J. Math. Soc. Japan
Vol. 14, No. 1, 1962

Homology groups and double complexes for arbitrary fields*)

By S. A. AMITSUR

(Received April 3, 1961)
(Revised July 27, 1961)

Let F be an algebraic extension of a field C and let $(F^n)^*$ be the multiplicative group of all invertible elements of the ring $F \otimes_C \cdots \otimes_C F$($n$-times). The author has introduced in [1] a complex structure $C^*(F/C)$ on the groups $(F^n)^*$ whose homology groups $H^n(F/C)$ were shown to be a generalization of the notion of the cohomology groups $H^n(G; F^*)$ for normal fields F with Galois groups G. This has been extended and simplified by Rosenberg and Zelinsky in [5].

In the present paper we introduce homology groups $H_n(F/C)$ for arbitrary commutative rings F which are finitely generated C-free modules. These again are obtained by a complex $C_*(F/C)$ obtained by the groups $(F^n)^*$ with a derivation $\mathfrak{N} : (F^n)^* \to (F^{n-1})^*$. The map $\mathfrak{N}$ is defined with the aid of the n different norms of the elements of F^n with respect to F^{n-1}. These groups are again isomorphic with the classical homology group $H_n(G; F^*)$ for normal field F with Galois groups G.

In section 2 we carry the notions of restriction, transfer and lift to the cohomology and homology groups of arbitrary fields which again is the generalization of the respective notion of the classical case.

These notions are used to prove that if $(F:C)=k$ then the order of the elements of $H^n(F/C)$ and $H_n(F/C)$ is a divisor of k. This together with the fact that $H^2(F/C)$ is isomorphic with the Brauer groups of all C-separable simple algebras ([5]) split by F, yields the result that the exponent of the algebras split by F divides k. The special feature of this proof is that it does not depend on the existence of normal separable splitting fields but rather on F itself.

A new notion of cohomology groups for two C-algebras F, K is introduced. This is done by considering the multiplicative groups $(F^n \otimes_C K^m)^*$ as a double complex. The first two cohomology groups of this complex are zero and $H^2(F, K)$ is isomorphic with the Brauer groups of all algebra split both by F

*) Part of this research has been done while the author was a member of a Summer Conference of the University of Chicago.

and K. In case $K \supseteq F$, these groups coincide with cohomology groups $H^n(F/C)$ introduced above. These groups have generalizations to any finite number $F_1, \cdots, F_r$ of C-algebras.

The double complex is then used to prove the fundamental exact sequence ([4]) for arbitrary field. It was shown in ([5]) that our groups $H^n(F/C)$ are naturally isomorphic with Adamson's cohomology groups for separable not necessarily normal extensions F of C. Now recently, the fundamental exact sequence has been shown to hold also for these groups (Nakayama [6, 7] and in a more general form by Hattori [8])*), it seems probable that our exact sequence (Theorem 4.2) coincides with Adamson-Nakayama's exact sequence. No attempt has been done to prove this fact in this paper; nevertheless, this has been carried out only for the classical case of normal separable fields and normal subfields. Namely it is proved that the Hochschild-Serre's exact sequence ([4]) coincides with the exact sequence of Theorem 4.2 under identical conditions.

The rest of the paper is devoted to show that the basic tools for the classical homology groups exist and work as well for our groups.

1. Homology groups.

Let C be a commutative ring with a unit, and let F be a commutative C-algebra containing C as a subalgebra (both F and C have the same unit). Put $F^n = F^n_C = F \otimes_C \cdots \otimes_C F$ (n-factors) and for $n=0$ set, $F^0 = C$ [1].

Following [1] and [5] we define the homomorphism $\varepsilon_i : F^n \to F^{n+1}$ $(n \geqq 1)$ by setting:

$$\varepsilon_i(a_1 \otimes \cdots \otimes a_n) = a_1 \otimes \cdots \otimes a_{i-1} \otimes 1 \otimes a_i \otimes \cdots \otimes a_n .$$

These homomorphisms satisfy the relation:

$$\varepsilon_i \varepsilon_j = \varepsilon_{j+1} \varepsilon_i \quad \text{for} \quad i \leqq j . \tag{1.1}$$

If R^* denotes the multiplicative groups of all invertible elements of a ring R, then the set of all groups $(F^n)^*$ form a cochain complex:

$$F^* \longrightarrow (F^2)^* \longrightarrow \cdots \longrightarrow (F^n)^* \longrightarrow (F^{n+1})^* \longrightarrow \cdots \tag{1.2}$$

with respect to the derivation $\Delta = \Delta^n = \sum_{i=1}^{n+1} (-1)^{i-1} \varepsilon_i$ (written additively) and to which we add the augmentation $\varepsilon : C^* \to F^*$ which is the injection of C^* in F^*. Thus:

$$\Delta(a) = [\varepsilon_1(a)\varepsilon_3(a) \cdots][\varepsilon_2(a)\varepsilon_4(a) \cdots]^{-1} . \tag{1.3}$$

*) The author is thankful to the referee for the remark.

1) All tensor product, henceforth, without a subscript will be with respect to C unless stated otherwise.

We denote this complex by $C^*(F/C)$ and its cohomology groups by $H^n(F/C)$ and $H^*(F/C)=\sum H^n(F/C)$.

