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On differentiable manifolds with contact metric structures

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§ 1. Introduction.

A $(2n+1)$-dimensional differentiable manifold M^{2n+1} of class C^∞ is said to have an almost contact structure or to be an almost contact manifold if the structural group of its tangent bundle reduces to $U(n)\times 1$, where $U(n)$ means the real representation of the unitary group of n complex variables (cf. [**1**]). On the other hand, a differentiable manifold M^{2n+1} of class C^∞ is said to have (ϕ, ξ, η)-structure if there exist three tensor fields ϕ^i_j, ξ^i and η_j satisfying the relations

$$\xi^i\eta_i = 1\,, \tag{1.1}$$

$$\operatorname{rank}(\phi^i_j) = 2n\,, \tag{1.2}$$

$$\phi^i_j\xi^j = 0\,, \tag{1.3}$$

$$\phi^i_j\eta_i = 0\,, \tag{1.4}$$

$$\phi^i_j\phi^j_k = -\delta^i_k + \xi^i\eta_k\,. \tag{1.5}$$

The notions of almost contact structure and (ϕ, ξ, η)-structure are equivalent in the sense that every almost contact manifold admits a (ϕ, ξ, η)-structure and every differentiable manifold with (ϕ, ξ, η)-structure is almost contact. (cf. [**2**]) So, in this paper we use the word *almost contact structure* in stead of (ϕ, ξ, η)-structure.

Now, every differentiable manifold M^{2n+1} with almost contact structure admits a Riemannian metric g which satisfies the relations

$$g_{ij}\xi^j = \eta_i\,, \tag{1.6}$$

$$g_{ij}\phi^i_h\phi^j_k = g_{hk} - \eta_h\eta_k\,. \tag{1.7}$$

We call g an *associated Riemannian metric* of the almost contact structure. Although we have called the (ϕ, ξ, η)-structure with associated Riemannian metric g as the (ϕ, ξ, η, g)-structure in [**2**], we shall call it an *almost contact metric structure* in this paper.

By virtue of (1.1) and (1.6), ξ^i is a unit vector field. The tensor

$$\phi_{ij} = g_{ih}\phi^h_j \tag{1.8}$$

is skew-symmetric and satisfies by (1.3)

$$\phi_{ij}\xi^j = 0. \tag{1.9}$$

From ϕ_{ij}, we may construct an exterior 2-form

$$\phi = \frac{1}{2}\phi_{ij}dx^i \wedge dx^j \tag{1.10}$$

which we shall call the *associated* 2-*form* of the almost contact metric structure in consideration.

A differentiable manifold M^{2n+1} of class C^∞ is said to have a contact structure or to be a contact manifold if there exists a 1-form η over M^{2n+1} such that

$$\eta \wedge \overbrace{d\eta \wedge \cdots \wedge d\eta}^{n} = 0, \tag{1.11}$$

where n operations $\wedge$ in the last equation mean exterior multiplication. In the paper [2], it is proved that, for any differentiable manifold M^{2n+1} with contact structure we can find four tensors ϕ_j^i, ξ^i, η_j and g_{ij} so that they define an almost contact metric structure such that the vector field η_j is the one given as the coefficients of η and satisfies (1.8), where we put

$$\phi_{ij} = \partial_i\eta_j - \partial_j\eta_i. \tag{1.12}$$

The last relation can also be written as

$$d\eta = \phi. \tag{1.13}$$

We shall call this structure as *contact metric structure* for brevity.

For almost contact structure, we can define four tensor fields N_{jk}^i, N_j^i, N_{jk} and N_j given later by (4.1)~(4.4) which are the analogue of the Nijenhuis' tensor in almost complex structure. In the paper [3], we have proved that if N_{jk}^i vanishes identically over M^{2n+1}, then N_j^i, N_{jk} and N_j also vanish identically.

The above considerations suggest us the following diagram of structures as the analogy of the diagrams: almost complex structure→almost Hermitian structure→almost Kählerian structure and complex structure→Hermitian structure→Kählerian structure.

$$\begin{tikzcd}
\boxed{\text{almost contact structure in the proper sense}} \arrow[rr] \arrow[d, shift left] & & \boxed{\text{contact structure in the proper sense}} \arrow[d, shift left] \\
\boxed{\text{almost contact structure in our sense}} \arrow[u, shift left] \arrow[d] \arrow[r, shift right] & \boxed{\text{almost contact metric structure}} \arrow[l, shift right] \arrow[r] \arrow[d] & \boxed{\text{contact metric structure}} \arrow[u, shift left] \arrow[d] \\
\boxed{\text{normal almost contact structure}} \arrow[r, shift right] & \boxed{\text{normal almost contact metric structure}} \arrow[l, shift right] \arrow[r] & \boxed{\text{normal contact metric structure}}
\end{tikzcd}$$

In this diagram, the adjective " normal " means that we have imposed the condition $N^i_{jk}=0$. We think that manifolds with normal contact structure correspond to Kählerian manifolds. So interesting theorems are expected for them.

In this paper, we shall give basic formulas and basic geometric properties of manifolds with general or normal contact metric structure. An example of normal contact metric structure and an application to the Lie algebra of infinitesimal contact transformations are given too.

§ 2. Intrinsic tensor fields of a contact structures.

By virtue of (1.13), the defining relation (1.11) can be written as

$$\eta_{[1}\phi_{23}\phi_{45}\cdots\phi_{2n\,2n+1]}\neq 0\,, \tag{2.1}$$

where [ ] means a determinant divided by the factorial of the number included in the bracket. In the paper [2], one of the authors has proved that the volume element of the Riemannian metric in consideration is a constant multiple of the form $\eta\wedge(d\eta)^n$. If we denote the determinant of the metric tensor g_{ij} by g, then the last statement can be expressed as

$$\sqrt{g}=c\lambda\,, \tag{2.2}$$

$$\lambda\equiv(2n+1)\eta_{[1}\phi_{23}\phi_{45}\cdots\phi_{2n\,2n+1]}\,, \tag{2.3}$$

$$c=(-1)^{\frac{n(n+1)}{2}}(2n-1)\,!\,!\,. \tag{2.4}$$

As $\sqrt{g}$ is a scalar density, we see that λ is a scalar density which is determined by the contact structure η.

Now, we define

$$\overset{2p}{\Phi}_{i_1\cdots i_{2p}}=(2p)\,!\,\phi_{[i_1i_2}\phi_{i_3i_4}\cdots\phi_{i_{2p-1}i_{2p}]}\,, \tag{2.5}$$

$$\overset{2p+1}{\Phi}_{i_1\cdots i_{2p+1}}=(2p+1)\,!\,\eta_{[i_1}\phi_{i_2i_3}\phi_{i_4i_5}\cdots\phi_{i_{2p}i_{2p+1}]} \tag{2.6}$$

for $p=1,2,\cdots,n$. Then, they are covariant $2p$-, and $(2p+1)$-vector fields over M^{2n+1}, which are induced by η. They satisfy the relation

$$\overset{2p+1}{\Phi}_{i_1\cdots i_{2p+1}}=S(\eta_{i_1}\overset{2p}{\Phi}_{i_2\cdots i_{2p+1}})\,, \tag{2.7}$$

where S means the sum of $2p+1$ terms which arise by cyclic permutation of indices $i_1,\cdots,i_{2p+1}$. In the sequel, we shall call tensors which are determined completely by η and do not depend upon the associated Riemannian metric g as *intrinsic tensors* for brevity. λ, η_i, ϕ_{ij} and the above tensors are intrinsic tensors. If we put

$$\overset{s}{\Phi}=\frac{1}{s\,!}\overset{s}{\Phi}_{i_1\cdots i_s}dx^{i_1}\wedge\cdots\wedge dx^{i_s} \tag{2.8}$$

for $s=2,3,\cdots,2n+1$, then we see that

$$\overset{2p}{\Phi}=2^p\phi^p \qquad (p=1,\cdots,n), \tag{2.9}$$

$$\overset{2q+1}{\Phi}=2^q\eta\wedge\phi^q \qquad (q=1,\cdots,n-1). \tag{2.10}$$

By virtue of (1.13), the following theorem is easily seen to be true.

