J. Math. Soc. Japan
Vol. 15, No. 4, 1963

On normal almost contact structures

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(Received April 30, 1963)
(Revised June 24, 1963)

Introduction

In the present paper we shall investigate some relations between almost contact structures and complex structures. An odd dimensional differentiable manifold is said to be an almost contact manifold if there exists a triple $\Sigma=(\phi, \xi, \eta)$, ϕ being (1.1)-tensor field, ξ a vector field and η 1-form on M satisfying certain conditions. We shall introduce in § 2 an almost complex structure on the product manifold of two almost contact manifolds. We shall call an almost contact structure on M to be integrable if the induced almost complex structure on $M\times M$ is integrable (i. e., complex structure). In § 2 we shall prove that the induced almost complex structure on the product of two almost contact manifolds is integrable if and only if both almost contact structures are integrable. This result generalises a theorem of Calabi-Eckmann which says that the product of two odd dimensional spheres admits a complex structure [3]. This result says also that the notion of the integrable almost contact structure coincides with that of the normal almost contact structure defined by Sasaki-Hatakeyama [8].

In § 3 we shall introduce the notion of isomorphism and automorphism of almost contact structures and we shall prove that the automorphism group of a compact almost contact manifold M is a Lie transformation group of M with respect to the compact-open topology, if the structure is integrable.

In § 4 we shall show examples of compact normal almost contact manifolds other than the odd dimensional spheres. In particular, we shall see that every compact simply connected homogeneous contact manifold studied by Boothby-Wang [2] has always a normal almost contact structure such that the automorphism group operates transitively.

In the last section we shall treat the left invariant normal almost contact structure on a Lie group and show that the problem can be reduced to a purely algebraic one in Lie algebras, and we shall prove that every compact Lie group of odd dimension admits a left invariant normal almost contact structure.

§ 1. Almost contact structures.

Let M be a differentiable manifold. ("differentiable" means always "differentiable of class C^∞", and in this paper vector fields, real valued functions and differential forms on M mean always differentiable ones.) We denote by $\mathfrak{B}(M)$ the Lie algebra of all vector fields on M, by $F(M)$ the $\boldsymbol{R}$-algebra of all real valued C^∞ functions on M, $\boldsymbol{R}$ being the real number field. As usual we use the following notations: for any $X \in \mathfrak{B}(M)$ and $f \in F(M)$ we mean $Xf \in F(M)$ and $fX \in \mathfrak{B}(M)$ as follows:

$$(Xf)(p) = X_p f,$$

$$(fX)_p = f(p) \cdot X_p$$

for $p \in M$. For any 1-form θ on M and for any $X \in \mathfrak{B}(M)$ we mean $\theta(X) \in F(M)$ as follows:

$$(\theta(X))(p) = \theta_p(X_p).$$

Let now M be a differentiable manifold of dimension $2n+1$ ($n \geqq 0$). An *almost contact structure* on M is, by definition, a triple $\Sigma = (\phi, \xi, \eta)$, where ϕ is a tensor field of type (1.1) on M, ξ is a vector field of M and η is a differential 1-form on M satisfying the following conditions:

$$\phi(\xi) = 0 \tag{1.1}$$

$$\eta \circ \phi = 0 \tag{1.2}$$

$$\eta(\xi) = 1 \tag{1.3}$$

$$\phi \circ \phi = -\mathbf{1} + \eta \cdot \xi, \tag{1.4}$$

where (1.4) means

$$\phi(\phi X) = -X + \eta(X) \cdot \xi$$

for all $X \in \mathfrak{B}(M)$.

A differentiable manifold of odd dimension with an almost contact structure is called an almost contact manifold.

It is to be noted that the conditions (1.1)~(1.4) imply the following

$$\operatorname{rank} \phi = 2n \quad \text{on } M \text{ everywhere}. \tag{1.5}$$

In fact, ϕ_p and η_p being the values of ϕ and η at $p \in M$, the linear map ϕ_p leaves invariant the subspace $V_p = \eta_p^{-1}(0)$ of the tangent space of M at p. Moreover, the restriction ϕ'_p of ϕ_p to V_p satisfies $\phi'_p \circ \phi'_p = -\mathbf{1}$. Hence, $\operatorname{rank} \phi'_p = 2n$. On the other hand, (1.1), (1.3) imply that $\operatorname{rank} \phi_p \leqq 2n$, whence $\operatorname{rank} \phi_p = 2n$.

It is also to be noted that we have the following

PROPOSITION 1. (*a*) *Let* (ϕ, ξ, η) *and* (ϕ, ξ', η) *be two almost contact structures on the same* M, *then we have* $\xi = \xi'$. (*b*) *Let* (ϕ, ξ, η) *and* (ϕ, ξ, η') *be two almost* *contact structures on the same M, then we have $\eta=\eta'$.*

The proof will be omitted.

§ 2. Product of almost contact manifolds.

Let M and $\bar{M}$ be differentiable manifolds of dimension $2n+1$ and $2m+1$ and let $\Sigma=(\phi, \xi, \eta)$ and $\bar{\Sigma}=(\bar{\phi}, \bar{\xi}, \bar{\eta})$ be almost contact structures on M and $\bar{M}$, respectively.

We can now introduce an almost complex structure J on the product manifold $M\times\bar{M}$ as follows: For any $X_p\in T_p(M)$ and $\bar{X}_q\in T_q(\bar{M})$, $T_p(M)$ being the tangent space of M at p, we define

$$J_{(p,q)}(X_p, \bar{X}_q)=(\phi(X_p)-\bar{\eta}(\bar{X}_q)\cdot\xi_p, \bar{\phi}(\bar{X}_q)+\eta(X_p)\cdot\bar{\xi}_q).$$

Then it is easily seen that $J\circ J=-\mathbf{1}$, $\mathbf{1}$ being the identity map, which shows that J is an almost complex structure on $M\times\bar{M}$. We call J the *induced almost complex structure* on $M\times\bar{M}$ by Σ and $\bar{\Sigma}$.

We state this elementary fact as

PROPOSITION 2. *Let M and $\bar{M}$ be almost contact manifolds. Then $M\times\bar{M}$ has an almost complex structure induced by the almost contact structures of M and $\bar{M}$.*

We shall now state the following

DEFINITION 1. Let $\Sigma=(\phi, \xi, \eta)$ be an almost contact structure on M. If the induced almost complex structure on $M\times M$ by Σ is integrable (i. e. complex structure), we call Σ is *integrable.*

DEFINITION 2. Let $\Sigma=(\phi, \xi, \eta)$ be an almost contact structure on M. Let $\bar{M}=\boldsymbol{R}$ be the real number space and let $\bar{\Sigma}=(\bar{\phi}, \bar{\xi}, \bar{\eta})=(0, \frac{d}{dt}, dt)$, t being the coordinate in $\boldsymbol{R}$. If the induced almost complex structure on $M\times\boldsymbol{R}$ by Σ and $\bar{\Sigma}$ is integrable, we call that Σ is *normal* (cf. [8] and [9]).

We shall prove later that these two apparently different definitions coincide with each other.