To define homology groups we use a similar procedure with the help of the Norm:

Let K be an arbitrary commutative ring and A be a finitely generated free K-algebra; the norm "Norm $(A/K; a)$" for $a\in A$ is defined to be determinant of the endomorphism: $a_R: x\to ax$ of A, when considered as a K-free module (e.g. [**2**, p. 133]). It is known (*ibid*) that if $PC(a;\lambda)=\lambda^n+c_{n-1}\lambda^{n-1}+\cdots+c_0$ is the characteristic polynomial of the endomorphism a_R then Norm $(A/K; a)=(-1)^n c_0$.

We assume, henceforth, that F is a finitely generated free C-module as well as a commutative ring. Thus, F^n is also a free $\varepsilon_i F^{n-1}$-module, so for all $x\in F^n$ we set

$$\nu_i(x)=\varepsilon_i^{-1}\operatorname{Norm}(F^n/\varepsilon_i F^{n-1}; x), \tag{1.4}$$

and we obtain a homomorphism $\nu_i: (F^n)^*\to(F^{n-1})^*$. (For $\nu_1: (F^1)^*\to(F^0)^*$, we also write ν.) Finally let $\mathfrak{N}: (F^n)^*\to(F^{n-1})^*$ be the homomorphism given by:

$$\mathfrak{N}(x)=[\nu_1(x)\nu_3(x)\cdots][\nu_2(x)\nu_4(x)\cdots]^{-1}. \tag{1.5}$$

That is: $\mathfrak{N}=\mathfrak{N}^n=\sum(-1)^{i-1}\nu_i$ (writing it additively), and note that $\mathfrak{N}^1: F^*\to C^*$ is the ordinary Norm$(F/C; *)$ map.

This procedure leads to a chain complex:

$$C^*=F^{0*}\longleftarrow F^*\longleftarrow\cdots\longleftarrow(F^{n-1})^*\longleftarrow\cdots \tag{1.6}$$

which we shall denote by $C_*(F/C)$. To this we add the augmentation $\mathfrak{N}^0: C^*\to 1$. It will be shown that $C_*(F/C)$ is a chain complex, i.e. $\mathfrak{N}^2=0$ and we then call its homology groups——*the homology groups of the extension F over C*—— and denote it by $H_n(F/C)$ and $H_*(F/C)=\sum H_n(F/C)$.

We remark that $H_0(F/C)=C^*/NF^*$ where NF^* is the subgroup of C^* containing all elements which are Norms of elements of F.

Furthermore, we can connect the complexes $C^*(F/C)$ and $C_*(F/C)$ to one complex $C(F/C)$ by setting: $F_n=(F^*)^{n+1}$ for $n\geqq 0$ and $F_{-n}=(F^*)^n$ for $-n<0$, then we get a complex

$$\longrightarrow F_{-n}\longrightarrow F_{-(n-1)}\longrightarrow\cdots\longrightarrow F_{-1}\longrightarrow F_0\longrightarrow F_2\longrightarrow\cdots\longrightarrow F_n\longrightarrow\cdots$$

with the derivation $d^n=\Delta: F_n\to F_{n+1}$ for $n\geqq 0$, and $d^n=\mathfrak{N}: F_n\to F_{n+1}$ for $n<0$ and $d^0(x)=\varepsilon\mathfrak{N}^0(x)$, i.e. the Norm $(F/C, *)$ followed by the injection of C^* into F^*.

The procedure is the same adopted for finite groups ([3]) as we shall note later for the case of finite normal Galois extensions.

To prove that $C_*(F/C)$ is a complex we need the following lemma:

LEMMA 1.1. *The $n+1$ homomorphisms $\nu_i: (F^{n+1})^*\to(F^n)^*$ satisfy the relations:*

$$\nu_i\nu_j = \nu_j\nu_{i+1} \quad for \quad i \geqq j. \tag{1.7}$$

Indeed, using the transitivity property of the Norm (e.g. [2, p. 142]); we get:

$$\begin{aligned}\nu_i\nu_j(x) &= \varepsilon_i^{-1}[\mathrm{Norm}\,(F^n/\varepsilon_i F^{n-1};\ \varepsilon_j^{-1}\,\mathrm{Norm}\,(F^{n+1}/\varepsilon_j F^n;\ x)] \\ &= \varepsilon_i^{-1}\varepsilon_j^{-1}\,\mathrm{Norm}\,(F^{n+1}/\varepsilon_j\varepsilon_i F^{n-1};\ x)\end{aligned}$$

and the rest follows from the relation among the ε_i given in (1.1).

To complete the proof that $C_*(F/C)$ is a complex we observe that: (writing the homomorphisms involved additively):

$$\begin{aligned}\mathfrak{N}^2 &= \sum(-1)^i\nu_i(-1)^j\nu_j = \sum_{i<j}(-1)^{i+j}\nu_i\nu_j + \sum_{i\geqq j}(-1)^{i+j}\nu_i\nu_j \\ &= \sum_{i<j}(-1)^{i+j}\nu_i\nu_j + \sum_{i\geqq j}(-1)^{i+j}\nu_j\nu_{i+1} = 0\end{aligned}$$

by replacing in the last sum $i+1$, j by j and i respectively. For $n=1$, we get $\mathfrak{N}^2 = \nu\nu_1 - \nu\nu_2 = 0$, since

$$\nu\nu_1(x) = \mathrm{Norm}\,[F/C;\ \varepsilon_1^{-1}\,\mathrm{Norm}\,(F^2/\varepsilon_1 F;\ x)] = \mathrm{Norm}\,(F^2/C;\ x) = \nu\nu_2(x).$$

The identification of these groups with the classical homology groups of normal fields will be dealt with in the last section.