THEOREM 1. *Suppose M^{2n+1} be a manifold with contact structure η. Then the relations*

$$d\overset{2p}{\Phi}=0 \qquad p=1,2,\cdots,n, \tag{2.11}$$

$$d\overset{2q+1}{\Phi}=\frac{1}{2}\overset{2q+2}{\Phi} \qquad q=1,\cdots,n-1 \tag{2.12}$$

hold good.

In the paper [2], we saw that the vector field ξ^i of the (ϕ,ξ,η,g)-structure induced by the contact structure η is given by

$$\left\{\begin{aligned} &\xi^1=\frac{1}{\lambda}\phi_{[23}\phi_{45}\cdots\phi_{2n\,2n+1]}, \\ &\xi^2=\frac{1}{\lambda}\phi_{[34}\phi_{56}\cdots\phi_{2n+1\,1]}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\xi^{2n+1}=\frac{1}{\lambda}\phi_{[12}\phi_{34}\cdots\phi_{2n-1\,2n]}. \end{aligned}\right. \tag{2.13}$$

Hence, ξ^i is an intrinsic vector field of the contact structure η.

Now, let $i_1,i_2,\cdots,i_{2n+1}$ be integers which are $\geqq 1$ and $\leqq 2n+1$ and define $\operatorname{sgn}(i_1\,i_2\cdots i_{2n+1})$ as follows: 1° if the ordered set $i_1,i_2,\cdots,i_{2n+1}$ is a permutation of $1,2,\cdots,2n+1$, then

$$\operatorname{sgn}(i_1\,i_2\cdots i_{2n+1})=\pm 1$$

according as the permutation is even or odd. 2° if $i_p=i_q$ for some pair p and q, then

$$\operatorname{sgn}(i_1\,i_2\cdots i_{2n+1})=0.$$

Making use of this notation, we put

$$\varepsilon^{i_1i_2\cdots i_{2n+1}}=\frac{1}{\lambda}\operatorname{sgn}(i_1\,i_2\cdots i_{2n+1}), \tag{2.14}$$

then we can easily see that $\varepsilon^{i_1i_2\cdots i_{2n+1}}$ is an intrinsic contravariant $(2n+1)$-vector field. Making use of such tensor we can write (2.13) as follows:

$$\xi^i=\frac{1}{(2n)!}\varepsilon^{i i_1 i_2\cdots i_{2n}}\phi_{[i_1i_2}\phi_{i_3i_4}\cdots\phi_{i_{2n-1}i_{2n}]}, \tag{2.13$'$}$$

where we suppose that the usual summation convention is applied.

We put also

(2.15) $$\xi^{ij}=\frac{1}{(2n-1)!}\varepsilon^{ijk_1\cdots k_{2n-1}}\eta_{[k_1}\phi_{k_2k_3}\cdots\phi_{k_{2n-2}k_{2n-1}]},$$

then ξ^{ij} is an intrinsic contravariant skew symmetric tensor field. For $i=j$, $\xi^{ij}=0$. For $i\neq j$, if we denote the complement of the pair i, j in the set of integers $(1, 2, \cdots, 2n+1)$ by $c_1, \cdots, c_{2n-1}$, then (2.15) can be written as

(2.15)′ $$\xi^{ij}=\varepsilon^{ijc_1\cdots c_{2n-1}}\eta_{[c_1}\phi_{c_2c_3}\cdots\phi_{c_{2n-2}c_{2n-1}]}$$

under the assumption that we do not sum for $c_1, \cdots, c_{2n-1}$.

In the same way we can define intrinsic tensor fields by

(2.16) $$\xi^{i_1\cdots i_{2p}}=\frac{1}{(2n-2p+1)!}\varepsilon^{i_1\cdots i_{2p}k_1\cdots k_{2n-2p+1}}\eta_{[k_1}\phi_{k_2k_3}\cdots\phi_{k_{2n-2p}k_{2n-2p+1}]},$$

(2.17) $$\xi^{i_1\cdots i_{2q+1}}=\frac{1}{(2n-2q)!}\varepsilon^{i_1\cdots i_{2q+1}k_1\cdots k_{2n-2q}}\phi_{[k_1k_2}\phi_{k_3k_4}\cdots\phi_{k_{2n-2q-1}k_{2n-2q}]}$$

for $p=1, \cdots, n$ and $q=1, \cdots, n-1$. And we can prove the following

THEOREM 2. *For the tensor fields ξ^i, $\xi^{i_1\cdots i_{2p}}$ and $\xi^{i_1\cdots i_{2p+1}}$ the following relation holds good.*

(2.18) $$\xi^{i_1\cdots i_{2p+1}}=(2n-2p+1)S(\xi^{i_1}\xi^{i_2\cdots i_{2p+1}}), \qquad (p=1, \cdots, n-1)$$

where S means the sum of $(2p+1)$ terms which are obtained by cyclic permutation of indices.

PROOF. It is sufficient to prove the case where $i_1, \cdots, i_{2p+1}$ are all distinct. Now, we denote the complement of the set $(i_1, \cdots, i_{2p+1})$ in the set $(1, 2, \cdots, 2n+1)$ by $a_1, \cdots, a_{2n-2p}$. Then, by definition we get

(2.17)′ $$\xi^{i_1\cdots i_{2p+1}}=\varepsilon^{i_1\cdots i_{2p+1}a_1\cdots a_{2n-2p}}\phi_{[a_1a_2}\cdots\phi_{a_{2n-2p-1}a_{2n-2p}]}.$$

By virtue of (1.1) and (1.9), this can be written also as

$$\begin{aligned}\xi^{i_1\cdots i_{2p+1}}&=(2n-2p+1)\varepsilon^{i_1\cdots i_{2p+1}a_1\cdots a_{2n-2p}}\xi^m\eta_{[m}\phi_{a_1a_2}\cdots\phi_{a_{2n-2p-1}a_{2n-2p}]}\\&=(2n-2p+1)\varepsilon^{i_1\cdots i_{2p+1}a_1\cdots a_{2n-2p}}\{\xi^{i_1}\eta_{[i_1}\phi_{a_1a_2}\cdots\phi_{a_{2n-2p-1}a_{2n-2p}]}\\&\qquad+\xi^{i_2}\eta_{[i_2}\phi_{a_1a_2}\cdots\phi_{a_{2n-2p-1}a_{2n-2p}]}\\&\qquad+\cdots\cdots\cdots\cdots\\&\qquad+\xi^{i_{2p+1}}\eta_{[i_{2p+1}}\phi_{a_1a_2}\cdots\phi_{a_{2n-2p-1}a_{2n-2p}]}\}\\&=(2n-2p+1)S(\xi^{i_1}\xi^{i_2\cdots i_{2p+1}}),\end{aligned}$$

which is to be proved.

Q. E. D.

§3. Lie derivatives of intrinsic tensor fields with respect to the vector field ξ^i.

We begin with the following

THEOREM 3. *Suppose M^{2n+1} be a manifold with contact structure η. Then,* *the relations*

$$\mathfrak{L}(\xi)\eta_j = 0\,, \tag{3.1}$$

$$\mathfrak{L}(\xi)\phi_{ij} = 0 \tag{3.2}$$

hold good, where $\mathfrak{L}(\xi)$ *means the operator of Lie derivative with respect to the vector field* ξ^i.