DEFINITION 3. Let $\Sigma=(\phi, \xi, \eta)$ be an almost contact structure on M. We define a tensor field Ψ of type (1.2) and a differential 2-form θ on M as follows.[1]

$$\Psi(X, Y)=\phi[X, Y]-[\phi X, Y]-[X, \phi Y]-\phi[\phi X, \phi Y] + \{(\phi X)\cdot(\eta(Y))-(\phi Y)\cdot(\eta(X))\}\cdot\xi \tag{2.1}$$

$$\theta(X, Y)=\eta([X, Y])-X\cdot\eta(Y)+Y\cdot\eta(X)-\eta([\phi X, \phi Y]) \tag{2.2}$$

1) Notations being as those of Sasaki-Hatakeyama [8], the components Ψ^i_{jk} of Ψ satisfy the following equalities:

$$\Psi^i_{jk}=\phi^i_h N^h_{jk}+\xi^i N_{jk} \qquad (i, j, k=1, 2, \cdots, 2n+1).$$

However, we shall not use these equalities.

for $X, Y\in\mathfrak{V}(M)$.

We now prove the following

PROPOSITION 3. *Let $\Sigma=(\phi, \xi, \eta)$ and $\bar{\Sigma}=(\bar{\phi}, \bar{\xi}, \bar{\eta})$ be almost contact structures on M and $\bar{M}$. Then the induced almost complex structure on $M\times\bar{M}$ by Σ and $\bar{\Sigma}$ is integrable if and only if the following conditions are satisfied:*

$$\Psi=0, \tag{2.3}$$

$$\bar{\Psi}=0, \tag{2.4}$$

where $\bar{\Psi}$ is the tensor fields of type (1.2) *on $\bar{M}$ corresponding to $\bar{\Sigma}$* (cf. Def. 3).

PROOF. We first identify $X\in\mathfrak{V}(M)$ with $\tilde{X}\in\mathfrak{V}(M\times\bar{M})$ such as

$$\tilde{X}_{(p,q)}=X_p+O_q$$

for $(p, q)\in M\times\bar{M}$, where O_q denotes the zero tangent vector of $\bar{M}$ at q. Similarly, we identify $\bar{X}\in\mathfrak{V}(\bar{M})$ with $\tilde{\bar{X}}\in\mathfrak{V}(M\times\bar{M})$ such as

$$\tilde{\bar{X}}_{(p,q)}=O_p+\bar{X}_q\,.$$

We also consider a function on M ($\bar{M}$ resp.) as a function on $M\times\bar{M}$ as usual. Then the integrability condition of the induced almost complex structure J on $M\times\bar{M}$ is as follows:

$$\begin{aligned}J[X+\bar{X}, Y+\bar{Y}]=[J(X+\bar{X}), Y+\bar{Y}]+[X+\bar{X}, J(Y+\bar{Y})]\\ +J[J(X+\bar{X}), J(Y+\bar{Y})]\end{aligned}$$

for all $X, Y\in\mathfrak{V}(M)$ and $\bar{X}, \bar{Y}\in\mathfrak{V}(\bar{M})$ (cf. [4] and [7]). By a direct calculation we see the above condition is equivalent to the following two conditions:

$$\begin{aligned}&\phi[X, Y]-\bar{\eta}([\bar{X}, \bar{Y}])\cdot\xi\\ &\quad=[\phi X-\bar{\eta}(\bar{X})\xi, Y]+\bar{Y}(\bar{\eta}(\bar{X}))\cdot\xi+[X, \phi Y-\bar{\eta}(\bar{Y})\cdot\xi]\\ &\qquad-\bar{X}(\bar{\eta}(\bar{Y}))\cdot\xi+\phi[\phi X-\bar{\eta}(\bar{X})\cdot\xi, \phi Y-\bar{\eta}(\bar{Y})\cdot\xi]\\ &\qquad-\bar{\eta}([\eta(X)\bar{\xi}+\bar{\phi}(\bar{X}), \eta(Y)\bar{\xi}+\bar{\phi}(\bar{Y})])\cdot\xi+\bar{f}(X, \bar{X}; Y, \bar{Y})\cdot\xi\,,\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\bar{\phi}[\bar{X}, \bar{Y}]+\eta([X, Y])\bar{\xi}\\ &\quad=[\bar{\phi}(\bar{X})+\eta(X)\cdot\bar{\xi}, \bar{Y}]-Y(\eta(X))\cdot\bar{\xi}+[\bar{X}, \bar{\phi}(\bar{Y})+\eta(Y)\bar{\xi}]\\ &\qquad+X(\eta(Y))\cdot\bar{\xi}+\bar{\phi}[\bar{\phi}(\bar{X})+\eta(X)\bar{\xi}, \bar{\phi}(\bar{Y})+\eta(Y)\bar{\xi}]\\ &\qquad+\eta([\phi X-\bar{\eta}(\bar{X})\xi, \phi Y-\bar{\eta}(\bar{Y})\xi])\cdot\bar{\xi}+f(X, \bar{X}; Y, \bar{Y})\bar{\xi}\,,\end{aligned} \tag{2.6}$$

where we define the functions f and $\bar{f}$ on $M\times\bar{M}$ as follows:

$$\begin{aligned}f(X, \bar{X};\ Y, \bar{Y})&=-\bar{\phi}(\bar{X})(\bar{\eta}(\bar{Y}))+\bar{\phi}(\bar{Y})(\bar{\eta}(\bar{X}))-\eta(X)\cdot\bar{\xi}(\bar{\eta}(\bar{Y}))+\eta(Y)\cdot\bar{\xi}(\bar{\eta}(\bar{X}))\,,\\ \bar{f}(X, \bar{X};\ Y, \bar{Y})&=-\phi(X)\eta(Y)+\phi(Y)(\eta(X))+\bar{\eta}(\bar{X})\cdot\xi(\eta(Y))-\bar{\eta}(\bar{Y})\cdot\xi(\eta(X))\,.\end{aligned}$$

Now, putting $\bar{X}=\bar{Y}=0$; $X=0$, $\bar{Y}=0$; $\bar{X}=0$, $Y=0$ or $X=Y=0$, we obtain the following eight conditions equivalent to (2.5) and (2.6):

$$\Psi(X, Y)=0, \tag{2.3}$$

$$\theta(X, Y)=0, \tag{2.7}$$

$$\bar{\Psi}(\bar{X}, \bar{Y})=0, \tag{2.4}$$

$$\bar{\theta}(\bar{X}, \bar{Y})=0, \tag{2.8}$$

$$\begin{aligned}[X, \bar{\eta}(\bar{Y})\xi]+\phi[\phi X, \bar{\eta}(\bar{Y})\xi]+\bar{\eta}([\eta(X)\cdot\bar{\xi}, \bar{\phi}(\bar{Y})])\xi \\ +\bar{\eta}(\bar{Y})\cdot\xi(\eta(X))\cdot\xi=0,\end{aligned} \tag{2.9}$$

$$\begin{aligned}[\bar{\eta}(\bar{X})\xi, Y]+\phi[\bar{\eta}(\bar{X})\cdot\xi, \phi Y]+\bar{\eta}([\bar{\phi}(\bar{X}), \eta(Y)\bar{\xi}])\cdot\xi \\ -\bar{\eta}(\bar{X})\cdot\xi(\eta(Y))\cdot\xi=0,\end{aligned} \tag{2.10}$$

