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Analytic-hypoelliptic differential operators of first order in two independent variables

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1. Introduction.

A differential operator L with analytic coefficients is called analytic-hypoelliptic, if a distribution solution f of the equation $Lf=g$ is analytic, whenever g is analytic. It is known that every elliptic operator with analytic coefficients is analytic-hypoelliptic. Recently Mizohata [**1**, Appendice] constructed an example showing that the converse is not generally true. In fact he proved that the operator

$$L=\frac{\partial}{\partial x}+ix^{k}\frac{\partial}{\partial y}, \quad k=0, 1, \cdots,$$

is analytic-hypoelliptic in the neighborhood of the origin if and only if k is even. His method can be applied to operators of the form

$$L=a(x, y)\frac{\partial}{\partial x}+b(x, y)\frac{\partial}{\partial y}. \tag{1.1}$$

Here we assume that the coefficients $a(x, y)$ and $b(x, y)$ are complex-valued analytic functions defined in an open set Ω in the (x, y)-plane, and that

$$|a(x, y)|+|b(x, y)|\neq 0.$$

In this paper we shall give a necessary and sufficient condition for an operator of this type to be analytic-hypoelliptic.

We denote the operator with complex conjugate coefficients by $\bar{L}$:

$$\bar{L}=\bar{a}(x, y)\frac{\partial}{\partial x}+\bar{b}(x, y)\frac{\partial}{\partial y}. \tag{1.2}$$

We define the kth commutator C_k by induction:

$$C_0=\bar{L},$$
$$C_k=[L, C_{k-1}]=LC_{k-1}-C_{k-1}L. \tag{1.3}$$

Let $k(x, y)$ denote the first value of k for which C_k is not proportional to L at the point (x, y). If C_k is proportional to L for all values of k, we define $k(x, y)$ to be ∞. Note that L is elliptic at (x, y), if and only if $k(x, y)=0$.

It is easily seen that $k(x, y)$ is independent of the particular local coordinates, and that it is invariant under multiplication of L by a non-vanishing factor.

Now the result obtained is:

THEOREM. *A differential operator L of the form* (1.1) *is analytic-hypoelliptic in Ω, if and only if the following condition holds in Ω.*

(AH) *At every point of Ω, $k(x, y)$ is finite and even.*

The form of the condition (AH) was suggested by that of the condition (P′) of Nirenberg and Treves [**2**].

I wish to express my gratitude to Professor T. Iwamura for his advice and encouragement.

2. Sufficiency of the condition (AH).

Our proof of sufficiency is based on the ideas of Mizohata [**1**].

Since the statement of the theorem is local, it is sufficient to prove that every point of Ω has an open neighborhood where L is analytic-hypoelliptic. Let (x_0, y_0) be a point of Ω. It is possible to introduce new local coordinates in a neighborhood of (x_0, y_0), so that the operator takes the form

$$L = a'(x, y)\Big(\frac{\partial}{\partial x} + ib'(x, y)\frac{\partial}{\partial y}\Big),$$

where $a'(x, y)$ is a non-vanishing complex-valued analytic function, and $b'(x, y)$ is a *real*-valued analytic function [**2**, pp. 332, 336]. Since $k(x, y)$ is invariant under multiplication of L by a non-vanishing factor, we may suppose that L has the form

$$L = \frac{\partial}{\partial x} + ib(x, y)\frac{\partial}{\partial y}, \tag{2.1}$$

where $b(x, y)$ is a *real*-valued analytic function.

For the operator L of the form (2.1), C_k becomes

$$C_k = \Big(-2i\frac{\partial^k b}{\partial x^k} + \sum_{j=0}^{k-1} c_{jk}\frac{\partial^j b}{\partial x^j}\Big)\frac{\partial}{\partial y}, \quad k \geqq 1,$$

where c_{jk} are analytic functions depending on $b(x, y)$. Hence $k(x, y)$ is the first value of k such that

$$\frac{\partial^k b(x, y)}{\partial x^k} \neq 0.$$

It follows therefore that, if the condition (AH) holds, the sign of the function $b(x, y)$ does not vary with x and y. We may thus suppose that $b(x, y) \geqq 0$. Then

$$\frac{\partial^k b(x, y)}{\partial x^k} > 0, \quad \text{if} \quad k = k(x, y). \tag{2.2}$$

LEMMA. *Let $w(x, y) = u(x, y) + iv(x, y)$ be a solution of the equation*

$$Lw = \frac{\partial w}{\partial x} + ib(x, y)\frac{\partial w}{\partial y} = 0, \tag{2.3}$$

which is analytic in a neighborhood of the point (x_0, y_0) and such that

$$\frac{\partial v(x, y)}{\partial y} > 0. \tag{2.4}$$

Then the transformation $(x, y) \to (u, v)$ is a homeomorphism of an open neighborhood U of (x_0, y_0) onto an open neighborhood $\tilde{U}$ of (u_0, v_0), $u_0 = u(x_0, y_0)$, $v_0 = v(x_0, y_0)$. Moreover, the sign of angles is conserved in this transformation.