PROOF. First, we see easily that

$$\begin{aligned}\mathfrak{L}(\xi)\eta_j &= \xi^h\partial_h\eta_j + (\partial_j\xi^h)\eta_h \\ &= \xi^h(\partial_h\eta_j - \partial_j\eta_h) \\ &= \phi_{hj}\xi^h = 0\,.\end{aligned}$$

Next, we see that

$$\begin{aligned}\mathfrak{L}(\xi)\phi_{ij} &= \xi^k\partial_k\phi_{ij} + \phi_{kj}\partial_i\xi^k + \phi_{ik}\partial_j\xi^k \\ &= (\partial_k\phi_{ij} + \partial_i\phi_{jk} + \partial_j\phi_{ki})\xi^k\,.\end{aligned}$$

Hence, by virtue of (1.12), we get

$$\mathfrak{L}(\xi)\phi_{ij} = 0\,.$$

More generally we get the following

THEOREM 4. *Suppose* M^{2n+1} *be a manifold with contact structure* η. *Then, the Lie derivatives of all intrinsic tensor fields with respect to the vector field* ξ^i *vanish identically.*

PROOF. By definition, intrinsic tensor fields are completely determined by the contact structure η. However, by virtue of (3.1), the structure η is invariant under the transformations of M^{2n+1} which are generated by the vector field (i. e. infinitesimal transformation) ξ^i. Hence, all intrinsic tensor fields are invariant under the same transformations. This fact shows that our assertion is true.

N. B. (3.2) is also a special case of Theorem 4.

THEOREM 5. *Suppose* M^{2n+1} *be a manifold with contact structure* η. *Then, for any contact metric structure associated to it,* ξ^i *is an incompressible vector field.*

PROOF. By Theorem 4, we see that

$$\mathfrak{L}(\xi)\lambda = 0\,. \tag{3.3}$$

However, $\sqrt{g}$ is equal to $c\lambda$. Hence, we get

$$\mathfrak{L}(\xi)\sqrt{g} = 0\,. \tag{3.4}$$

On the other hand, we have

$$\mathfrak{L}(\xi)\sqrt{g} = \sqrt{g}\,\xi^i{}_{,i} \tag{3.5}$$

where $\xi^i{}_{,i}$ is the divergence of ξ^i with respect to the associated Riemannian metric g. Therefore we get

(3.6) $$\xi^i{}_{,i}=0\,.$$

Q. E. D.

As an application of (3.1), we shall prove the following

THEOREM 6. *Suppose M^{2n+1} be a differentiable manifold with contact structure η. Then, all trajectories of the associated vector field ξ^i are geodesics of any associated Riemannian metric g.*

PROOF. By virtue of (3.1) and (1.1) we see that

$$\begin{aligned}0=\mathfrak{L}(\xi)\eta_j&=(\partial_k\eta_j)\xi^k+(\partial_j\xi^k)\eta_k\\&=(\partial_k\eta_j-\partial_j\eta_k)\xi^k\\&=(\eta_{j,k}-\eta_{k,j})\xi^k\,.\end{aligned}$$

However, ξ^k is a unit vector field, we see that

(3.7) $$\eta_{k,j}\xi^k=0$$

holds good. Hence, we get

(3.8) $$\eta_{j,k}\xi^k=0$$

which shows that our assertion is true.

Q. E. D.

§ 4. The four tensors N^i_{jk}, N_{jk}, N^i_j and N_j.

In the paper [3], we have defined for any almost contact structure the following four tensors:

(4.1) $$N^i_{jk}=\phi^h_k(\partial_h\phi^i_j-\partial_j\phi^i_h)-\phi^h_j(\partial_h\phi^i_k-\partial_k\phi^i_h)+(\partial_j\xi^i)\eta_k-(\partial_k\xi^i)\eta_j\,,$$

(4.2) $$N_{jk}=\phi^h_k(\partial_j\eta_h-\partial_h\eta_j)-\phi^h_j(\partial_k\eta_h-\partial_h\eta_k)\,,$$

(4.3) $$N^i_j=\xi^h\partial_h\phi^i_j+(\partial_j\xi^h)\phi^i_h-(\partial_h\xi^i)\phi^h_j\,,$$

(4.4) $$N_j=\xi^h(\partial_j\eta_h-\partial_h\eta_j)\,,$$

and proved the following formulas:

(4.5) $$N^i_j=\mathfrak{L}(\xi)\phi^i_j\,,$$

(4.6) $$N_j=-\mathfrak{L}(\xi)\eta_j\,.$$

In the same paper, we have proved also the following Theorem 7. However, the proof was given as a corollary of some general theorems, so we shall give here a direct proof.

THEOREM 7. *Suppose M^{2n+1} be a manifold with contact structure η. Then, for any contact metric structure associated to it, the relations*

(4.7) $$N_{jk}=0\,,$$

(4.8) $$N_j=0$$

hold good.

PROOF. (4.8) follows from (3.1) and (4.6). To prove (4.7) we put (1.12) into (4.2), then we see that

$$N_{jk}=\phi_k^h\phi_{hj}-\phi_j^h\phi_{hk}=0\,.$$

Q. E. D.

For the tensor field N_j^i we can now prove the following formulas

$$N_j^i\xi^j=0\,, \tag{4.9}$$

$$N_j^i\eta_i=0\,. \tag{4.10}$$

To prove (4.10), we take the Lie derivative of (1.4) with respect to ξ^i, then we get

$$[\mathfrak{L}(\xi)\phi_j^i]\eta_i+\phi_j^i\mathfrak{L}(\xi)\eta_i=0\,.$$

Hence, by virtue of (3.1) and (4.5), we see that (4.10) is true. (4.9) can also be proved in the same way.

Theorem 7 shows that in any differentiable manifold with contact metric structure associated to a contact structure only the tensors N_{jk}^i and N_j^i among four tensors N_{jk}^i, N_j^i, N_{jk} and N_j are essential. For components of the tensors N_{jk}^i and N_j^i there exist the relations ([3], (3.4), (3.6))

$$\tag{4.11} \left\{\begin{array}{l} \phi_h^iN_{jk}^h+N_{jh}^i\phi_k^h-N_j^i\eta_k=0\,, \\ N_{jh}^i\phi_k^h-N_{hk}^i\phi_j^h-N_j^i\eta_k-N_k^i\eta_j=0\,, \\ N_{jh}^i\xi^h-N_h^i\phi_j^h=0\,, \\ N_{jh}^i\xi^h+\phi_h^iN_j^h=0\,, \\ \eta_hN_{jk}^h=0\,, \\ \phi_h^iN_j^h+N_h^i\phi_j^h=0\,. \end{array}\right.$$

THEOREM 8. *Suppose M^{2n+1} be a manifold with contact structure η. If the tensor field N_j^i of the associated contact metric structure vanishes, then*

$$\tag{4.12} \left\{\begin{array}{l} \mathfrak{L}(\xi)\phi_j^i=0\,, \\ \mathfrak{L}(\xi)(\phi_j^i+\xi^i\eta_j)=0\,. \end{array}\right.$$

PROOF. These relations follow easily from (4.5), (3.1) and the Leibnitz' rule for Lie derivation.

THEOREM 9. *Suppose M^{2n+1} be a manifold with contact structure η. If the tensor field N_j^i of the associated contact metric structure vanishes identically, then ξ^i is a Killing vector field of the Riemannian metric and vice versa.*

PROOF. By virtue of (1.8)

$$\phi_j^i=g^{ih}\phi_{hj}\,. \tag{4.13}$$

Applying Lie derivation with respect to ξ^i for both sides of the last equation, and making use of (3.2) and (4.5) we get

$$[\mathfrak{L}(\xi)g^{ih}]\phi_{hj}=0\,.$$

On the other hand, we see that

$$[\mathfrak{L}(\xi)g^{ih}]\eta_h\eta_j=[\mathfrak{L}(\xi)\xi^i]\eta_j=0$$

holds good. However, as the matrix

$$\phi_{ij}+\eta_i\eta_j=g_{ik}(\phi^k_j+\xi^k\eta_j)$$

is non-singular, we see that

$$\mathfrak{L}(\xi)g^{ih}=0\,. \tag{4.14}$$

Hence, ξ is a Killing vector field. The converse is easily seen to be true.

Q. E. D.

COROLLARY. *Under the same assumption as in Theorem* 9, *suppose* $N^i_j=0$. *Then the relation*

$$\eta_{i,j}=-\frac{1}{2}\phi_{ij} \tag{4.15}$$

holds good.

Now, let us prove $(4.11)_3$ directly.