We begin with introducing the idea of "restriction ' and "transfer" in the (co-) homology group for arbitrary F which will correspond to their classical counterpart.

2. Restriction, transfer and splitting of cycles.

If F is not C-free or not finitely generated then the results to follow will still hold, but only for the complex $C^*(F/C)$ and for the cohomology groups $H^*(F/C)$.

In addition to F and C dealt with above, we consider an arbitrary commutative ring R which is a C-algebra, and denote by $C(F/C)\otimes R$ ($C^*(F/C)\otimes R$, $C_*(F/C)\otimes R$) the complex

$$\longrightarrow (F^n\otimes_C R)^* \longrightarrow (F^{n+1}\otimes_C R)^* \longrightarrow \cdots$$

with the derivation $\Delta_F = \sum(-1)^{i-1}(\varepsilon_i\otimes 1)$ (written additively) and $\mathfrak{N}_F = \sum(-1)^i\nu_{Fi}$ with $\nu_{Fi} = (\varepsilon_i\otimes 1)^{-1}\ \mathrm{Norm}\,(F^{n+1}\otimes R/(\varepsilon_i\otimes 1)(F^n\otimes R);\ *)$. Clearly $C(F/C)\otimes R$ is again a complex and one readily establishes the isomorphism:

Theorem 2.1. $C(F/C)\otimes R \cong C(F\otimes R/R)$[2].

Indeed, $C(F\otimes R/R)$ consists of the groups and the morphisms: $[(F\otimes R)_R^n]^* \to [(F\otimes R)_R^{n+1}]^*$. The isomorphism of the theorem is given by the map:

$$(a_1\otimes\cdots\otimes a_n)\otimes r \longrightarrow [a_1\otimes 1]\otimes_R\cdots\otimes_R[a_n\otimes r] \quad \text{for} \quad a_i\in F,\ r\in R$$

whose inverse is given by:

2) Similar results hold for $C_*(F/C)\otimes R$ and $C^*(F/C)\otimes R$.

$$[a_1\otimes r_1]\otimes_R\cdots\otimes_R[a_n\otimes r_n]\longrightarrow(a_1\otimes\cdots\otimes a_n)\otimes(r_1r_2\cdots r_n).$$

It follows readily that these maps commutes with the ε_i and consequently, that they form a complex isomorphism.

Now let $\rho=\rho^n: F^n\to F^n\otimes R$ be the isomorphism given by $\rho(x)=x\otimes 1$. As ρ will be shown to be a complex isomorphism of $C(F/C)$ into $C(F\otimes R/R)$ one obtains:

THEOREM 2.2. *ρ induces a morphism $\rho^*: H(F/C)\to H(F\otimes R/R)$.*

The proof for the cohomology groups is evident since $\rho\varepsilon_i=(\varepsilon_i\otimes 1)\rho$. For the homology case, let $a_1,\cdots,a_m$ be a C-base of F then (for $i=1$) $a_1\otimes 1,\cdots,a_m\otimes 1$ will be ε_1F^{n-1}-base of F^n as well as an $(\varepsilon_1\otimes 1)(F^{n-1}\otimes R)$-base of $F^n\otimes R$. Consequently, one readily observes that for $a\in F^n$, one gets $(\det a_R)\otimes 1=\det[(a\otimes 1)_R]$ in $\varepsilon_1F^{n-1}\otimes R$, where a_R is the endomorphism: $x\to ax$ in F^n and $(a\otimes 1)_R: x\to(a\otimes 1)x$ in $F^n\otimes R$, from which we conclude, in view of definition of Norm $(F^n/\varepsilon_1F^{n-1};*)$ and Norm $(F^n\otimes R/(\varepsilon_1\otimes 1)(F^{n-1}\otimes R);*)$, that ρ commutes with the respective ν_i's and, therefore, it is a complex morphism.

DEFINITION 2.1. We shall refer to ρ^* as the *restriction* homomorphism; and a cycle $a\in H(F/C)$ will be said to be *split* by R if $\rho^*(a)=1$ in $H(F\otimes R/R)$.

It will be shown that the restriction map defined above, though not exactly the known restriction homomorphism for the cases where these cohomology groups coincide with the classical ones, nevertheless, it seems that this definition is more appropriate and its relation with the classical definition will be pointed out later.