The existence of a solution $w(x, y)$ having the required properties is established by the Cauchy-Kowalewski theorem, since $x = x_0$ is non-characteristic for L.

PROOF. By differentiating the equation (2.3) k times with respect to x, we obtain

$$\frac{\partial^j w}{\partial x^j} = 0,\ j = 1, \cdots, k = k(x, y);\ \frac{\partial^{k+1} w}{\partial x^{k+1}} = -i\frac{\partial^k b}{\partial x^k}\frac{\partial w}{\partial y}. \tag{2.5}$$

We call v-curves the curves in the (x, y)-plane defined by the equations $v(x, y) = \text{constant}$. Since $\partial v/\partial y \neq 0$, v-curves are smooth and we may choose x as parameter on these curves. We denote by d/dx the differentiation along v-curves with respect to the parameter x. Then we have

$$\frac{d^j y}{dx^j} = 0,\ j = 1, \cdots, k = k(x, y);\ \frac{d^{k+1} y}{dx^{k+1}} = -\frac{\partial^{k+1} v}{\partial x^{k+1}} \Big/ \frac{\partial v}{\partial y}.$$

Hence

$$\frac{d^j u}{dx^j} = 0, \quad j = 1, \cdots, k = k(x, y);$$

$$\frac{d^{k+1} u}{dx^{k+1}} = \frac{\partial^{k+1} u}{\partial x^{k+1}} - \frac{\partial u}{\partial y}\frac{\partial^{k+1} v}{\partial x^{k+1}} \Big/ \frac{\partial v}{\partial y}$$

$$= \frac{\partial^k b}{\partial x^k}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \Big/ \frac{\partial v}{\partial y}.$$

Hence, by (2.2) and (2.4), the first non-vanishing derivative of u along v-curves is of odd order and positive. It follows therefore that u is strictly increasing on v-curves and the correspondence $(x, y) \leftrightarrow (u, v)$ is one to one.

In this transformation, an open set in the (x, y)-plane given by

$$p_1 < x < p_2, \quad q_1 < v(x, y) < q_2,$$

is transformed into an open set in the (u, v)-plane given by

$$r_1(v) < u < r_2(v), \quad q_1 < v < q_2,$$

where $r_i(v)$, $i=1, 2$, is a continuous function defined by

$$r_i(v) = u(p_i, y)$$
$$v = v(p_i, y).$$

Hence the correspondence is a homeomorphism of an open neighborhood of (x_0, y_0) onto an open neighborhood of (u_0, v_0).

It is clear that the sign of angles is conserved in this transformation. The proof is thus complete.

By the Cauchy-Kowalewski theorem, we can find an analytic function f' in a neighborhood of (x_0, y_0) such that $Lf' = g$. Then $L(f-f') = 0$. We may thus suppose without loss of generality that f is a solution of the homogeneous equation $Lf = 0$.

PROPOSITION 1. *Let $f(x, y)$ be a C^1-solution of the equation $Lf = 0$ in a neighborhood of (x_0, y_0). Let $w(x, y)$ be as in the lemma. If $\tilde{f}(w)$ is defined by the equation*

$$\tilde{f}(w(x, y)) = f(x, y),$$

then $\tilde{f}(w)$ is a holomorphic function of w in a neighborhood of $w_0 = w(x_0, y_0)$.

PROOF. By the above lemma, $\tilde{f}(w)$ is a single-valued continuous function of w. Let $\tilde{C}$ be a rectangle in the w-plane with its sides parallel to the axes, and C the closed contour in the (x, y)-plane corresponding to $\tilde{C}$ under the transformation $w = w(x, y)$. Then, by Stokes formula,

$$\int_{\tilde{C}} \tilde{f}(w)dw = \int_C f(x, y)dw(x, y) = \iint df(x, y) \wedge dw(x, y).$$

Since df and dw are linearly dependent, the last integral vanishes. Hence, by Morera's theorem, $\tilde{f}(w)$ is a holomorphic function of w. Q. E. D.

From the above result, it is sufficient to show that L is hypoelliptic. To prove this, we construct a very regular left elementary kernel for L (noyau élémentaire à gauche très régulier). For the theory of kernels, we refer to [**3**, Chap. V, § 6].