We notice first

$$N^i_{jk}=\phi^h_k(\phi^i_{j,h}-\phi^i_{h,j})-\phi^h_j(\phi^i_{k,h}-\phi^i_{h,k})+\xi^i{}_{,j}\eta_k-\xi^i{}_{,k}\eta_j\,, \tag{4.16}$$

$$N^i_h=\xi^k(\phi^i_{h,k}-\phi^i_{k,h})-\phi^k_h\xi^i{}_{,k}\,. \tag{4.17}$$

Then, we can easily verify that both sides of $(4.11)_3$ reduce to

$$\phi^h_j(\phi^i_{h,k}-\phi^i_{k,h})\xi^k+\xi^i{}_{,j}-\xi^i{}_{,k}\xi^k\eta_j\,.$$

Hence, $(4.11)_3$ holds good.

The relation $(4.11)_3$ implies an important result:

THEOREM 10. *Suppose* M^{2n+1} *be a manifold with contact structure* η. *If the tensor field* N^i_{jk} *of the associated contact metric structure vanishes, then* N^i_j, N_{jk}, *and* N_j *vanish too.*

PROOF. The vanishing of N_{jk} and N_j is independent upon N^i_{jk} and was already proved in Theorem 7. To prove the vanishing of N^i_j we notice the relations

$$N^i_h\phi^h_j=0\,,$$

$$N^i_h\xi^h\eta_j=0$$

which follow from $(4.11)_3$ and (4.9). Then, since the matrix $\phi^h_j+\xi^h\eta_j$ is non-singular, we see that

$$N^i_h=0\,.$$

Q. E. D.

THEOREM 11. *Suppose* M^{2n+1} *be a manifold with normal contact metric structure. Then, the relation*

$$2\phi_{ij,k}=\eta_i g_{jk}-\eta_j g_{ik} \tag{4.18}$$

holds good.

PROOF. First, we put

(4.19) $$N_{ijk}=g_{ih}N^h_{jk},$$

then we get

(4.20) $$N_{ijk}=\phi^h_k(\phi_{ij,h}-\phi_{ih,j})-\phi^h_j(\phi_{ik,h}-\phi_{ih,k})+\eta_{i,j}\eta_k-\eta_{i,k}\eta_j.$$

However, as the form ϕ defined by (1.10) is closed, we have

(4.21) $$\phi_{ij,k}+\phi_{jk,i}+\phi_{ki,j}=0.$$

Hence, (4.20) can be written as

$$\begin{aligned}N_{ijk}&=\phi^h_k\phi_{hj,i}-\phi^h_j\phi_{hk,i}+\eta_{i,j}\eta_k-\eta_{i,k}\eta_j\\&=(\phi^h_k\phi_{hj})_{,i}-2\phi^h_j\phi_{hk,i}+\eta_{i,j}\eta_k-\eta_{i,k}\eta_j.\end{aligned}$$

Therefore, we get

(4.22) $$N_{ijk}=-2\phi^h_j\phi_{hk,i}-\eta_j(\eta_{i,k}+\eta_{k,i})+\eta_k(\eta_{i,j}-\eta_{j,i}).$$

Now, as $N^i_{jk}=0$ by assumption, we know by Theorems 9 and 10 that ξ^i is a Killing vector field. Hence from (4.22), we get

$$\phi^h_j\phi_{hk,i}=\eta_k\eta_{i,j}.$$

On the other hand, we have easily

$$\xi^h\eta_j\phi_{hk,i}=-\eta_j\eta_{h,i}\phi^h_k.$$

Adding these two equations sides by sides, we get

$$(\phi^h_j+\xi^h\eta_j)\phi_{hk,i}=-\eta_j\eta_{h,i}\phi^h_k+\eta_k\eta_{i,j}.$$

As $(\phi^h_j+\xi^h\eta_j)$ is non-singular, we can solve the last equation by $\phi_{hk,i}$ and get

$$\phi_{lk,i}=-\eta_l\eta_{h,i}\phi^h_k-\eta_k\eta_{i,j}\phi^j_l.$$

Putting (4.15) in the last equation, we finally get

$$\phi_{lk,i}=\frac{1}{2}(\eta_l g_{ik}-\eta_k g_{il})$$

which is to be proved.

Q. E. D.

In an almost complex manifold of class C^∞, the vanishing of the Nijenhuis' tensor is equivalent to the possibility of introducing complex structure such that its real representation is the given almost complex structure. In our case of contact structure, we saw that if N^i_{jk} vanishes, then all the tensors N^i_j, N_{jk} and N_j vanish. So to find some other properties which characterize the vanishing of N^i_{jk} geometrically in M^{2n+1} (not in $M^{2n+1}\times R$) seems to be an important question [4].

§ 5. S^{2n+1} as a manifold with normal contact metric structure.

Let E^{2n+2} be a Euclidean space with rectangular coordinates $x^A(A, B, C=1, 2, \cdots, 2n+2)$ and S^{2n+1} be a hypersphere defined by

$$\sum (x^A)^2 = 4\,. \tag{5.1}$$

We put

$$z^\alpha = x^\alpha + i x^{m+\alpha}\,, \qquad (m = n+1,\ \alpha, \beta, \gamma = 1, 2, \cdots, m)$$

then z^α's define a complex structure in E^{2n+2}. Its almost complex structure is given by the matrix

$$J = (J_B^A) = \begin{pmatrix} 0 & E_m \\ -E_m & 0 \end{pmatrix}, \tag{5.2}$$

where E_m is the unit matrix of dimension m. The Euclidean metric

$$g = (g_{AB}) = \begin{pmatrix} E_m & 0 \\ 0 & E_m \end{pmatrix} \tag{5.3}$$

is almost Hermitian with respect to the almost complex structure, that is, the relation

$${}^tJgJ = g \tag{5.4}$$

holds good. From the last relation we get

$${}^tJg = -gJ\,, \tag{5.5}$$

hence $gJ = (J_{AB})$ is skew-symmetric.

Now, suppose a point x belongs to S^{2n+1}. We may consider x as a vector from the origin O to the point x. So we may operate J to x. If we put

$$\xi^A = \frac{1}{2} J_B^A x^B\,, \tag{5.6}$$

then ξ is a unit vector orthogonal to the vector x as we may easily verify it by virtue of (5.1) and (5.4).

We put

$$\eta_A = g_{AB}\xi^B\,, \tag{5.7}$$

then the 1-form η over S^{2n+1} defined by

$$\eta = \eta_A dx^A = \frac{1}{2} \sum_\alpha (x^{m+\alpha} dx^\alpha - x^\alpha dx^{m+\alpha}) \tag{5.8}$$

is a contact form of S^{2n+1}.

Let $T_x(E^{2n+2})$ and $T_x(S^{2n+1})$ be tangent spaces of E^{2n+2} and S^{2n+1} at a point $x \in S^{2n+1}$. Then there exists the natural orthogonal projection

$$\pi:\ T_x(E^{2n+2}) \longrightarrow T_x(S^{2n+1})\,.$$

We put

$$\phi = -\pi \cdot J\,. \tag{5.9}$$

THEOREM 12. *The vectors ξ, η defined by* (5.6), (5.7) *for x restricted to S^{2n+1} and the tensors ϕ and g defined by* (5.9), (5.3) *and restricted to operate for tangent vectors of S^{2n+1} determine a contact metric structure for S^{2n+1}.*

PROOF. First we see easily that

(5.10) $$\xi^A\eta_A = \frac{1}{4}\,{}^t(Jx)g(Jx) = 1\,,$$

(5.11) $$\phi\xi = -\frac{1}{2}\pi\cdot J(Jx) = \frac{1}{2}\pi\cdot x = 0\,.$$

Next, for any vector $v \in T_x(S^{2n+1})$, we have by definition

(5.12) $$\phi v = -Jv + \frac{1}{4}({}^t xgJv)x\,.$$

Therefore, we see that

$$\eta\phi v = \frac{1}{2}\,{}^t(Jx)g\Big\{-Jv + \frac{1}{4}({}^t xgJv)\,x\Big\} = 0\,,$$

as each term vanishes by itself. So we know that

(5.13) $$\eta\phi = 0$$

is true. As

$${}^t xgJ\cdot\Big\{-Jv + \frac{1}{4}({}^t xgJv)\,x\Big\} = 0\,,$$

we see also that

$$\begin{aligned}\phi^2 v &= -J\cdot\Big\{-Jv + \frac{1}{4}({}^t xgJv)\,x\Big\}\\ &= -v + \xi(\eta v)\,.\end{aligned}$$

Hence, when we operate ϕ for vectors of $T_x(S^{2n+1})$, we get

(5.14) $$\phi^2 = -1 + \xi\eta\,.$$

Equations (5.10), (5.11), (5.13) and (5.14) show that the set of tensors ϕ, ξ and η constitutes an almost contact structure for S^{2n+1}.