$$\begin{aligned}[\bar{X}, \eta(Y)\bar{\xi}]+\bar{\phi}[\bar{\phi}(\bar{X}), \eta(Y)\bar{\xi}]-\eta([\bar{\eta}(\bar{X})\xi, \phi Y])\cdot\bar{\xi} \\ +\eta(Y)\cdot\bar{\xi}(\bar{\eta}(\bar{X}))\cdot\bar{\xi}=0,\end{aligned} \tag{2.11}$$

$$\begin{aligned}[\eta(X)\bar{\xi}, \bar{Y}]+\bar{\phi}[\eta(X)\bar{\xi}, \bar{\phi}(\bar{Y})]-\eta([\phi X, \bar{\eta}(\bar{Y})\xi])\cdot\bar{\xi} \\ -\eta(X)\cdot\bar{\xi}(\bar{\eta}(\bar{Y}))\cdot\bar{\xi}=0,\end{aligned} \tag{2.12}$$

for all $X, Y\in\mathfrak{V}(M)$ and $\bar{X}, \bar{Y}\in\mathfrak{V}(\bar{M})$.

We shall now prove that the conditions (2.3) and (2.4) imply (2.7)$\sim$(2.12). First, it is easy to see that (2.9) and (2.10) are equivalent and also (2.11) and (2.12) are equivalent. We now verify that (2.9) is implied by (2.3) and (2.4). For this purpose we first remark that

$$\phi[X, \xi]=[\phi X, \xi] \tag{2.13}$$

for all $X\in\mathfrak{V}(M)$. In fact, putting $Y=\xi$ in (2.3) we obtain (2.13) by virtue of (1.1) and (1.3). Now the left hand side of (2.9) is equal to

$$\begin{aligned}&\bar{\eta}(\bar{Y})\cdot[X, \xi]+\bar{\eta}(\bar{Y})\phi[\phi X, \xi]+\eta(X)\cdot\bar{\eta}([\bar{\xi}, \bar{\phi}(\bar{Y})])\cdot\xi+\bar{\eta}(\bar{Y})\cdot\xi(\eta(X))\cdot\xi \\ &=\bar{\eta}(\bar{Y})\{[X, \xi]+[\phi^2 X, \xi]+\xi(\eta(X))\cdot\xi\}+\eta(X)\cdot\bar{\eta}(\bar{\phi}[\bar{\xi}, \bar{Y}])\cdot\xi \\ &=\bar{\eta}(\bar{Y})\{[X, \xi]+[-X+\eta(X)\cdot\xi, \xi]+\xi(\eta(X))\cdot\xi\}=0.\end{aligned}$$

In the same way (2.11) is implied by (2.3) and (2.4).

It is now sufficient to prove that (2.3) implies (2.7), since (2.8) is implied by (2.4) in the same way. First, by operating ϕ on both hand sides of (2.3) we have

$$\phi^2[X, Y]=\phi[\phi X, Y]+\phi[X, \phi Y]+\phi^2[\phi X, \phi Y]. \tag{2.14}$$

Again by (2.3) we can calculate the first term of the right hand side of (2.14) as follows:

$$\begin{aligned}\phi[\phi X, Y]&=[\phi^2 X, Y]+[\phi X, \phi Y]+\phi[\phi^2 X, \phi Y]-\{(\phi^2 X)(\eta(Y))-(\phi Y)\cdot(\eta(\phi X))\}\xi \\ &=[-X+\eta(X)\cdot\xi, Y]+[\phi X, \phi Y]+\phi[-X+\eta(X)\xi, \phi Y]-(\phi^2 X)(\eta(Y))\cdot\xi.\end{aligned}$$

Inserting this into (2.14) we obtain

$$-[X, Y]+\eta([X, Y])\cdot\xi=[-X+\eta(X)\xi, Y]+[\phi X, \phi Y]+\phi[-X+\eta(X)\xi, \phi Y]$$
$$-(\phi^2X)(\eta(Y))\cdot\xi+\phi[X, \phi Y]-[\phi X, \phi Y]+\eta([\phi X, \phi Y])\cdot\xi\,.$$

Hence we have

$$\begin{aligned}\eta([X, Y])\cdot\xi &= [\eta(X)\xi, Y]+\phi[\eta(X)\xi, \phi Y]-(-X+\eta(X)\xi)(\eta(Y))\cdot\xi+\eta([\phi X, \phi Y])\cdot\xi\\ &= \eta(X)[\xi, Y]-Y(\eta(X))\cdot\xi+\phi\{\eta(X)[\xi, \phi Y]-(\phi Y)(\eta(X))\cdot\xi\}\\ &\quad +X(\eta(Y))\xi-\eta(X)\xi(\eta(Y))\cdot\xi+\eta([\phi X, \phi Y])\cdot\xi\\ &= \eta(X)[\xi, Y]-Y(\eta(X))\cdot\xi+\eta(X)[\xi, \phi^2Y]\\ &\quad +X(\eta(Y))\xi-\eta(X)\xi(\eta(Y))\cdot\xi+\eta([\phi X, \phi Y])\xi\\ &= \{X(\eta(Y))-Y(\eta(X))+\eta([\phi X, \phi Y])\}\cdot\xi\,,\end{aligned}$$

where we have used (2.13). Thus we have proved (2.7) and Proposition 3 is proved.

In the case $M=\bar{M}$ and $\Sigma=\bar{\Sigma}$ we have the following

THEOREM 1. *Let $\Sigma=(\phi, \xi, \eta)$ be an almost contact structure on M. Then Σ is integrable if and only if the following condition is satisfied:*

$$\Psi=0\,. \tag{2.3}$$

Combining Proposition 3 and Theorem 1 we obtain

THEOREM 2. *Let $\Sigma=(\phi, \xi, \eta)$ and $\bar{\Sigma}=(\bar{\phi}, \bar{\xi}, \bar{\eta})$ be almost contact structures on M and $\bar{M}$. Then the induced almost complex structure on $M\times\bar{M}$ is integrable if and only if Σ and $\bar{\Sigma}$ are both integrable.*

THEOREM 3. *An almost contact structure $\Sigma=(\phi, \xi, \eta)$ is integrable if and onlf if Σ is normal. In particular, Σ is normal if and only if the condition* (2.3) *is satisfied.*

In fact, taking $\bar{M}=\boldsymbol{R}$ and $\bar{\Sigma}=(\bar{\phi}, \bar{\xi}, \bar{\eta})=(0, \frac{d}{dt}, dt)$, Theorem 3 follows from Theorem 2.

Now it is known that spheres of odd dimension have normal almost contact structures (cf. [8]), whence we can apply Theorem 2, and obtain the following theorem due to Calabi-Eckmann [3].

COROLLARY. *The direct product of two spheres of odd dimension has a complex structure.*

We shall find later many examples of normal almost contact manifolds other than spheres of odd dimension.

§ 3. Automorphism groups of almost contact manifolds.