PROPOSITION 2. *Let $w(x, y)$ be as before. Then*

$$K(x', y'; x, y) = \frac{1}{2\pi i} \frac{1}{w(x', y') - w(x, y)} \frac{\partial w(x, y)}{\partial y}$$

is a very regular left elementary kernel for L.

PROOF. (i) We begin by proving that $K(x', y'; x, y)$ is locally summable for a fixed (x', y').

By (2.5), if we write $k' = k(x', y')$, we have

$$w(x, y)-w(x', y')=\frac{\partial w(x', y')}{\partial y}\Big\{\Big[-\frac{i}{(k'+1)!}\frac{\partial^{k'}b(x', y')}{\partial x^{k'}}+o(1)\Big](x-x')^{k'+1}$$
$$+[1+o(1)](y-y')\Big\}.$$

Hence, if $|x-x'|$ and $|y-y'|$ are sufficiently small, say $<\delta$,

$$|w(x, y)-w(x', y')|\geqq M(|x-x'|^{k'+1}+|y-y'|).$$

Integrating over $|x-x'|<\delta$, $|y-y'|<\delta$, we have

$$\iint|K(x', y'; x, y)|dxdy\leqq M'\iint\frac{1}{|x-x'|^{k'+1}+|y-y'|}dxdy,$$

and the substitution $s=(x-x')^{k'+1}/(y-y')$, $t=y-y'$ gives

$$\leqq\frac{4M'}{k'+1}\int_0^\delta\frac{dt}{t^{k'/(k'+1)}}\int_0^{\frac{\delta^{k'+1}}{t}}\frac{ds}{(1+s)s^{k'/(k'+1)}}<\infty.$$

(ii) Next we show that K is a very regular kernel. Since K is analytic except when $x'=x$ and $y'=y$, we have only to prove that K is a regular kernel.

Let

$$l_m(w)=\frac{1}{m!}w^m(\log w+c_m),\quad m=0, 1, \cdots,$$

where $c_m=-1-1/2-\cdots-1/m$. If we make a cut on the domain $U_{x',y'}\times U_{x,y}$ along $\{(x', y'; x, y); x'=x \text{ and } y'\geqq y\}$, we obtain a simply connected domain V. Since $w(x', y')-w(x, y)\neq 0$ on V, we can choose a branch of $\log(w(x', y')-w(x, y))$ in V.

Now consider the function

$$L_m(x', y'; x, y)=\frac{1}{2\pi i}l_m(w(x', y')-w(x, y)).$$

It is easily seen that, if $m\geqq 1$, L_m is locally bounded, and that for every ϕ in $C_0^\infty(U)$,

$$\phi_m(x', y')=\iint L_m(x', y'; x, y)\phi(x, y)dxdy$$

is a continuous function. If we differentiate this function with respect to x' and y', we have

$$\frac{\partial}{\partial x'}\phi_m(x', y')=\frac{\partial}{\partial x'}w(x', y')\phi_{m-1}(x', y')$$
$$+\frac{1}{m!}\int^{y'}(w(x', y')-w(x', y))^m\phi(x', y)dy, \tag{2.6}$$

$$\frac{\partial}{\partial y'}\phi_m(x', y')=\frac{\partial}{\partial y'}w(x', y')\phi_{m-1}(x', y'). \tag{2.7}$$

Since the second term on the right of (2.6) is indefinitely differentiable, we see that $\psi_m(x', y')$ is an $m-1$ times continuously differentiable function.

Let

$$\Phi(x', y') = \iint K(x', y'\,;\, x, y)\varphi(x, y)dxdy\,.$$

Integrating by parts, we obtain

$$\Phi(x', y') = \iint L_m(x', y'\,;\, x, y)\Big[\frac{\partial}{\partial y}\Big(\frac{\partial w}{\partial y}\Big)^{-1}\Big]^m \frac{\partial \varphi}{\partial y}\, dxdy\,.$$

Hence, for every $\varphi(x, y)$ in $C_0^\infty(U)$, $\Phi(x', y')$ is $m-1$ times continuously differentiable. Making $m \to \infty$, it follows therefore that $\Phi(x', y')$ is indefinitely differentiable.

A similar argument shows that

$$\iint \varphi(x', y')K(x', y'\,;\, x, y)dx'dy'$$

is also indefinitely differentiable. Hence K is a regular kernel.

(iii) It remains to prove that K is a left elementary kernel for L.