To show that the metric g restricted to S^{2n+1} is an associated Riemannian metric for the almost contact structure in consideration, we first remark that

(5.15) $$\phi = -J + \frac{1}{4}x({}^t xgJ)$$

follows from (5.12). Then we get

$${}^t\phi g\phi = \Big\{-{}^tJ + \frac{1}{4}\,{}^t({}^t xgJ){}^t x\Big\}g\Big\{-J + \frac{1}{4}x({}^t xgJ)\Big\}\,.$$

By virtue of (5.4) and (5.5), this reduces to

$$\begin{aligned}{}^t\phi g\phi &= g - \frac{1}{4}({}^tJgx)({}^t xgJ)\\ &= g - \frac{1}{4}(gJx)({}^t x\,{}^tJg)\,.\end{aligned}$$

So we have

(5.16) $${}^t\phi g\phi = g - {}^t\eta\eta\,.$$

Equations (5.7) and (5.16) show that the metric g restricted to S^{2n+1} is an associated Riemannian metric for our almost contact structure.

Now, from (15.8) we see that

$$\eta=\frac{1}{2}{}^{t}(dx)gJx,$$

so we have

$$d\eta=-\frac{1}{2}{}^{t}(dx)gJ(dx).$$

On the other hand, by virtue of (5.15), we get

$${}^{t}(dx)g\phi(dx)=-{}^{t}(dx)gJ(dx)$$

as x and dx are orthogonal on S^{2n+1}. Hence, we get

$$d\eta=\frac{1}{2}{}^{t}(dx)g\phi(dx), \tag{5.17}$$

which shows that our almost contact structure is the one associated with the contact structure η.

THEOREM 13. *The contact metric structure of* S^{2n+1} *defined in Theorem* 12 *is normal.*

PROOF. We denote by R^{+} a real half line which corresponds to the interval $0<r<\infty$ of radius r. Then we may regard $S^{2n+1}\times R^{+}$ as identical with $E^{2n+2}-O$. In the same way as we have shown in [3],

$$F=\begin{pmatrix}\phi & -\xi\\ \eta & 0\end{pmatrix} \tag{5.18}$$

is an almost complex structure for $S^{2n+1}\times R^{+}$ and its Nijenhuis tensor N_{BC}^{A} is nothing but the set of tensors N_{jk}^{i}, N_{j}^{i}, N_{jk} and N_{j} of our almost contact structure. Therefore, if we can show that F is identical with the almost complex structure $-J$ of E^{2n+2}, then N_{BC}^{A} vanishes identically and hence the four tensors in consideration all vanish identically.

To prove F coincides with $-J$, we take an arbitrary vector of $T_{x}(E^{2n+2})$ at $x\in E^{2n+2}-O$. The component v_t of v orthogonal to x and the component v_n of v parallel to x are given by

$$\begin{cases} v_t=v-\dfrac{({}^{t}xgv)}{|x|^2}x, \\ v_n=\dfrac{{}^{t}xgv}{|x|} \end{cases} \tag{5.19}$$

respectively. Therefore, the components of

$$Fv=\begin{pmatrix}\phi & -\xi\\ \eta & 0\end{pmatrix}\begin{pmatrix}v_t\\ v_n\end{pmatrix} \tag{5.20}$$

orthogonal to x is given by

$$\phi v_t - \xi v_n = \left\{-J + x\frac{({}^t xgJ)}{|x|^2}\right\}\left\{v - \frac{({}^t xgv)}{|x|^2}x\right\} - \frac{Jx}{|x|}\cdot\frac{({}^t xgv)}{|x|}$$
$$= -Jv + x\cdot\frac{({}^t xgJv)}{|x|^2}$$

and the components of Fv parallel to x is given by

$$\eta v_t = \frac{{}^t(gJx)}{|x|}\left\{v - \frac{({}^t xgv)}{|x|^2}x\right\}$$
$$= \frac{{}^t x {}^t Jgv}{|x|} = -\frac{{}^t xgJv}{|x|}$$

respectively. Therefore we see that

$$Fv = (\phi v_t + \xi v_n) + \eta v_t \cdot \frac{x}{|x|}$$
$$= -Jv,$$

which shows that F coincides with $-J$ in $E^{2n+2}-O$. Hence, N^i_{jk}, N^i_j, N_{jk} and N_j vanish identically. Consequently, our contact structure is normal.

§ 6. The tensor field ϕ^{ij}.

Let us define a skew-symmetric tensor field ϕ^{ij} by

(6.1) $$\phi^{ij} = \phi^i_k g^{kj} = g^{ih}\phi_{hk}g^{kj}.$$

Apparently, ϕ^{ij} depends upon the metric g. However, we can prove that it does not depend upon g and is an intrinsic tensor field.

THEOREM 14. *Suppose ξ^{ij} be the tensor field defined by* (2.15). *Then, the relation*

(6.2) $$\phi^{ij} = \xi^{ij}$$

holds good.

PROOF. By definition, we see that

(6.3) $$\phi^{ij}\phi_{jk} = \phi^i_j\phi^j_k = -\delta^i_k + \xi^i\eta_k,$$

(6.4) $$\phi^{ij}\eta_j\eta_k = \phi^i_j\xi^j\eta_k = 0.$$

So, we have

$$\phi^{ij}(\phi_{jk} + \eta_j\eta_k) = -\delta^i_k + \xi^i\eta_k.$$

As the matrix $(\phi_{jk} + \eta_j\eta_k)$ is non-singular, we can solve the last equation with respect to ϕ^{ij}. On the other hand, $(\phi_{jk} + \eta_j\eta_k)$ and $-\delta^i_k + \xi^i\eta_k$ are both intrinsic. Hence, we can see that ϕ^{ij} is intrinsic too.

As ϕ^{ij} can be solved uniquely by (6.3) and (6.4), to prove (6.2) it is sufficient to show

(6.5) $$\xi^{ij}\phi_{jk} = -\delta^i_k + \xi^i\eta_k,$$

(6.6) $$\xi^{ij}\eta_j = 0.$$

First, from the definition (2.15), (6.6) is evident. To prove (6.5), we fix i and k and assume first that $i \neq k$. Denoting the complement of the set (i, k) in $(1, 2, \cdots, 2n+1)$ by $c_1, \cdots, c_{2n-1}$, we get

$$\begin{aligned}
\xi^{ij}\phi_{jk} = \frac{1}{(2n-1)!}\{&\varepsilon^{ic_1c_2\cdots c_{2n-1}k}\eta_{[c_2}\phi_{c_3c_4}\cdots\phi_{c_{2n-1}k]}\phi_{c_1k} \\
&+\varepsilon^{ic_2c_1c_3\cdots c_{2n-1}k}\eta_{[c_1}\phi_{c_3c_4}\cdots\phi_{c_{2n-1}k]}\phi_{c_2k} \\
&+\cdots\cdots\cdots\cdots\cdots \\
&+\varepsilon^{ic_{2n-1}c_1\cdots c_{2n-2}k}\eta_{[c_1}\phi_{c_2c_3}\cdots\phi_{c_{2n-2}k]}\phi_{c_{2n-1}k}\} \\
= \varepsilon^{ic_1c_2\cdots c_{2n-1}k}\{&\phi_{c_1k}\eta_{[c_2}\phi_{c_3c_4}\cdots\phi_{c_{2n-1}k]} \\
&-\phi_{c_2k}\eta_{[c_1}\phi_{c_3c_4}\cdots\phi_{c_{2n-1}k]} \\
&+\cdots\cdots\cdots\cdots\cdots \\
&+\phi_{c_{2n-1}k}\eta_{[c_1}\phi_{c_2c_3}\cdots\phi_{c_{2n-1}k]}\}\,.
\end{aligned}$$