We now define the isomorphisms of almost contact structures as follows:

DEFINITION 4. Let M and M' be differentiable manifolds having almost contact structures $\Sigma=(\phi, \xi, \eta)$ and $\Sigma'=(\phi', \xi', \eta')$, respectively. A diffeomorphism f of M onto M' is called an *isomorphism* of M onto M' if the following conditions are satisfied:

$$\phi' \circ f = f \circ \phi \tag{3.1}$$

$$f(\xi)=\xi' \tag{3.2}$$

where we denote the differential of f by the same letter f. If, moreover, $M=M'$ and $\Sigma=\Sigma'$, f is called an *automorphism* of M. The set of all automorphisms of M forms a group of transformations of M. This group is denoted by $A(M)=A_\Sigma(M)$.

LEMMA. *If $f \in A(M)$, then*

$$f^*\eta=\eta\,. \tag{3.3}$$

PROOF. Put $\eta'=f^*\eta$, then $\Sigma'=(\phi, \xi, \eta')$ is an almost contact structure on M, for since the conditions (1.1)$\sim$(1.3) for Σ' is clear, it is sufficient to prove that (1.4) holds. Take $X \in \mathfrak{B}(M)$, then we have

$$\begin{aligned}\phi\circ\phi X &= \phi\circ\phi(f^{-1}fX)=f^{-1}\phi^2(fX)=f^{-1}(-fX+\eta(fX)\cdot\xi)\\ &= -X+(f^*\eta)(X)\cdot f^{-1}(\xi)=-X+\eta'(X)\cdot\xi\,,\end{aligned}$$

which proves (1.4). Since Σ' is an almost contact structure we see that $\eta=\eta'$ by Proposition 1, which proves the Lemma.

DEFINITION 5. Let $\tilde{M}$ be an almost complex manifold. We denote by $A(\tilde{M})$ the group of all diffeomorphisms f of M which leaves invariant the almost complex structure of $\tilde{M}$. Such diffeomorphism as f is called an *automorphism* of $\tilde{M}$.

DEFINITION 6. Let M be a differentiable manifold. We denote by $D(M)$ the group of all diffeomorphisms of M onto itself. For any manifolds M and $\bar{M}$, we can define a homomorphism H of $D(M)\times D(\bar{M})$ into $D(M\times\bar{M})$ by

$$H(f, g)=f\times g$$

for $f\in D(M)$ and $g\in D(\bar{M})$, i.e.

$$(f\times g)(p, q)=(f(p), g(q))$$

for $p\in M$ and $q\in\bar{M}$.

THEOREM 4. *Let M and $\bar{M}$ be almost contact manifolds. Then*

$$H(A(M)\times A(\bar{M}))\subset A(M\times\bar{M})\,,$$

where $M\times\bar{M}$ is considered as an almost complex manifold with the induced almost complex structure by M and $\bar{M}$.

PROOF. For $(f, g)\in A(M)\times A(\bar{M})$, put $H=H(f, g)$. Let (ϕ, ξ, η) and $(\bar{\phi}, \bar{\xi}, \bar{\eta})$ be the almost contact structures on M and $\bar{M}$ respectively and let J the induced almost complex structure of $M\times\bar{M}$. Denoting the differential of diffeomorphisms by the same letter, we have, for any tangent vectors $X_p \in T_p(M)$ and $\bar{X}_q \in T_q(\bar{M})$,

$$\begin{aligned} JH(X_p, \bar{X}_q) &= J_{(f(p), g(q))}(f(X_p), g(\bar{X}_q)) \\ &= (\phi f(X_p) - \bar{\eta}(g(\bar{X}_q)) \cdot \xi_{f(p)}, \bar{\phi}(g(\bar{X}_q)) + \eta(f(X_p)) \cdot \bar{\xi}_{g(p)}) \\ &= (f\phi(X_p) - g^*\bar{\eta}(\bar{X}_q) \cdot f(\xi_p), g\bar{\phi}(\bar{X}_q) + f^*\eta(X_p) \cdot g(\bar{\xi}_q)) \\ &= (f\phi(X_p) - \bar{\eta}(\bar{X}_q) \cdot f(\xi_p), g\bar{\phi}(\bar{X}_q) + \eta(X_p) \cdot g(\bar{\xi}_q)) \\ &= (f(\phi(X_p) - \bar{\eta}(\bar{X}_q) \cdot \xi_p), g(\bar{\phi}(\bar{X}_q) + \eta(X_p) \cdot \bar{\xi}_q)) \\ &= HJ(X_p, \bar{X}_q). \end{aligned}$$

Hence $J \circ H = H \circ J$, which proves that $H \in A(M \times \bar{M})$.

COROLLARY 1. *Let M and $\bar{M}$ be almost contact manifolds. If $A(M)$ ($A(\bar{M})$ resp.) operates transitively on M ($\bar{M}$ resp.), then $A(M \times \bar{M})$ operates transitively on $M \times \bar{M}$.*

In fact, $H(A(M) \times A(\bar{M}))$, and hence $A(M \times \bar{M})$ operates transitively on $M \times \bar{M}$.

COROLLARY 2. *Let M be a normal almost contact manifold. If $A(M)$ operates transitively on M, then $M \times M$ is a homogeneous complex manifold.*

In fact, this is an immediate consequence of Corollary 1 and Theorem 2.

THEOREM 5. *Let M be a compact, normal almost contact manifold. Then $A(M)$ is a Lie transformation group of M with respect to the compact-open topology.*

PROOF. First, we may suppose that M is connected. For any compact subset K ($\tilde{K}$ resp.) and an open set U ($\tilde{U}$ resp.) containing K ($\tilde{K}$ resp.) of M ($M \times M$ resp.) we denote by $W(K, U)$ ($\tilde{W}(\tilde{K}, \tilde{U})$ resp.) the set of all automorphisms f ($\tilde{f}$ resp.) of M (of $M \times M$ resp.) such that $f(K) \subset U$, and $f^{-1}(K) \subset U$ ($\tilde{f}(\tilde{K}) \subset \tilde{U}$, $\tilde{f}^{-1}(\tilde{K}) \subset \tilde{U}$ resp.). Now it is known (cf. [1]) that the automorphism group of a compact complex manifold is a Lie transformation group with respect to the compact-open topology. Hence we can find a finite number of compact sets $\tilde{K}_i$ and open sets $\tilde{U}_i \supset \tilde{K}_i$ such that the neighborhood $\tilde{W} = \bigcap_{i=1}^{N} \tilde{W}(\tilde{K}_i, \tilde{U}_i)$ of the identity in $A(M \times M)$ is relatively compact in $A(M \times M)$. We see readily that $\tilde{K}_i$ and $\tilde{U}_i$ may be chosen such as $\tilde{K}_i = K_i \times L_i$, $\tilde{U}_i = U_i \times V_i$, where K_i, L_i are compact sets in M and U_i, V_i are open in M and $K_i \subset U_i$, $L_i \subset V_i$ for $i = 1, 2, \cdots, N$. Now put $W = \bigcap_{i=1}^{N} W(K_i, U_i) \cap \bigcap_{i=1}^{N} W(L_i, V_i)$. Then W is a neighborhood of the identity in $A(M)$.