Let L^* be the adjoint of L. Then we see that

$$L^*_{x,y}K(x', y'\,;\, x, y) = 0$$

except when $x = x'$ and $y = y'$. Also, if ψ and φ are C^1-functions,

$$d(\psi\varphi(-ibdx+dy)) = (\psi(L\varphi) - (L^*\psi)\varphi)dxdy\,.$$

Let $\tilde{C}_\varepsilon$ be a rectangle containing $w' = w(x', y')$ which tends to the point w' as $\varepsilon \to 0$, and C_ε the closed contour in the (x, y)-plane corresponding to $\tilde{C}_\varepsilon$. Then, by Stokes formula,

$$\begin{aligned} I &= \iint K(x', y'\,;\, x, y)L\varphi(x, y)dxdy \\ &= -\lim_{\varepsilon\to 0}\int_{C_\varepsilon} K(x', y'\,;\, x, y)\varphi(x, y)(-ib(x, y)dx+dy)\,, \end{aligned}$$

for every φ in $C_0^\infty(U)$. Since $dw = \partial w/\partial y\ (-ibdx+dy)$, it follows that

$$\begin{aligned} I &= \lim_{\varepsilon\to 0}\frac{-1}{2\pi i}\int_{\tilde{C}_\varepsilon}\frac{\tilde{\varphi}(w)}{w'-w}dw \\ &= \tilde{\varphi}(w') = \varphi(x', y')\,. \end{aligned}$$

Hence K is a left elementary kernel for L, and Proposition 2 is therefore proved.

3. Necessity of the condition (AH).

Suppose that the condition (AH) does not hold in Ω. Then there is a point (x_0, y_0) in Ω such that $k(x_0, y_0)$ is either finite and odd, or infinite. We now try to find a solution of the equation $Lf=0$ which is not analytic in the neighborhood of the point (x_0, y_0).

Let $w(x, y)$ be a solution of the equation

$$Lw=\frac{\partial w}{\partial x}+ib(x, y)\frac{\partial w}{\partial y}=0$$

which is analytic in a neighborhood of (x_0, y_0) and such that $w(x_0, y)=y-y_0$. If $\tilde{f}(w)$ is an analytic function of w, then $f(x, y)=\tilde{f}(w(x, y))$ is also a solution of the equation $Lf=0$. Consider, for example, the function $\tilde{f}(w)=w^{\alpha}$, where α is a complex number. We shall prove that there exists a single-valued branch of the function $f(x, y)$.

If $k_0=k(x_0, y_0)$ is finite and odd, we have

$$w(x, y)=\Big[-\frac{i}{(k_0+1)\,!}\frac{\partial^{k_0}b(x_0, y_0)}{\partial x^{k_0}}+o(1)\Big](x-x_0)^{k_0+1} \\ +[1+o(1)](y-y_0)\,. \tag{3.1}$$

Let $A(p, q)$ denote the angular region in the w-plane given by

$$p\pi \leqq \arg w \leqq q\pi\,.$$

If (x, y) is sufficiently close to (x_0, y_0), the first term on the right of (3.1) takes its values in $A(1/2-\varepsilon, 1/2+\varepsilon)$ or $A(3/2-\varepsilon, 3/2+\varepsilon)$, according as $\partial^{k_0}b(x_0, y_0)/\partial x^{k_0}$ is negative or positive. The second term takes its values in $A(-\varepsilon, +\varepsilon) \cup A(1-\varepsilon, 1+\varepsilon)$. Hence the values of the function $w(x, y)$ are in $A(-\varepsilon, 1+\varepsilon)$ or $A(1-\varepsilon, 2+\varepsilon)$.

If $k_0=k(x_0, y_0)$ is infinite, we have

$$w(x, y)=[1+o(1)](y-y_0)\,.$$

Hence the values of the function $w(x, y)$ are in $A(-\varepsilon, +\varepsilon)\cup A(1-\varepsilon, 1+\varepsilon)$.

In either case we can choose a single-valued branch of $f(x, y)$ in a neighborhood of (x_0, y_0). If $\operatorname{Re}\alpha \geqq m, f(x, y)$ is a C^m-solution of $Lf=0$, and if α is not an integer, $f(x, y)$ is not analytic in the neighborhood of (x_0, y_0). We have thus proved the necessity of the condition (AH).

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Bibliography

[1] S. Mizohata, Solutions nulles et solutions non analytiques, J. Math. Kyoto Univ., **1** (1962), 271-302.

[2] L. Nirenberg and F. Treves, Solvability of a first order linear partial differential equation, Comm. Pure Appl. Math., **16** (1963), 331-351.

[3] L. Schwartz, Théorie des distributions, Tome I, 2e éd., Hermann, Paris, 1957.