However, as

$$\begin{aligned}
&\phi_{c_1k}\eta_{[c_2}\phi_{c_3c_4}\cdots\phi_{c_{2n-1}k]} \\
&\quad= \frac{2n-2}{2n-1}\phi_{c_1k}\eta_{[c_2}\phi_{c_3c_4}\cdots\phi_{c_{2n-1}|k|]} + \frac{1}{2n-1}\phi_{c_1k}\eta_k\phi_{[c_2c_3}\cdots\phi_{c_{2n-2}c_{2n-1}]}\,,
\end{aligned}$$

the last equation reduces to

$$\begin{aligned}
\xi^{ij}\phi_{jk} &= \varepsilon^{ic_1\cdots c_{2n-1}k}\{(2n-2)\phi_{[c_1|k|}\eta_{c_2}\phi_{c_3c_4}\cdots\phi_{c_{2n-1}|k|]} \\
&\qquad+\eta_k\phi_{[c_1|k|}\phi_{c_2c_3}\cdots\phi_{c_{2n-2}c_{2n-1}]}\} \\
&= \varepsilon^{ikc_1\cdots c_{2n-1}}\eta_k\phi_{[kc_1}\phi_{c_2c_3}\cdots\phi_{c_{2n-2}c_{2n-1}]} \\
&= \xi^i\eta_k\,.
\end{aligned}$$

In the second place, we assume that $i = k$. Denoting the complement of i in the set $(1, 2, \cdots, 2n+1)$ by $c_1, \cdots, c_{2n}$ and assuming that we do not sum for i, we get

$$\begin{aligned}
\sum_j \xi^{ij}\phi_{ji} &= \varepsilon^{ic_1\cdots c_{2n}}\phi_{[c_1|i|}\eta_{c_2}\phi_{c_3c_4}\cdots\phi_{c_{2n-1}c_{2n}]} \\
&= \varepsilon^{ic_1\cdots c_{2n}}\{\phi_{[c_1i}\eta_{c_2}\phi_{c_3c_4}\cdots\phi_{c_{2n-1}c_{2n}]} + \phi_{[c_1c_2}\phi_{c_3c_4}\cdots\phi_{c_{2n-1}c_{2n}]}\eta_i\} \\
&= \varepsilon^{ic_1\cdots c_{2n}}\,\mathrm{sgn}\,(c_2c_1ic_3\cdots c_{2n-1}c_{2n})\lambda + \xi^i\eta_i \\
&= -1 + \xi^j\eta_i
\end{aligned}$$

which is to be proved.

In the next place, we shall prove the following

THEOREM 15. *Suppose M^{2n+1} be a manifold with contact structure η and (ϕ, ξ, η, g) be its associated contact metric structure. Then, $\eta_{m,i}\phi^m_j$ and $\eta_{i,m}\phi^m_j$ are both symmetric tensor fields and the relation*

(6.7) $$\eta_{m,n}\phi^m_j\phi^n_k=-\eta_{k,j}$$
holds good.

PROOF. By virtue of Theorems 4 and 14, we see that

$$0=\mathfrak{L}(\xi)\phi^{ij}=\mathfrak{L}(\xi)(g^{ih}\phi_{hk}g^{kj})$$
$$=[\mathfrak{L}(\xi)g^{ih}]\phi_{hk}g^{kj}+g^{ih}\phi_{hk}\mathfrak{L}(\xi)g^{kj}.$$

However, as

$$\mathfrak{L}(\xi)g^{ih}=-\xi^i{}_{,m}g^{mh}-\xi^h{}_{,m}g^{im},$$

the last equation can be transformed to

$$(\xi^i{}_{,m}g^{mh}+\xi^h{}_{,m}g^{im})\phi_{hk}g^{kj}$$
$$+g^{ih}\phi_{hk}(\xi^k{}_{,m}g^{mj}+\xi^j{}_{,m}g^{km})=0.$$

If we lower the indices i and j, the last equation is easily seen to be equivalent to

$$(\eta_{i,m}+\eta_{m,i})\phi^m_j-(\eta_{j,m}+\eta_{m,j})\phi^m_i=0.$$

However, as $N_{ij}=0$ by Theorem 7, we get

$$(\eta_{i,m}-\eta_{m,i})\phi^m_j-(\eta_{j,m}-\eta_{m,j})\phi^m_i=0.$$

Adding or subtracting the last two equations we get

(6.8) $$\eta_{i,m}\phi^m_j=\eta_{j,m}\phi^m_i,$$

(6.9) $$\eta_{m,i}\phi^m_j=\eta_{m,j}\phi^m_i.$$

Hence, the first part of the theorem is proved.

To prove the second part, we contract ϕ^i_k to (6.9) and get

$$\eta_{m,i}\phi^m_j\phi^i_k=\eta_{m,j}(-\delta^m_k+\xi^m\eta_k).$$

From the last equation (6.7) follows immediately.

Q. E. D.

N. B. The relation (6.7) is a better relation than

(6.10) $$\phi_{mn}\phi^m_j\phi^n_k=\phi_{jk},$$

as the last one follows immediately from (6.7).

As a corollary, we get the following

THEOREM 16. *Suppose M^{2n+1} be a manifold with contact structure. Then, with respect to its associated contact metric structure we have*

(6.11) $$\phi_{ij,m}\xi^m=0,$$

(6.12) $$\phi^{ij}{}_{,m}\xi^m=0.$$

PROOF. Writing (3.2) explicitly, we have

$$\phi_{ij,m}\xi^m+\xi^m{}_{,i}\phi_{mj}+\xi^m{}_{,j}\phi_{im}=0.$$

However, the second and the third terms of the left hand side cancel to each other, as $\eta_{m,i}\phi^m_j$ is symmetric with respect to i and j. Hence, (6.11) is proved. (6.12) follows from (6.11) as the metric g_{ij} is covariant constant.

In the third place we shall prove the

THEOREM 17. *Suppose M^{2n+1} be a manifold with contact structure η and (ϕ, ξ, η, g) be its associated contact metric structure. Then, the relation*

$$\phi^{m[i}\phi^{jk]}{}_{,m} = \xi^{[i}\phi^{jk]} \tag{6.13}$$

holds good.

N. B. Apparently, the above relation involves the Riemannian metric g. However, the Christoffel's symbols cancel with each other. So, it is an intrinsic equation.

PROOF. We put

$$3P^{ijk} = \phi^{m[i}\phi^{jk]}{}_{,m} - \xi^{[i}\phi^{jk]} \tag{6.14}$$

and first show that

$$3P^{ijk}\eta_k = 0 . \tag{6.15}$$

This is easily done, because

$$\begin{aligned} 3P^{ijk}\eta_k &= -\phi^{mi}\phi^{jk}\eta_{k,m} + \phi^{mj}\phi^{ik}\eta_{k,m} - \phi^{ij} \\ &= -\phi^{mi}\phi^{jn}\phi_{mn} - \phi^{ij} \\ &= (-\delta^i_n + \xi^i\eta_n)\phi^{jn} - \phi^{ij} \\ &= 0 . \end{aligned}$$

In the next place, we shall show that

$$3P^{ijk}\phi_{i\alpha}\phi_{j\beta}\phi_{k\gamma} = 0 . \tag{6.16}$$

To show it, we notice first

$$3\xi^{[i}\phi^{jk]}\phi_{i\alpha}\phi_{j\beta}\phi_{k\gamma} = 0 . \tag{6.17}$$