We shall prove that W is relatively compact in $A(M)$. For this purpose take a sequence $\{f_\nu\}_{\nu=1}^{\infty}$ in W. Consider the image $H_\nu = H(f_\nu, f_\nu)$ in $A(M \times M)$ by the homomorphism H. Then H_ν is clearly contained in $\tilde{W}$. Since $\tilde{W}$ is relatively compact in $A(M \times M)$, we can suppose that H_ν converges to an element H_0 in $A(M\times M)$. Now for any point $p\in M, H_0(p,p)=(q,q)$, since $H_0(p,p)=\lim H_\nu(p,p)=\lim(f_\nu(p),f_\nu(p))$. Put $q=f(p)$, then f is clearly a differentiable map of M into itself. We shall see that f is a diffeomorphism of M. In fact, the convergence of H_ν to H_0 being uniform convergence (since the topology of $A(M\times M)$ is the compact-open topology), $\{f_\nu\}$ converges uniformly to f on M. For the sequence $\{H_\nu^{-1}\}$ we can apply the same argument and we find a differentiable map g of M into itself such that f_ν^{-1} converges uniformly to g on M. Then $f\circ g=g\circ f=id.$ on M. This proves that f is a diffeomorphism of M. On the other hand, again by the uniform convergence of H_ν to H_0, the differential H_ν' of H_ν converges uniformly to the differential H_0' of H_0 since H_ν are holomorphic maps of the complex manifold $M\times M$. Hence we see that f_ν' also converges uniformly to f' on M. Then since f_ν' leaves invariant the almost contact structure of M, f' also leaves invariant the almost contact structure of M. Hence $f\in A(M)$, and so f_ν converges to f in $A(M)$, which proves that W is relatively compact in $A(M)$, for $A(M)$ satisfies the second axiom of countability.

Next we prove that an element f of $A(M)$ which leaves fixed any point of a non empty open set U in M is necessarily the identity map of M. In fact, in this case the holomorphic map $H_1=H(f,f)$ leaves fixed any point of $U\times U$ and so H_1 is the identity map of $M\times M$ and hence f is also the identity map of M. Thus we have proved that the group $A(M)$ is locally compact and that any element of $A(M)$ leaving fixed non-empty open set in M is the identity map of M. Hence by a theorem of Bochner-Montgomery [**1**], $A(M)$ is a Lie transformation group of M. Thus Theorem 5 is proved.

We may conjecture that in the case where (ϕ,ξ,η) is not necessarily normal Theorem 5 will also be true, but we have not succeeded to prove it.[2)]

Let again $\Sigma=(\phi,\xi,\eta)$ and $\bar\Sigma=(\bar\phi,\bar\xi,\bar\eta)$ be almost contact structures on M and $\bar M$. Then $M\times\bar M$ has an almost complex structure J induced by Σ and $\bar\Sigma$. We denote by $\mathfrak{a}(M\times\bar M)$ the Lie algebra of all infinitesimal automorphism of $M\times\bar M$. The homomorphism H induces a homomorphism of $\mathfrak{V}(M)\times\mathfrak{V}(\bar M)$ into $\mathfrak{V}(M\times\bar M)$ i.e.

$$H(X,\bar X)=X+\bar X$$

for $X\in\mathfrak{V}(M)$ and $\bar X\in\mathfrak{V}(\bar M)$, where we have identified X ($\bar X$ resp.) with $\tilde X$ defined in the proof of Proposition 3. We now want to determine the inverse image of $\mathfrak{a}(M\times\bar M)$ by the homomorphism H.

2) By using a recent result of Boothby-Kobayashi-Wang (Ann. of Math., 77 (1963)) we can prove, in the same way as the proof of Theorem 5, that $A_\Sigma(M)$ is always a Lie transformation group with respect to a somewhat stronger topology than the compact-open one, if M is compact.

PROPOSITION 4. *Notations being as above, $X+\bar{X}$ is contained in $\mathfrak{a}(M\times\bar{M})$ if and only if the following conditions are satisfied:*

$$\phi[X, Y]=[X, \phi Y] \tag{3.4}$$

$$\bar{\phi}[\bar{X}, \bar{Y}]=[\bar{X}, \bar{\phi}\bar{Y}] \tag{3.5}$$

$$\eta([X, \xi])=\bar{\eta}([\bar{X}, \bar{\xi}])=const. \tag{3.6}$$

for all $Y\in\mathfrak{V}(M)$ and $\bar{Y}\in\mathfrak{V}(\bar{M})$.

PROOF. Suppose that $X+\bar{X}\in\mathfrak{a}(M\times\bar{M})$. Then by definition of $\mathfrak{a}(M\times\bar{M})$ the following conditions are satisfied:

$$[X+\bar{X}, J(Y+\bar{Y})]=J[X+\bar{X}, Y+\bar{Y}] \tag{3.7}$$

for all $Y\in\mathfrak{V}(M)$ and $\bar{Y}\in\mathfrak{V}(\bar{M})$. By the definition of J, the left hand side of (3.7) is equal to

$$[X, \phi Y]+[\bar{X}, \bar{\phi}\bar{Y}]-\bar{\eta}(\bar{Y})[X, \xi]+\eta(Y)[\bar{X}, \bar{\xi}]+X(\eta(Y))\cdot\bar{\xi}-\bar{X}(\bar{\eta}(\bar{Y}))\cdot\xi. \tag{3.8}$$

The right hand side of (3.7) is equal to

$$\phi[X, Y]-\bar{\eta}([\bar{X}, \bar{Y}])\cdot\xi+\bar{\phi}[\bar{X}, \bar{Y}]+\eta([X, Y])\cdot\bar{\xi}. \tag{3.9}$$

Comparing (3.8) with (3.9) we obtain the following four conditions equivalent to (3.7):

$$\phi[X, Y]=[X, \phi Y], \tag{3.4}$$

$$\bar{\phi}[\bar{X}, \bar{Y}]=[\bar{X}, \bar{\phi}\bar{Y}], \tag{3.5}$$

$$\bar{\eta}([\bar{X}, \bar{Y}])\cdot\xi=\bar{\eta}(\bar{Y})[X, \xi]+\bar{X}(\bar{\eta}(\bar{Y}))\cdot\xi, \tag{3.10}$$

$$\eta([X, Y])\cdot\bar{\xi}=\eta(Y)[\bar{X}, \bar{\xi}]+X(\eta(Y))\cdot\bar{\xi}, \tag{3.11}$$

for all $Y\in\mathfrak{V}(M)$ and $\bar{Y}\in\mathfrak{V}(\bar{M})$.

Putting $\bar{Y}=\bar{\xi}$ in (3.10) we have $\bar{\eta}([\bar{X}, \bar{\xi}])\cdot\xi=[X, \xi]$, hence $\bar{\eta}([\bar{X}, \bar{\xi}])=\eta([X, \xi])$, which shows that $\eta([X, \xi])$ and $\bar{\eta}([\bar{X}, \bar{\xi}])$ are both constant functions and proves (3.6).