To justify the definition of the splitting of cycles we prove the following:

A homomorphism $\tau: H^2(F/C)\to\mathscr{B}(F/C)$ was defined in [**5**], where $\mathscr{B}(F/C)$ is the Brauer group of all equivalent central separable C-algebras. Now the correspondence $\mathfrak{A}\to\mathfrak{A}\otimes R$ defines a homomorphism $\sigma:\mathscr{B}(F/C)\to\mathscr{B}(F\otimes R/R)$, and we wish to show that:

THEOREM 2.3. *If R is C-free then $\sigma\tau=\tau\rho^*$.*

Indeed, recall the definition of τ:

Let η_i $(i=2,3)$ be the two possible maps of $\mathrm{End}_{1\otimes F}(F\otimes F)\to\mathrm{End}_{1\otimes F^2}(F\otimes F\otimes F)$[3], and for $t\in F^3$ let $L(t)\in\mathrm{End}_{1\otimes F^2}(F\otimes F\otimes F)$ be the endomorphism defined by multiplication by t. Consider the algebra $\mathfrak{A}(t)=\{a\in\mathrm{End}_{1\otimes F}(F\otimes F);\ L(t)\eta_2(a)L(t)^{-1}=\eta_3(a)\}$, then τ is given by: $\tau(\bar t)=\mathfrak{A}(t)$, where $\bar t\in H^2(F/C)$ is the class of cocycles determined by t.

Now we get $\sigma\tau(t)=\mathfrak{A}(t)\otimes R=\tau(\rho^*t)$, as we show that

$$\mathfrak{A}(\rho t)=\{a\in\mathrm{End}_{1\otimes F\otimes R}(F\otimes F\otimes R),\ L(\rho t)\eta_2(a)L(\rho t)^{-1}=\eta_3(a)\}\cong\mathfrak{A}(t)\otimes R.$$

Indeed, first note that $\rho: F^n\to F^n\otimes R$ is actually a $1\otimes F^{n-1}$-linear injection and clearly it induces an injection $\rho:\mathrm{End}_{1\otimes F^{n-1}}(F\otimes F^{n-1})\to\mathrm{End}_{1\otimes F^{n-1}\otimes R}(F\otimes F^{n-1}\otimes R)=[\mathrm{End}_{1\otimes F^{n-1}}(F\otimes F^{n-1})]\otimes R$; namely, $\rho(a)=a\otimes 1$. It is now not

3) $\mathrm{End}_R(V)$ denotes the ring of all R-endomorphism of an R-module V.

difficult to show that $\rho\eta_i=\eta_i\rho$. Consequently, $\mathfrak{A}(\rho t)\supseteqq\rho\mathfrak{A}(t)\cong\mathfrak{A}(t)$ since for $a\in\mathfrak{A}(t)$, $L(\rho t)\eta_2(\rho a)L(\rho t)^{-1}=\rho[L(t)\eta_2(a)L(t)^{-1}]=\rho\eta_3(a)=\eta_3(\rho a)$.

It remains now to show that $\rho\mathfrak{A}(t)$ and R are in tensor product relation in $\mathfrak{A}(\rho t)$ (cf. [**5**, p. 336]). Now $\rho\mathfrak{A}(t)\subseteqq \mathrm{End}_{1\otimes F}(F\otimes F)\otimes 1$ Hence, $\rho\mathfrak{A}(t)\otimes R\subseteqq \mathrm{End}_{1\otimes F}(F\otimes F)\otimes R$ which yields that $\rho\mathfrak{A}(t)$ and R are in tensor product relation in $\mathrm{End}_{1\otimes F\otimes R}(F^2\otimes R)$ and clearly in $\mathfrak{A}(\rho t)$. To conclude the proof we show that $\rho\mathfrak{A}(t)\otimes R=\mathfrak{A}(\rho t)$: Indeed, let $a\in\mathfrak{A}(\rho t)\subseteqq \mathrm{End}_{1\otimes F}(F\otimes F)\otimes R$, so $a=\sum a_i\otimes r_i$ for a C-base $\{r_i\}$ of R, with $a_i\in \mathrm{End}_{1\otimes F}(F\otimes F)$. Since $a\in\mathfrak{A}(\rho t)$, we have $\eta_3(a)=L(\rho t)\eta_2(a)L(\rho t)^{-1}=L(\rho t)(\sum\eta_2(a_i)\otimes r_i)L(\rho t)^{-1}=\sum L(t)\eta_2(a_i)L(t)^{-1}\otimes r_i=\sum\eta_3(a_i)\otimes r_i$ Consequently, $L(t)\eta_2(a_i)L(t)^{-1}=\eta_3(a_i)$ i.e. $a_i\in\mathfrak{A}(t)$.

REMARK. If R is not C-free we know only that $\mathfrak{A}(t)R\subseteqq\mathfrak{A}(\rho t)$.

An immediate consequence of Theorems 2.3 is the fact that a cycle $t\in H^2(F/C)$ is split by R if and only if the corresponding algebra $\mathfrak{A}(t)$ is split by R, which is the justification for the notion of splitting a cocycle.

Next we deal with the *transfer*: To this end we consider a finitely generated free C-algebra R and a homomorphism $\tau^*: H(F\otimes R/R)\to H(F/C)$ which replaces the transfer homomorphism.

Let $\tau^n: F^n\otimes R\to F^n$ be given by $\tau^n(x)=\mathrm{Norm}\,(F^n\otimes R/F^n\,;\,x)$.

THEOREM 2.4. *$\tau: C(F\otimes R/R)\to C(F/C)$ is a complex isomorphism and thus induces a homomorphism τ^* $H(F\otimes R/R)\to H(F/C)$.*

PROOF. For arbitrary F, τ commutes with ε_i: Indeed, let $r_1,\cdots,r_m$ be a C-base of R, and let $x\in F^n\otimes R$ then since $\{1\otimes r_i\}$ is an F^n-base of $F^n\otimes R$, it follows that $x(1\otimes r_i)=\sum t_{ik}\otimes r_k$ and $\tau(x)=\det(t_{ik})$. Now $\varepsilon_j(1\otimes r_i)=1\otimes r_i$ and ε_j is an isomorphism; hence we get $(\varepsilon_j x)(1\otimes r_i)=\sum\varepsilon_j t_{ik}\otimes r_k$ so that $\tau(\varepsilon_j x)=\det(\varepsilon_j t_{ik})=\varepsilon_j\det(t_{ik})=\varepsilon_j\tau(x)$. q.e.d.