Now, making use of the relation (6.12), i. e.,

$$\phi^{jk}{}_{,m}\xi^m = 0 ,$$

we see that

$$\phi^{mi}\phi^{jk}{}_{,m}\phi_{i\alpha}\phi_{j\beta}\phi_{k\gamma} = (-\delta^m_\alpha + \xi^m\eta_\alpha)\phi^{jk}{}_{,m}\phi_{j\beta}\phi_{k\gamma} = -\phi^{jk}{}_{,\alpha}\phi_{j\beta}\phi_{k\gamma} .$$

On the other hand we have

$$\begin{aligned} -\phi^{jk}{}_{,\alpha}\phi_{j\beta}\phi_{k\gamma} &= [(\xi^k{}_{,\alpha}\eta_\beta + \xi^k\eta_{\beta,\alpha}) + \phi^{jk}\phi_{j\beta,\alpha}]\phi_{k\gamma} \\ &= \eta_\beta\eta_{k,\alpha}\phi^k_\gamma + (-\delta^j_\gamma + \xi^j\eta_\gamma)\phi_{j\beta,\alpha} . \end{aligned}$$

Hence, we get

$$\phi^{mi}\phi^{jk}{}_{,m}\phi_{i\alpha}\phi_{j\beta}\phi_{k\gamma} = -\phi_{\gamma\beta,\alpha} + \eta_\beta\eta_{k,\alpha}\phi^k_\gamma - \eta_\gamma\eta_{k,\alpha}\phi^k_\beta .$$

Making use of this relation, (6.9), (6.17) and noticing that the form ϕ is closed, we can easily see that (6.16) is true.

Now, by virtue of (6.15) and (6.16), we can see that

$$3P^{ijk}(\phi_{i\alpha} + \eta_i\eta_\alpha)(\phi_{j\beta} + \eta_j\eta_\beta)(\phi_{k\gamma} + \eta_k\eta_\gamma) = 0.$$

However, as the matrix $(\phi_{i\alpha}+\eta_i\eta_\alpha)$ is non-singular, we know that

$$P^{ijk}=0\,.$$

Q. E. D.

Lowering the indices i, j and k, we get the following

COROLLARY. *Under the same assumption as Theorem 17, we have the relation*

$$\phi^m_{[i}\phi_{jk],m}=\eta_{[i}\phi_{jk]}\,. \tag{6.18}$$

Next, we shall give here two formulas which are equivalent to Gray's ones.

THEOREM 18. *Let M^{2n+1} be a differentiable manifold with contact structure. Then, we have the relations*

$$[\phi^{ik}U_k,\, V\xi^i]=\phi^{jk}V_jU_k\xi^i-V\phi^{ik}(U_j\xi^j)_{,k}\,, \tag{6.19}$$

$$[\phi^{ij}U_j,\, \phi^{ik}V_k]=-\phi^{im}(\phi^{jk}U_jV_k)_{,m} -(\phi^{jk}\xi^i+\phi^{ki}\xi^j+\phi^{ij}\xi^k)U_jV_k\,, \tag{6.20}$$

where U, V are arbitrary differentiable functions over M^{2n+1} and $U_i=\partial_iU$, $V_i=\partial_iV$.

PROOF. First we notice that for any two vector fields X and Y over M^{2n+1}, we have

$$[X,Y]^i=X^jY^i{}_{,j}-Y^jX^i{}_{,j}\,.$$

Then, we can easily see that (6.19) and (6.20) are immediate consequences of (6.12) and (6.13) respectively.

§7. The forms $\delta\eta$ and $\delta\phi$.

Suppose $*$ be a dual operator for differential forms over M^{2n+1} with (ϕ,ξ,η,g)-structure associated to a contact structure η and δ be the operator of codifferentiation, then by definition

$$\delta\eta=-*d*\eta\,.$$

However, it is known that

$$\delta\eta=-\xi^i{}_{,i}\,. \tag{7.1}$$

Hence, by virtue of Theorem 5, we get the following

THEOREM 19. *Suppose M^{2n+1} be a manifold with contact structure η. Then, with respect to the associated contact metric structure, the relation*

$$\delta\eta=0 \tag{7.2}$$

holds good.

Theorem 19 is also an immediate consequence of the following

THEOREM 20. *Suppose M^{2n+1} be a manifold with contact structure η. Then, with respect to the associated contact metric structure, the relation*

$$\delta\phi=n\eta \tag{7.3}$$

holds good.

PROOF. By definition

$$\delta\phi = * d * \phi .$$

However, it is well known that

$$\delta\phi = -\phi^{i}_{j,i} dx^{j} . \tag{7.4}$$

So, to prove (7.3), it is sufficient to show

$$\phi^{i}_{j,i} = -n\eta_{j} . \tag{7.5}$$

Now, multiplying $3\phi^{ki}$ to both sides of (6.18) and making use of the relations

$$\phi^{ki}\phi_{kj} = \delta^{i}_{j} - \xi^{i}\eta_{j} ,$$

$$\phi^{ki}\phi_{ki} = 2n ,$$

$$\phi^{ki}\phi^{m}_{i} = g^{km} - \xi^{k}\xi^{m}$$

and (6.11), we can easily transform it to (7.5). Hence, the theorem is proved.

More generally, we shall prove the following

THEOREM 21. *Suppose M^{2n+1} be a manifold with contact structure. If (ϕ, ξ, η, g) be the contact metric structure associated to it, then the relations*

$$\delta\overset{2p}{\Phi} = (n-p+1)\overset{2p-1}{\Phi} \qquad (p = 2, 3, \cdots, n), \tag{7.6}$$

$$\delta\overset{2q+1}{\Phi} = 0 \qquad (q = 1, 2, \cdots, n) \tag{7.7}$$

hold good.

PROOF. To prove (7.6), we first note that

$$\begin{aligned}
(\delta\overset{2p}{\Phi})_{i_1\cdots i_{2p-1}} &= -\overset{2p}{\Phi}_{ji_1\cdots i_{2p-1},k} g^{jk} \\
&= -(2p)!\, \phi_{[ji_1}\phi_{i_2 i_3} \cdots \phi_{i_{2p-2} i_{2p-1}],k} g^{jk} \\
&= -(2p-1)!\, \phi_{j[i_1}\phi_{i_2 i_3} \cdots \phi_{i_{2p-2} i_{2p-1}],k} g^{jk} \\
&= -(2p-1)!\, \{ \phi^{m}_{[i_1, |m|}\phi_{i_2 i_3} \cdots \phi_{i_{2p-2} i_{2p-1}]} \\
&\quad + \phi^{m}_{[i_1}\phi_{i_2 i_3, |m|} \cdots \phi_{i_{2p-2} i_{2p-1}]} \\
&\quad + \cdots\cdots\cdots\cdots\cdots \\
&\quad + \phi^{m}_{[i_1}\phi_{i_2 i_3} \cdots \phi_{i_{2p-2} i_{2p-1}, |m|]} \} .
\end{aligned}$$

Putting (7.5) and (6.11) in the right hand side of the last equation, we can easily get

$$(\delta\overset{2p}{\Phi})_{i_1\cdots i_{2p-1}} = (n-p+1)\overset{2p-1}{\Phi}_{i_1\cdots i_{2p-1}} .$$

Hence, we have

$$\delta\overset{2p}{\Phi} = (n-p+1)\overset{2p-1}{\Phi} .$$

To prove (7.7) for $q=1,2,\cdots,n-1$, we apply (7.6) for $p=q+1$. Then, we get

$$\delta\overset{2q+1}{\Phi}=\frac{1}{n-q}\delta\delta\overset{2q+2}{\Phi}=0\,.$$

The case $q=n$ is an immediate consequence of the theorem that $\eta\wedge(d\eta)^n$ is a constant multiple of the volume element of the Riemannian metric in consideration.

Q. E. D.

§ 8. Lie algebras of infinitesimal contact transformations.

Let M^{2n+1} be a differentiable manifold with contact structure η. Every diffeomorphism f of M^{2n+1} onto itself is by definition a contact transformation of M^{2n+1} if it satisfies

$$f^*\eta=\rho\eta\,, \tag{8.1}$$

where f^* is the dual map of rings of differential forms over M^{2n+1} which is induced by the diffeomorphism f and ρ is a function defined over M^{2n+1} which does not vanish at any point of M^{2n+1}. Especially, if $\rho\equiv 1$, i.e. if f satisfies

$$f^*\eta=\eta\,, \tag{8.2}$$

then f is said to be a strict contact transformation. It is clear that the following theorem is true.