Conversely, suppose that $(X, \bar{X})\in\mathfrak{V}(M)\times\mathfrak{V}(\bar{M})$ satisfies (3.4)~(3.6). Putting $\bar{Y}=\bar{\xi}$ in (3.5) we have $\bar{\phi}[\bar{X}, \bar{\xi}]=0$, hence $[\bar{X}, \bar{\xi}]=-\bar{\phi}^2[\bar{X}, \bar{\xi}]+\bar{\eta}([\bar{X}, \bar{\xi}])\cdot\bar{\xi}=\eta([X, \xi])\cdot\bar{\xi}$. Therefore to prove (3.11) it is now sufficient to verify

$$\eta([X, Y])=X(\eta(Y))+\eta(Y)\eta([X, \xi]) \tag{3.12}$$

for all $Y\in\mathfrak{V}(M)$. Now (3.12) is clear for $Y=\xi$. Take $Y\in\mathfrak{V}(M)$ satisfying $\eta(Y)=0$. Then $Y=-\phi^2(Y)$. Hence

$$\eta([X, Y])=\eta([X, -\phi^2 Y])=\eta(\phi[X, -\phi Y])=0,$$

where we have used (3.4). Therefore, (3.12) is true for Y such that $\eta(Y)=0$. For an arbitrary Y, Y can be written as $Y=Y-\eta(Y)\xi+\eta(Y)\xi=Y_1+\eta(Y)\cdot\xi$, where $\eta(Y_1)=0$. This shows that (3.12) is also true for all $Y\in\mathfrak{V}(M)$. The condition (3.10) is verified in the same way. Thus $X+\bar{X}$ is contained in $\mathfrak{a}(M\times\bar{M})$, which proves Proposition 4.

It may be an interesting problem to investigate the case when $\mathfrak{a}(M\times\bar{M})$ is spanned by $X+\bar{X}$ satisfying (3.4)~(3.6).

§ 4. Homogeneous contact manifolds.

In this section we shall prove that the bundle space of a principal circle bundle over a complex manifold satisfying certain conditions admit a normal almost contact structure and prove that any simply connected compact homogeneous contact manifold M (in the sense of Boothby-Wang [2]) has a normal almost contact structure such that $A(M)$ operates transitively on M. This will show the existence of many examples of compact normal almost contact manifolds other than spheres of odd dimension as stated in § 2.

We shall show another kind of normal almost contact manifolds in the next section.

THEOREM 6. *Let $\tilde{M}$ be the bundle space of a principal circle bundle over a complex manifold M of complex dimension n (i.e., the structure group is the multiplicative group $U(1)$ of complex numbers of absolute value 1). The Lie algebra of $U(1)$ may be identified with the real number space $\boldsymbol{R}$. Let there exist a connection (form) η on $\tilde{M}$ such that $d\eta=\pi^*\Omega$. Here π is the projection of $\tilde{M}$ onto M and Ω is a 2-form on M satisfying the following condition:*

$$\Omega(JX, JY)=\Omega(X, Y)$$

for $X, Y\in\mathfrak{V}(M)$, J being the complex structure of M.

Then we can find a (1.1)-tensor field ϕ on $\tilde{M}$ and a vector field ξ on $\tilde{M}$ such that (ϕ, ξ, η) is a normal almost contact structure on $\tilde{M}$.

PROOF. For any tangent vector $\tilde{X}$ of $\tilde{M}$ at $\tilde{p}$ we write $\tilde{X}=h\tilde{X}+v\tilde{X}$, where $h\tilde{X}$ ($v\tilde{X}$ resp.) is the horizontal (vertical resp.) component of $\tilde{X}$ with respect to the connection η. And for any tangent vector X of M at p ($\pi(\tilde{p})=p$), we denote by $X^*_{\tilde{p}}$ the lift of X at $\tilde{p}$ with respect to the connection η (for these terminologies see [7]). We now define the endomorphism $\phi_{\tilde{p}}$ of the tangent space $T_{\tilde{p}}(\tilde{M})$ of $\tilde{M}$ at $\tilde{p}$ as follows:

$$\phi_{\tilde{p}}(\tilde{X})=(J\pi' h\tilde{X})^*_{\tilde{p}}, \tag{4.1}$$

π' being the differential of π. By the identification of the Lie algebra $\mathfrak{u}$ of $U(1)$ with $\boldsymbol{R}$, we find an element A of $\mathfrak{u}$ such that $\eta(A^*)=1$, where A^* is the fundamental vector field corresponding to A (cf. [7]). Put $\xi=A^*$.

We shall verify the conditions (1.1)~(1.4) for (ϕ, ξ, η). First, $\phi(\xi)=0$, since ξ is vertical. $\eta\circ\phi(\tilde{X})=0$, since $\phi(\tilde{X})$ is horizontal. Next $\phi^2=-\mathbf{1}+\eta\cdot\xi$. In fact, since $v\tilde{X}=\eta(\tilde{X})\cdot\xi$ for all $\tilde{X}\in T_{\tilde{p}}(\tilde{M})$ we see that

$$\begin{aligned}\phi_{\tilde{p}}(\phi_{\tilde{p}}(\tilde{X})) &= \phi_{\tilde{p}}((J\pi' h\tilde{X})^*_{\tilde{p}}) = (J\pi'(J\pi' h\tilde{X})^*_{\tilde{p}})^*_{\tilde{p}} \\ &= (J(J\pi' h\tilde{X}))^*_{\tilde{p}} = -(\pi' h\tilde{X})^*_{\tilde{p}} = -h\tilde{X} \\ &= -\tilde{X} + v\tilde{X} = -\tilde{X} + \eta(\tilde{X}) \cdot \xi\,.\end{aligned}$$

Hence (ϕ, ξ, η) is an almost contact structure on $\tilde{M}$.

To prove that (ϕ, ξ, η) is normal it is sufficient to prove the condition (2.3). First, we have by the definition of ϕ the following equality

$$\phi(X^*) = (JX)^* \tag{4.2}$$

for all $X \in \mathfrak{V}(M)$, where X^* means the lift of X with respect to the connection η. Secondly, we know the following equality

$$h[X^*, Y^*] = [X, Y]^* \tag{4.3}$$

for all $X, Y \in \mathfrak{V}(M)$ (cf. [7]). Using (4.2) and (4.3) and the fact that the almost complex structure J of M is integrable we can now calculate $\phi[X^*, Y^*]$ for $X, Y \in \mathfrak{V}(M)$ as follows:

$$\begin{aligned}\phi[X^*, Y^*] &= (J\pi h[X^*, Y^*])^* = (J\pi[X, Y]^*)^* = (J[X, Y])^* \\ &= ([JX, Y] + [X, JY] + J[JX, JY])^* \\ &= [JX, Y]^* + [X, JY]^* + (J[JX, JY])^* \\ &= h[(JX)^*, Y^*] + h[X^*, (JY)^*] + \phi[JX, JY]^* \\ &= h[\phi X^*, Y^*] + h[X^*, \phi Y^*] + \phi(h[(JX)^*, (JY)^*]) \\ &= [\phi X^*, Y^*] - \eta([\phi X^*, Y^*]) \cdot \xi \\ &\quad + [X^*, \phi Y^*] - \eta([X^*, \phi Y^*])\xi + \phi[\phi X^*, \phi Y^*]\,.\end{aligned} \tag{4.4}$$