This leads to the fact that $\Delta\tau=\tau\Delta$. To prove the relation $\mathfrak{N}\tau=\tau\mathfrak{N}$ we show that τ commutes with each ν_i. Indeed, by the transitivity of the Norm one gets,

$$\begin{aligned}\tau\nu_i(x)&=\mathrm{Norm}\,[F^n\otimes R/F^n\,;\,(\varepsilon_i\otimes 1)^{-1}\,\mathrm{Norm}\,(F^{n+1}\otimes R/(\varepsilon_i\otimes 1)(F^n\otimes R)\,;\,x)]\\&=\varepsilon_i^{-1}\,\mathrm{Norm}\,[F^{n+1}\otimes R/\varepsilon_i F^n\,;\,x]\\&=\varepsilon_i^{-1}\,\mathrm{Norm}\,[F^{n+1}/\varepsilon_i F^n\,;\,\mathrm{Norm}\,(F^{n+1}\otimes R/F^{n+1}\,;\,x)]=\nu_i\tau(x)\,,\end{aligned}$$

from which the rest of theorem follows.

DEFINITION 2.2. The map τ^* will be called the *transfer* map.

An immediate consequence of the definition of τ^* and ρ^* is the fact that:

COROLLARY 2.5. *$\tau^*\rho^*=$ dimension of R over C.*

Comparing these definitions with classical definitions of restriction and transfer as will be given in Section 4, one would wish that for a subalgebra $R\subset F$ the restriction should be a map: $H(F/C)\to H(F/R)$ and the transfer a map $H(F/R)\to H(F/C)$. The first can be easily achieved whereas the latter can be obtained only in special cases which cover the known ones.[4)]

To deal with the first case we start from a subalgebra $R \subseteqq F$, and here let $\mu: F \otimes R \to F$ be the homomorphism: $\mu(a \otimes r) = ar$. Then μ clearly induces a homomorphism $\mu = \mu^n : (F \otimes R)^n_R \to F^n_R$ and we have:

THEOREM 2.6. *μ is a complex homorphism: $C(F \otimes R/R) \to C(F/R)$ and thus induces a homomorphism $\mu^* : H(F \otimes R/R) \to H(F/R)$.*

PROOF. Clearly μ commutes with the ε_i, hence it commutes with the derivation Δ. To prove that it commutes with ν_i and therefore, with $\mathfrak{N}$- we note first that for $a \in (F \otimes R)^{n+1}_R)^*$

$$\nu_j(a) = \varepsilon_j^{-1} \operatorname{Norm}[(F \otimes R)^{n+1}_R / \varepsilon_j (F \otimes R)^n ; a] = \varepsilon_i^{-1} \det(u_{ik})$$

with $u_{ik} \in \varepsilon_i (F \otimes R)^n$ i.e. $u_{ik} = \varepsilon_j v_{ik}$ so that $\nu_i(x) = \det(v_{ik})$. Now μ is a homomorphism, hence $\mu\nu_j(a) = \det(\mu v_{ik}) = \nu_j \mu(a)$. Since $\det(u_{ik})$ was the determinant of the endomorphism $a_R : x \to ax$, but $\mu(a)_R : \mu(x) \to \mu(a)\mu(x)$ and as $\mu((F \otimes R^{n+1})) = F^{n+1}$ it follows that the determinant of $\mu(a)_R$ is $\det(\mu v_{ik})$ as required.

One readily verifies that the composite map $\sigma^* = \mu^* \rho^* : H(F/C) \to H(F/R)$ is simply the induced morphism of the map $\sigma(a_1 \otimes_C \cdots \otimes_C a_n) = a_1 \otimes_R \cdots \otimes_R a_n$.

We do obtain a similar result for the transfer, only if we restrict ourself to cohomology groups and even then only under certain conditions:

Let $\psi, \varphi : F \to K$ be a C-homomorphism of F into a commutative C-algebra K preserving the unit: Let $\Phi = \varphi \otimes \cdots \otimes \varphi$, $\Psi = \psi \otimes \cdots \otimes \psi$ the complex homomorphism: $C^*(F/C) \to C^*(K/C)$ given by:

$$\Phi(a_1 \otimes \cdots \otimes a_n) = \varphi(a_1) \otimes \cdots \otimes \varphi(a_n) \quad \text{and} \quad \Psi(a_1 \otimes \cdots \otimes a_n) = \psi(a_1) \otimes \cdots \otimes \psi(a_n)$$

induced by the injections $F \to K$. Then:

LEMMA 2.7. *Φ and Ψ are homotopic and thus $\Phi^* = \Psi^* : H(F/C) \to H(K/C)$.*

PROOF. Define $u_i : (F^n)^* \to (K^{n-1})^*$, $i = 1, 2, \cdots n$, by setting $u_i(a_1 \otimes \cdots \otimes a_n) = \varphi(a_1) \otimes \cdots \otimes \varphi(a_{i-1}) \otimes \varphi(a_i)\psi(a_{i+1}) \otimes \psi(a_{i+2}) \otimes \cdots \otimes \psi(a_n)$.

