leqq r^2 + r < (\sigma - \varepsilon) r^2 < \chi^*(r)$ for some $\varepsilon > 0$, $r > R$.

Form $f(w) = \sum_k a_{n_k} w^{n_k} / n_k !$ by the method described in Theorem 6. Then, for this $f(w)$, we can estimate the series in a manner analogous to (5.6) by replacing r by $|w|$ and $\chi(r)$ by $\chi^*(r)$. For example, (5.7) becomes

$$\sum_{i<k} \frac{a_{n_i}}{n_i !} \cdot |w|^{n_i} < |w|^{\sqrt{|w|+1}} < \frac{1}{3} e^{|w|} < \frac{1}{3} e^{\chi^*(r)} . \tag{5.7$'$}$$

Similarly, all other terms of the series analogous to (5.6) are also estimated as less than (1/3) $e^{\chi^*(r)}$, and we have $M(r) < e^{\chi^*(r)}$.

COROLLARY. *Every transcendental Hurwitz function at* 0 *and* 1 *which is entire in every p-adic field is at least of order* 2, *maximal type. There exist such Hurwitz functions satisfying $M(r) < e^{\chi^*(r)}$, $r > R$ for any $\chi^*(r)$ with $\chi^*(r)/r^2 \to +\infty$.*

§ 6. Highly algebraic valued functions.

Our main interest so far has been in the fact that there are non-trivial lower bounds for the rate of growth of the functions in question when we restrict the set of functions by certain arithmetic conditions on their values at one or more points.

One might ask whether these results can be extended to cases in which the values of the functions satisfy arithmetic restrictions to values which do not form a discrete set of numbers. In this section we show that some forms of discreteness condition, such as restrictions on the denominators (of rational values) or heights (of algebraic values) are needed to get analogous results.

The necessity of such restrictions was already shown in [**3**], [**13**] where a transcendental entire function whose derivatives of all orders at any algebraic points are algebraic numbers is constructed with arbitrary rate of growth compatible with transcendentality.

We now show that even the addition of p-adic restrictions gives no non-trivial lower bounds on the rate of growth.

THEOREM 8. *There exist series of polynomials with rational coefficients that converge uniformly in every bounded region of every completion of an algebraic number field to a transcendental function whose derivatives of all orders at any algebraic point α lie in the field $Q(\alpha)$. The functions can be* *constructed to be of arbitrarily slow rate of growth compatible with transcendentality.*

PROOF. Let $P_1(z)=z,\ P_2(z), \cdots$ be an enumeration of all irreducible polynomials with integral coefficients so that, say, $\deg(P_n(z))\leqq n$ and height $(P_n(z))\leqq n$.

Let

$$Q_n(z)=z^{n^4}(P_1(z)\cdot P_2(z)\cdots P_n(z))^n\,. \tag{6.1}$$

We now construct a sequence c_n of rationals so that $c_n=(p_1\cdots p_n)^N/p_M$ where p_k is the k-th prime and N, M are functions of n of sufficiently rapid rate of increase so that N is sufficiently large that for any prime p_k, $k\leqq n$, we have for $|z|_{p_k}<n$ that

$$|c_nQ_n(z)|_{p_k}<p_k^{-N}n^{2n^4} \tag{6.2}$$

goes to 0 as rapidly as desired, while M is chosen so large that for $|z|<n$

$$|c_nQ_n(z)|<(p_1\cdots p_n)^N n^{2(n+1)n^3}/p_M \tag{6.3}$$

goes to 0 as rapidly as desired.

The function $f(z)=\sum_{n=1}^{\infty}c_nQ_n(z)$ now satisfies the condition of our theorem. To see that it is transcendental, we only have to observe that the $Q_n(z)$ are sections of the Taylor series of $f(z)$ since the term of lowest degree in z in $Q_n(z)$ is of degree

$$n^4>(n-1)^4+n^3>\deg Q_{n-1}\,. \tag{6.4}$$

If α is an algebraic number, then $P_k(\alpha)=0$ for some k. Hence $Q_n^{(m)}(\alpha)=0$ for all $n>\max\{k, m\}$ and

$$f^{(m)}(\alpha)=\sum_{n=1}^{\max\{k,m\}}Q_n^{(m)}(\alpha) \tag{6.5}$$

lies in the field of α.

PROBLEMS. Finally, we would like to propose problems which are related to this paper.

Does there exist a transcendental entire function which (together with all its derivatives) has algebraic values at every algebraic point and has transcendental values at every transcendental point? [13]

This question can be interpreted as refering to the entire functions of a complex variable or a p-adic variable. If we restrict our attention to functions whose derivatives at $z=0$ belong to a fixed algebraic number field K, then we can ask for the existence of such functions which are algebraic at algebraic points and transcendental at transcendental points in several (even all) completions of K.

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