On the other hand, we have

$$\begin{aligned}\eta([\phi X^*, Y^*]) &= \phi X^*(\eta(Y^*)) - Y^* \cdot (\eta(\phi X^*)) - d\eta(\phi X^*, Y^*) \\ &= -\pi^*\Omega(\phi X^*, Y^*) \\ &= -\pi^*\Omega((JX)^*, Y^*) \\ &= -\Omega(JX, Y)\,.\end{aligned} \tag{4.5}$$

In the same way, we have

$$\eta([X^*, \phi Y^*]) = -\Omega(X, JY) \tag{4.6}$$

Hence by the assumption for Ω, we obtain from (4.4)$\sim$(4.6)

$$\phi[X^*, Y^*] = [\phi X^*, Y^*] + [X^*, \phi Y^*] + \phi[\phi X^*, \phi Y^*]\,.$$

From this we easily see that

$$\Psi(X^*, Y^*) = 0\,.$$

Now it is clear that $[X^*, \xi] = 0$ for all $X \in \mathfrak{V}(M)$ since X^* is right invariant vector field on $\tilde{M}$. We see also $[\phi X^*, \xi] = [(JX)^*, \xi] = 0$. Hence we easily verify that

$$\Psi(X^*, \xi)=0\,.$$

We have thus proved that (2.3) holds good for the lifts of vector fields on M or the vertical vector fields. Since Ψ is a tensor field, (2.3) holds good identically, Thus Theorem 6 is proved.

COROLLARY. *Let M be a compact simply connected homogeneous contact manifold* (in the sense of Boothby-Wang [**2**]) *then M has a normal almost contact structure such that $A(M)$ operates transitively on M.*

PROOF. By a theorem of Boothby-Wang [**2**], M has a bundle structure having the properties stated in Theorem 6 and there exists a compact transitive Lie transformation group K of M which is fibre-preserving and leaves η and ξ invariant and moreover any element of K induces a holomorphic transformation of the base space of M. From these facts we see that any element of K leaves ϕ invariant. This proves that K is a subgroup of $A(M)$. Since K operates transitively on M, $A(M)$ is also transitive on M, which proves Corollary.

§ 5. Almost contact group manifold.

Let G be a connected Lie group. For any element $a \in G$, we denote by L_a the left translation of G defined by

$$L_a(x)=a\cdot x$$

for $x \in G$. An almost contact structure $\Sigma=(\phi, \xi, \eta)$ on G will be called *left invariant* if $L_a \in A_\Sigma(G)$ for all $a \in G$.

In this section we shall show that the problem to find a left invariant normal almost contact structure on a group manifold is reduced to a purely algebraic problem in the Lie algebra.

DEFINITION 7. Let $\mathfrak{g}$ be a real Lie algebra, and let ϕ_0 be a linear map of $\mathfrak{g}$ into itself, ξ_0 be an element of $\mathfrak{g}$ and η_0 be a linear function on $\mathfrak{g}$. The triple $\Sigma_0=(\phi_0, \xi_0, \eta_0)$ is called a *contact structure on* $\mathfrak{g}$ if the following conditions are satisfied:

(5.1) $$\phi_0(\xi_0)=0\,, \qquad \eta_0\circ\phi_0=0\,, \qquad \eta_0(\xi_0)=1\,, \qquad \phi_0^2=-\mathbf{1}+\eta_0\cdot\xi_0$$

(5.2) $$\phi_0[X, Y]=[\phi_0X, Y]+[X, \phi_0Y]+\phi_0[\phi_0X, \phi_0Y]\,,$$

for $X, Y \in \mathfrak{g}$.

REMARK. If $\mathfrak{g}$ is a Lie algebra having a contact structure (ϕ_0, ξ_0, η_0) satisfying (5.1) and (5.2), then the following condition is automatically satisfied:

(5.3) $$\eta_0([X, Y])=\eta_0([\phi_0X, \phi_0Y])$$

for $X, Y \in \mathfrak{g}$.

In fact, (5.3) is implied by (5.1) and (5.2) in the same way as the implication of (2.7) from (2.3) in the proof of Proposition 3. We shall not repeat the proof in detail.

We shall call a Lie algebra with a contact structure a *contact Lie algebra.*

THEOREM 7. *Let G be a connected Lie group of odd dimension. Then G admits a left invariant normal almost contact structure if and only if the Lie algebra $\mathfrak{g}$ of the left invariant vector fields on G is a contact Lie algebra.*

PROOF. Suppose that G admits a left invariant normal almost contact structure $\Sigma=(\phi, \xi, \eta)$. For any $X\in\mathfrak{g}$, we have $\phi X\in\mathfrak{g}$, since $L_a\phi X=\phi L_a X=\phi X$ for any $a\in G$. Hence the restriction ϕ_0 of ϕ to $\mathfrak{g}$ maps $\mathfrak{g}$ into $\mathfrak{g}$. Take $X\in\mathfrak{g}$, then $\eta(X)$ is a constant on G, since η and X are left invariant. Hence the restriction η_0 of η to $\mathfrak{g}$ is a linear function on $\mathfrak{g}$. On the other hand, it is clear that $\xi\in\mathfrak{g}$. Hence by putting $\xi_0=\xi$, $\Sigma_0=(\phi_0, \xi_0, \eta_0)$ satisfies (5.1). Now we can readily see that (2.3) implies (5.2), since $\eta(X)$ is constant for $X\in\mathfrak{g}$, which proves that $\mathfrak{g}$ is a contact Lie algebra.

Conversely, suppose that $\mathfrak{g}$ has a contact structure $\Sigma_0=(\phi_0, \xi_0, \eta_0)$ satisfying (5.1), (5.2). Let $X_1, X_2, \cdots, X_{2n+1}$ be a basis of $\mathfrak{g}$ over $\boldsymbol{R}$. Then for any $X\in\mathfrak{B}(G)$, we can find $2n+1$ functions $\alpha_1, \alpha_2, \cdots, \alpha_{2n+1}$ on G such that X can be written uniquely

$$X=\sum_{i=1}^{2n+1}\alpha_i X_i .$$

Now define $\Sigma=(\phi, \xi, \eta)$ as follows:

$$\phi(X)=\sum_{i=1}^{2n+1}\alpha_i\phi_0(X_i),$$

$$\eta(X)=\sum_{i=1}^{2n+1}\alpha_i\eta_0(X_i),$$

and $\xi=\xi_0$. Then clearly Σ satisfies (1.1)$\sim$(1.4), hence Σ is an almost contact structure on G. On the other hand, by (5.2) we have

$$\Psi(X_i, X_j)=0,$$

for $i, j=1, 2, \cdots, 2n+1$. Since Ψ is a tensor field on G, Ψ vanishes identically, which proves that Σ is normal. Thus Theorem 7 is proved.

REMARK. Every connected Lie group G of odd dimension admits a left invariant almost contact structure. In fact, the proof of Theorem 7 shows that the existence of (ϕ_0, ξ_0, η_0) satisfying (5.1) for the Lie algebra $\mathfrak{g}$ of G implies the existence of a left invariant almost contact structure on G. (The existence of such (ϕ_0, ξ_0, η_0) is evident, since the condition (5.1) has a concern with $\mathfrak{g}$ only as a real vector space of odd dimension.)