Writing the morphism additively we define

$$u = u^n = \sum_{i=1}^{n-1} (-1)^i u_i$$

and we add a homomorphism $u^0 : F^* \to C^*$ by setting $u^*(a) = 1$.

Considering the complex $C^*(F/C)$ and $C^*(K/C)$ we show that

$$\Psi - \Phi = u\Delta + \Delta u\,.$$

Indeed, first one verifies that

$$u_i \varepsilon_j = \begin{cases} \varepsilon_{j-1} u_i & \text{for} \quad i < j-1 \\ \sigma_j & \text{for} \quad i = j,\ j-1 \\ \varepsilon_j u_{i-1} & \text{for} \quad i > j \end{cases}$$

4) Noting that F is C-free.

where $\sigma_i(a_1\otimes\cdots\otimes a_n)=\varphi(a_1)\otimes\cdots\otimes\varphi(a_{i-1})\otimes\psi(a_i)\otimes\cdots\otimes\psi(a_n)$. In particular $\rho_1=\Psi$ and $\rho_{n+1}=\Phi$.

Thus for $n\geqq 1$:

$$\begin{aligned} u\Delta &= \sum_{i=1}^{n}\sum_{j=1}^{n+1}(-1)^{i+j}u_i\varepsilon_j \sum_{i<j-1}+\sum_{i>j}+\sum_{i=1}^{n}u_i\varepsilon_i-\sum_{i=1}^{n}u_j\varepsilon_{i+1} \\ &= \sum_{i<j-1}(-1)^{i+j}\varepsilon_{j-1}u_i+\sum_{i>j}(-1)^{i+j}\varepsilon_j u_{i-1}+\sigma_1-\sigma_{n+1}=\Delta u+\Psi-\Phi\,. \end{aligned}$$

For $n=0$, we have only $u\Delta=u_1(\varepsilon_1-\varepsilon_2)=\sigma_1-\sigma_2=\Psi-\Phi$. q. e. d.

REMARK. This last result enables us to define $H^n(F/C)$ for infinite extensions of C as $\varinjlim H^n(K/C)$ where K ranges over all C-finitely generated C-subalgebras K of F, since for $K\subset K'$ the map $H^n(K/C)\to H^n(K'/C)$ does not depend on the embedding of K in K'.

Our extension for the transfer can be stated as follows:

THEOREM 2.8. *Let $R\subseteqq F$, if there exists a homomorphism $\vartheta: F\to F\otimes R$ such that $\vartheta(r)=1\otimes r$ for $r\in R$, then $H^*(F/R)\cong H^*(F\otimes R/R)$.*

REMARK. In this case, we shall refer to the composite map $H^*(F/R)\to H^*(F\otimes R/R)\to H^*(F/C)$ as the transfer map.

PROOF. Consider the two morphisms $\mu\vartheta: F\to F$, and $\vartheta\mu: F\otimes R\to F\otimes R$. The first is an R-homomorphism, and the latter is an $1\otimes R$-homomorphism. Indeed, $(\mu\vartheta)(r)=\mu(1\otimes r)=r$ and $\vartheta\mu(1\otimes r)=\vartheta(r)=1\otimes r$, by the property of ϑ. Now, both induce homomorphism $(\mu\vartheta)^*: H^*(F/R)\to H^*(F/R)$ and $[\vartheta\mu]^*$: $H^*(F\otimes R/R)\to H^*(F\otimes R/R)\to H^*(F\otimes R/R)$. But by the previous theorem it follows that $\mu^*\vartheta^*=(\mu\vartheta)^*=$ identity and similarly $\vartheta^*\mu^*=[\vartheta\mu]^*=$ identity. Consequently, $\mu^*: H^*(F\otimes R/R)\to H^*(F/R)$ is an isomorphism, as required.

An example where this situation exists will be given in the last section.

The justification of the "splitting" is given in the following result which may be simply stated that the cocycles $H^n(F/C)$ are split by all $R\supseteqq F$. More precisely.

THEOREM 2.9. *If $F\subseteqq R$ then $H^*(F\otimes R/R)=0$.*

PROOF. Let $u: F^{n+1}\otimes R\to F^n\otimes R$ be given by: $u(a_1\otimes\cdots\otimes a_{n+1}\otimes r)=a_1\otimes\cdots\otimes a_n\otimes a_{n+1}r$ which is well defined since $a_{n+1}r\in R$.

One readily verifies that $u\varepsilon_i=\varepsilon_i u$ for $i=1,\cdots,n$, but $u\varepsilon_{n+1}=$ identity. Thus,

$$u\Delta=\sum_{i=1}^{n+1}(-1)^i u\varepsilon_i=\sum_{i=1}^{n}(-1)^i\varepsilon_i u+(-1)^{n+1}u\varepsilon_{n+1}=\Delta u+(-1)^{n+1}1$$

from which one readily shows that the identity of $C^*(F\otimes R/R)$ is homotopic with zero.