THEOREM 22. *The set of all (strict) contact transformations over any differentiable manifold M^{2n+1} with contact structure η constitutes a group under the natural rule of composition.*

In order to study such group of (strict) contact transformations, we start with infinitesimal (strict) contact transformations.

A vector field X over M^{2n+1} is said to be an infinitesimal contact transformation if it satisfies

$$\mathfrak{L}(X)\eta=\sigma\eta\,, \tag{8.3}$$

where σ is a function defined over M^{2n+1}. Especially, if σ vanishes identically, i.e., if X satisfies

$$\mathfrak{L}(X)\eta=0\,, \tag{8.4}$$

X is said to be an infinitesimal strict contact transformation.

THEOREM 23. *The set of all infinitesimal (strict) contact transformations (^sL) L over M^{2n+1} constitutes a Lie algebra with respect to the usual bracket operation.*

PROOF. Suppose X and Y belong to L and satisfy

$$\mathfrak{L}(X)\eta=\rho\eta\,,\qquad \mathfrak{L}(Y)\eta=\sigma\eta\,.$$

Then, we can see easily from

$$\mathfrak{L}([X, Y])\eta = \{\mathfrak{L}(X)\mathfrak{L}(Y) - \mathfrak{L}(Y)\mathfrak{L}(X)\}\eta$$
$$= \{\mathfrak{L}(X)\sigma - \mathfrak{L}(Y)\rho\}\eta ,$$

that our assertion is true.

THEOREM 24. *Let X be an infinitesimal contact transformation over M^{2n+1}. Then there exists a function U defined over M^{2n+1} such that*

$$X^i = U\xi^i - \phi^{ik}U_k \qquad (U_k = \partial_k U). \tag{8.5}$$

Conversely, every vector field X of the form (8.5) *is an infinitesimal contact transformation. Especially, X is an infinitesimal strict contact transformation if and only if*

$$\mathfrak{L}(\xi)U \equiv \xi^k U_k = 0 . \tag{8.6}$$

PROOF. Suppose X be an infinitesimal contact transformation and X satisfies

$$\mathfrak{L}(X)\eta_i = \rho\eta_i .$$

This equation can be rewritten as

$$\eta_{i,k}X^k + X^k{}_{,i}\eta_k = \sigma\eta_i , \tag{8.7}$$

where commas denote covariant differentiations with respect to the Riemannian metric g of the associated contact metric structure of the contact structure in consideration. If we put

$$U = \eta_i X^i , \tag{8.8}$$

then (8.7) is transformed to

$$U_i + \phi_{ki}X^k = \sigma\eta_i .$$

Contracting both sides of the last equation by ϕ^i_h we get

$$\phi^i_h U_i + (-g_{hk} + \eta_h\eta_k)X^k = 0 .$$

The last equation is easily solved for X giving (8.5).

Conversely, suppose a vector field X is given by (8.5). Then, making use of the relations (1.1), (3.7), (3.8), (6.3) and (6.4) we can easily see that

$$\mathfrak{L}(X)\eta_i = (\xi^k U_k)\eta_i$$

holds good, whence the theorem follows.

Q. E. D.

We say that the function $U \equiv \eta_i X^i$ is the characteristic function of the infinitesimal contact transformation X.

THEOREM 25. *Let X and Y be infinitesimal contact transformations over M^{2n+1} and U and V be characteristic functions of X and Y respectively. Then, the characteristic function of $[X, Y]$ is given by*

$$[U, V] = \phi^{ij}U_i V_j + U\cdot V_j\xi^j - V\cdot U_j\xi^j . \tag{8.9}$$

PROOF. We put

$$X^i = U\xi^i - \phi^{ik}U_k ,$$
$$Y^i = V\xi^i - \phi^{ik}U_k$$

into

$$\eta_i[X, Y]^i = \eta_i(Y^i{}_{,j}X^j - X^i{}_{,j}Y^j) ,$$

and making use of the relations (1.1), (1.12), (3.7), (3.8), (6.3) and (6.4), we can see that

$$\eta_i[X, Y]^i = \phi^{ij}U_iV_j + U\cdot V_j\xi^j - V\cdot U_j\xi^j .$$

Q. E. D.

Now, we denote by F the set of differentiable functions defined over M^{2n+1}. Then we get the following

THEOREM 26. *If we define a bracket operation for functions of F by*

$$[U, V] = \phi^{ij}U_iV_j + U\cdot V_j\xi^j - V\cdot U_j\xi^j , \qquad (U, V\in F)$$

then F constitutes a Lie algebra with respect to it.

PROOF. As we can see easily that

$$[U, V] = -[V, U] ,$$
$$[U, V+W] = [U, V] + [U, W]$$

hold good, we only need to show that the Jacobi identity

$$[U, [V, W]] + [V, [W, U]] + [W, [U, V]] = 0$$

holds good too. To show it, we classify the terms on the left hand side of the last equation into the following four classes:

(i) The first class consists of terms which are quadratic with respect to the set of non differentiated functions U, V and W. The terms of this class cancel with each other.

(ii) The second class consists of terms which are linear with respect to the set of non differentiated functions U, V and W. The terms of this class vanish by virtue of the relation (3.8).

(iii) The third class consists of terms which contain two of U_i, V_i, W_i as factors and the covariant derivative of the remaining one as a factor. The terms of this class cancel with each other.

(iv) The fourth class consists of terms which are trilinear with respect to U_i, V_i and W_i. The terms of this class vanish by virtue of (6.13).

Now, if we denote by ${}^sF$, the set of functions whose Lie derivatives with respect to ξ^i vanish, then we get the following

COROLLARY. *If we define a bracket operation for functions of ${}^sF$ by*

$$[U, V] = \phi^{ij}U_iV_j , \qquad (U, V\in {}^sF)$$

then ${}^sF$ constitutes a Lie algebra with respect to it.

PROOF. We only need to show that $[U, V]$ belongs to ${}^sF$. By virtue of

$$\xi^i U_i = 0\,, \qquad \xi^i V_i = 0$$

and (3.8), (6.12), we can see that

$$\xi^k [U, V]_k = -U_i V_j (\phi^{hj} \xi^i{}_{,h} + \phi^{ih} \xi^j{}_{,h})\,.$$

The right hand side of the last equation vanishes by virtue of Theorem 4 and (6.12). Hence our assertion is true.

THEOREM 27. *If we define a map* $h: F \to L$ $({}^s h: {}^s F \to {}^s L)$ *by*

$$U \longrightarrow U\xi^i - \phi^{ik} U_k\,,$$

then $h({}^s h)$ *is an isomorphism of* F *onto* L $({}^s F$ *onto* ${}^s L)$.

PROOF. First, it is clear that h is a homomorphism of F onto L if we regard them merely as additive groups. So, to prove that h is a homomorphism of the Lie algebra F onto the Lie algebra L, it is sufficient to show

$$h[U, V] = [hU, hV]\,.$$

By virtue of (6.19) and (6.20), the right hand side of the last equation can be easily calculated. The coefficient of ξ^i in it is easily verified to be $[U, V]$ and the remaining terms are easily verified to be $-\phi^{im}[U, V]_m$. Hence, the right hand side is equal to $h[U, V]$. So h is a homomorphism.

Now, the kernel of the homomorphism h is equal to zero, because

$$U\xi^i - \phi^{ik} U_k \equiv 0$$

implies $U = 0$. Hence, h is an isomorphism of F onto L (and ${}^s F$ onto ${}^s L$).

Q. E. D.

COROLLARY. *Let* M^{2n+1} *be a differentiable manifold with contact structure* η. *Then the Lie algebra* L *of all infinitesimal contact transformations of* M^{2n+1} *is infinite dimensional.*

PROOF. As the Lie algebras F and L are isomorphic and dim F is infinity, so dim L is equal to infinity.

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