THEOREM 8. *Let $\Sigma=(\phi, \xi, \eta)$ be a contact structure on a Lie algebra $\mathfrak{g}$. Let $\tilde{\mathfrak{g}}=\mathfrak{g}\oplus\mathfrak{a}$ be the direct sum of $\mathfrak{g}$ and 1-dimensional Lie algebra $\mathfrak{a}$. Define the* *linear map J of $\tilde{\mathfrak{g}}$ into itself by*

$$J(X, a)=(\phi X-a\xi, \eta(X))$$

for $X\in\mathfrak{g}$ and $a\in\mathfrak{a}$. Then J satisfies the following conditions:

$$J^2=-\mathbf{1}, \tag{5.4}$$

$$J[\tilde{X}, \tilde{Y}]=[J\tilde{X}, \tilde{Y}]+[\tilde{X}, J\tilde{Y}]+J[J\tilde{X}, J\tilde{Y}], \tag{5.5}$$

for all $\tilde{X}, \tilde{Y}\in\tilde{\mathfrak{g}}$.

PROOF. Let $\tilde{X}=(X, a)\in\tilde{\mathfrak{g}}$. Then we have

$$\begin{aligned} J^2(\tilde{X})=J(\phi X-a\cdot\xi, \eta(X))&=(\phi^2X-\eta(X)\cdot\xi, -a)\\ &=(-X, -a)=-\tilde{X}. \end{aligned}$$

Hence $J^2=-\mathbf{1}$. For $\tilde{X}=(X, a)$ and $\tilde{Y}=(Y, b)\in\tilde{\mathfrak{g}}$, we have

$$\begin{aligned} J[\tilde{X}, \tilde{Y}]&=J([X, Y], 0)=(\phi[X, Y], \eta([X, Y]))\\ &=([\phi X, Y]+[X, \phi Y]+\phi[\phi X, \phi Y], \eta([X, Y])). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} [J\tilde{X}, \tilde{Y}]&=[(\phi X-a\cdot\xi, \eta(X)), (Y, b)]\\ &=([\phi X-a\cdot\xi, Y], 0), \end{aligned} \tag{5.6}$$

$$[\tilde{X}, J\tilde{Y}]=([X, \phi Y-b\cdot\xi], 0), \tag{5.7}$$

and

$$\begin{aligned} J[J\tilde{X}, J\tilde{Y}]&=J[(\phi X-a\xi, \eta(X)), (\phi Y-b\xi, \eta(Y))]\\ &=J([\phi X-a\xi, \phi Y-b\xi], 0)\\ &=(\phi[\phi X-a\xi, \phi Y-b\xi], \eta([\phi X-a\xi, \phi Y-b\xi])). \end{aligned}$$

Here we shall use the equality

$$[\phi X, \xi]=\phi[X, \xi]$$

which is implied by (5.2) by putting $Y=\xi$. Then we have

$$\begin{aligned} J[J\tilde{X}, J\tilde{Y}]&=(\phi[\phi X, \phi Y]-\phi^2[X, b\xi]-\phi^2[a\xi, Y], \eta([X, Y]))\\ &=(\phi[\phi X, \phi Y]+[X, b\xi]-\eta([X, b\xi])\cdot\xi\\ &\qquad +[a\xi, Y]-\eta([a\xi, Y])\xi, \eta([X, Y])). \end{aligned} \tag{5.8}$$

Now from (5.3) we obtain $\eta([X, \xi])=0$ by putting $Y=\xi$. Adding (5.6)$\sim$(5.8), we obtain (5.5), which proves Theorem 8.

Conversely, we have the following

THEOREM 9. *Let $\tilde{\mathfrak{g}}=\mathfrak{g}\oplus\mathfrak{a}$ be the direct sum of a Lie algebra $\mathfrak{g}$ and 1-dimensional Lie algebra $\mathfrak{a}$. Let J be a linear map of $\tilde{\mathfrak{g}}$ into itself satisfying (5.4) and (5.5). Suppose that there exist an element $\xi\in\mathfrak{g}$ and a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{m}\oplus\{\xi\}$ (direct sum of vector spaces), $J(\xi)\in\mathfrak{a}$ and $J(\mathfrak{m})\subset\mathfrak{m}$.*

Then there exist a linear map ϕ of $\mathfrak{g}$ into itself and a linear function η on

$\mathfrak{g}$ *such that* $\Sigma=(\phi, \xi, \eta)$ *is a contact structure on* $\mathfrak{g}$.

PROOF. We see first that we can suppose $J(\xi)=1\in\mathfrak{a}$. Now any element $X\in\mathfrak{g}$ can be written uniquely in the form

$$X=X_1+a\cdot\xi,$$

where $X_1\in\mathfrak{m}$ and $a\in\boldsymbol{R}$. Define ϕ and η as follows

$$\phi(X)=J(X_1) \quad\text{and}\quad \eta(X)=a.$$

Now we shall verify that $\Sigma=(\phi, \xi, \eta)$ satisfies (5.1) and (5.2). First, we shall verify (5.1): the conditions $\eta\circ\phi=0$, $\eta(\xi)=1$ and $\phi(\xi)=0$ are clearly satisfied. For $X=X_1+a\cdot\xi\in\mathfrak{g}$, $X_1\in\mathfrak{m}$, $a\in\boldsymbol{R}$, we have

$$\begin{aligned}\phi^2(X)=\phi(J(X_1))=J^2(X_1)=-X_1&=-(X_1+a\cdot\xi)+a\cdot\xi\\&=-X+\eta(X)\cdot\xi,\end{aligned}$$

which proves (5.1). Concerning (5.2) (and (5.3)), we first remark that

$$J(X, \lambda)=(\phi X-\lambda\cdot\xi, \eta(X)) \tag{5.9}$$

is satisfied for $X\in\mathfrak{g}$ and $\lambda\in\mathfrak{a}$. For, both hand sides of (5.9) are equal to $(JX_1-\lambda\cdot\xi, a)$. Now we calculate both hand sides of

$$J([X, Y])=[JX, Y]+[X, JY]+J[JX, JY]$$

for $X, Y\in\mathfrak{g}$. Using (5.9) we obtain the following

$$(\phi([X, Y]), \eta([X, Y]))=([\phi X, Y]+[X, \phi Y]+\phi[\phi X, \phi Y], \eta([\phi X, \phi Y])).$$

Therefore, we have proved (5.2). Thus Theorem 9 is proved.

COROLLARY 1. *Let G be a reductive Lie group of odd dimension (i.e., the Lie algebra of G be the direct sum of a semi-simple Lie algebra and an abelian Lie algebra). Then G admits a left invariant normal almost contact structure.*

In fact, the Lie algebra $\tilde{\mathfrak{g}}=\mathfrak{g}\oplus\mathfrak{a}$ ($\mathfrak{a}$ is 1-dimensional Lie algebra) has a structure stated in Theorem 9 (for this fact see [**5**]). Then by Theorem 9 and 7, we conclude that G admits a left invariant normal almost contact structure.

COROLLARY 2. *Every compact connected Lie group G of odd dimension has a left invariant normal almost contact structure.*

In fact, since the Lie algebra of G is reductive, we can apply Corollary 1.

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