The "lift" map is quite evident: Let $C\subseteqq F\subset K$, then the injection $\lambda: F\to K$ yields a complex homomorphism λ: $C^*(F/C)\to C^*(K/C)$. Namely λ: $(F^n)^*\to(K^n)^*$ and one readily verifies that $\varepsilon_i\lambda=\lambda\varepsilon_i$. Consequently λ induces a homomorphism $\lambda^*: H^*(F/C)\to H^*(K/C)$.

DEFINITION 2.3. λ^* will be called the *lift* morphism.

We note that Lemma 2.7 implies that λ^* depends only on F and K and not on the different embeddings of F and K. Furthermore the lift map is defined only for the cohomology groups.

We conclude with the result that:

THEOREM 2.10. *If F is C-free and of dimension m then the groups $H(F/C)$ are torsion groups consisting of elements of orders dividing m.*

PROOF. Consider the isomorphism $\delta:(F^n)^*\to(F^n)^*$: (written additively)

$$\delta=\Delta\nu_n-\nu_{n+1}\Delta=\sum_{i=1}^{n}(-1)^i\varepsilon_i\nu_n-\sum_{i=1}^{n+1}(-1)^i\nu_{n+1}\varepsilon_i\,.$$

Now to compute $\nu_{n+1}\varepsilon_i$ we choose a base $c_1,\cdots,c_m$ of F over C and denote by $c_\lambda^{(n+1)}=1\otimes\cdots\otimes c_\lambda\in F^{n+1}$. Thus, $\nu_{n+1}\varepsilon_i(x)=\varepsilon_{n+1}^{-1}\operatorname{Norm}(F^{n+1}/\varepsilon_{n+1}F^n\,;\,\varepsilon_i x)=\det(\alpha'_{\lambda\rho})$ where $\varepsilon_i(x)c_\lambda^{(n+1)}=\sum_\rho(\varepsilon_{n+1}\alpha'_{\lambda\rho})c_\rho^{(n+1)}$.

For $i\leqq n$, $c_\lambda^{(n+1)}=\varepsilon_i c_\lambda^{(n)}$, so to compute $\alpha'_{\lambda\rho}$ we can start with $xc_\lambda^{(n)}=\sum(\varepsilon_n\alpha_{\lambda\rho})c_\rho^{(n)}$ and applying ε_i on both sides and we get: $(\varepsilon_i x)c_\lambda^{n+1}=\sum(\varepsilon_i\varepsilon_n)(\alpha_{\lambda\rho})c_\rho^{(n+1)}=\sum(\varepsilon_{n+1}\varepsilon_i\alpha_{\lambda\rho})c_\rho^{(n+1)}$ so that $\varepsilon_{n+1}\alpha'_{\lambda\rho}=\varepsilon_{n+1}\varepsilon_i\alpha_{\lambda\rho}$ and, therefore

$$\nu_{n+1}\varepsilon_i(x)=\det(\varepsilon_i\alpha_{\lambda\rho})=\varepsilon_i\det(\alpha_{\lambda\rho})=\varepsilon_i\nu_n(x)$$

which yields: $\Delta\nu_n-\nu_{n+1}\Delta=(-1)^{n+1}\nu_{n+1}\varepsilon_{n+1}=(-1)^{n+1}m$. For, $\nu_{n+1}\varepsilon_{n+1}(x)=x^m$ from which one readily proves that the elements of $H^*(F/C)$ satisfy $\bar{x}^m=1$.

Next consider the isomorphism $d:(F^n)^*\to(F^n)^*$ (written additively):

$$d=\mathfrak{N}\varepsilon_{n+1}-\varepsilon_n\mathfrak{N}=\sum_{i=1}^{n+1}(-1)^i\nu_i\varepsilon_{n+1}-\sum_{i=1}^{n}(-1)^i\varepsilon_n\nu_i\,.$$

To compute $\varepsilon_n\nu_i(x)$ for $x\in F^n$ we choose a C-base $\{c_i\}$ of F and denote $c_\lambda^{(i)}=1\otimes\cdots\otimes c_\lambda\otimes 1\otimes\cdots\otimes 1$ where c_λ stands in the i-th place. Then $\nu_i(x)=\det(\alpha_{\lambda\rho})$ where $xc_\lambda^{(i)}=\sum\varepsilon_i\alpha_{\lambda\rho}c_\rho^{(i)}$.

Apply ε_n, and we get $(\varepsilon_{n+1}x)(\varepsilon_{n+1}c_\lambda^{(i)})=\sum(\varepsilon_{n+1}\varepsilon_i\alpha_{\lambda\rho})(\varepsilon_{n+1}\rho)$. But for $i=1,2\cdots,n$ $\varepsilon_{n+1}c_\lambda^{(i)}=c_\lambda^{(i)}$ and $\varepsilon_{n+1}\varepsilon_i=\varepsilon_i\varepsilon_n$. So that $\nu_i(\varepsilon_{n+1}x)=\det(\varepsilon_n\alpha_{\lambda\rho})=\varepsilon_n\det(\alpha_{\lambda\rho})=\varepsilon_n\nu_i(x)$. Thus $\mathfrak{N}\varepsilon_{n+1}=\varepsilon_n\mathfrak{N}=(-1)^{n+1}\nu_{n+1}\varepsilon_{n+1}=(-1)^{n+1}m$ since $\nu_{n+1}\varepsilon_{n+1}(x)=x^m$ for $x\in F^n$, and the proof is concluded.

Since $H^2(F/C)\cong B(F/C)$ where $B(F/C)$ is the Brauer group of F over C, we have:

COROLLARY. *The index of the central separable algebras split by F divide the dimension of F over C.*

We would like to point out that the present proof of the finiteness of the index of simple algebras does not depend on the existence of separable splitting fields as the classical proof does.

3. Double complexes.

Let F and K be two C-algebras. We define a double complex $\{A^{m,n}=(F^m\otimes K^n)^*\}$ $m,n\geqq 0$, and its two derivations: $\Delta_F: A^{m,n}\rightarrow A^{m+1,n}$ given by $\Delta_F=\Delta_F^{n,m}=\sum_{i=1}^{m+1}(-1)^i(\varepsilon_i\otimes 1)$, and $\Delta_K: A^{m,n}\rightarrow A^{m,n+1}$ given by $\Delta_K=\Delta_K^{n,m}=\sum_{i=1}^{n+1}(-1)^{m+i}(1\otimes\varepsilon_i)$ with the $\varepsilon_i=\varepsilon_i^n$ acting on K. In the first case $\varepsilon_i=\varepsilon_i^F$ acts really only the F^m part of $(F^m\otimes K^n)^*$ and in the second case only on the K^n part.

Our first result is:

THEOREM 3.1. *The groups $\{A^{m,n}\}$ constitute a double complex with respect to derivation $\Delta=\Delta_F+\Delta_K$.*

In view of [3, p. 60] it suffices to show that $\Delta_F^2=\Delta_K^2=0$ and $\Delta_F\Delta_K+\Delta_K\Delta_F=0$. The first two have been proved, and for the last relation we have:

$$\Delta_F\Delta_K=\sum_{i=1}^{m+1}\sum_{j=1}^{n+1}(-1)^{m+i+j}(\varepsilon_j^F\otimes 1)(1\otimes\varepsilon_j^K)=-\Delta_K\Delta_F\,,$$

since one readily observes that $(\varepsilon_i^F\otimes 1)(1\otimes\varepsilon_j^K)=(1\otimes\varepsilon_j^K)(\varepsilon_i^F\otimes 1)$. but $(1\otimes\varepsilon_i^K)$ acts now on $(F^{m+1}\otimes K^n)^*$.

The double complex obtained by F and K will be denoted by $C^*(F,K/C)$ and its cohomology group by $H^n(F,K/C)$.

For the interpretation of the cohomology groups we show.

THEOREM 3.2. *If K and F are C-free and let $H^1(F/C)=H^1(K/C)=H^1(F\otimes K^2/K^2)=H^1(F\otimes K/K)=0$ then $H^0(F,K/C)=H^1(F,K/C)=0$ and $H^2(F,K/C)\cong H^1(C^*(F/C)\otimes K/C^*(F/C))$, $(\cong H^1(C^*(F/C)\otimes K/C^*(K/C)))$ and the cohomology group of the quotient complex is isomorphic with $B(F/C)\cap B(K/C)$, i.e. with the Brauer group of all algebras split both by F and K.*

PROOF. To compute the first cohomology groups of the complex $C^*(F,K/C)$ we have to consider the sequence

$$(C^*\longrightarrow)F^*\otimes K^*\longrightarrow[(F^2)^*\oplus(F\otimes K)^*\oplus(K^2)^*]\longrightarrow\sum_{m+n=3}(F^m\otimes K^n)^*\longrightarrow.$$

Let $a_{m,n}$ denote an element in $(F^m\otimes K^n)^*$ so that $a_r=\sum a_{m,n}$ is a cocycle if and only if the following relations are valid:

$$(\Delta_K a_{m,n})(\Delta_F a_{m-1,n+1})=1\,,\qquad 1\leqq m<r \tag{3.1}$$

$$\Delta_F a_{r0}=\Delta_K a_{0r}=1\,.$$

REMARK 3.3. It is interesting to note that for the elements $a_{r0}\in(F^r\otimes K^0)^*=(F^r)^*$, Δ_F is actually the derivation in the complex $C^*(F/C)$, while $\Delta_K a_{r0}=a_{r0}\otimes 1$ which means that Δ_K is the restriction map defined in the previous section.

Thus, $H^0(F,K/C)=0$, for $\Delta a_1=1$, with $a_1=a_{10}+a_{01}$. In this case $a_{10}\in F^*$, $a_{01}\in K^*$, so $\Delta_F a_{10}=1$, $\Delta_K a_{01}=1$ yields a_{10} and $a_{01}\in C^*$. Finally, one readily observes in view of the last remark that $(\varDelta_K a_{10})(\varDelta_F a_{01})=1$ yields $a_{10}=a_{01}=c\in C^*$ since $\varDelta_K a_{10}=a_{10}^{-1}$ and $\varDelta_F a_{01}=a_{01}$.

To compute $H^1(F, K/C)$, we start with $a_2=a_{20}+a_{11}+a_{02}$ which represents a cocycle. By (3.1) it follows that $\varDelta_F a_{20}=1$ i.e. a_{20} represents a cocycle in $H^1(F/C)$ but the latter $=0$, so $a_{20}=\varDelta_F a_{10}$. Similarly, $a_{02}=\varDelta_K a_{01}$. By considering the $a_2\varDelta(a_{10}^{-1}+a_{01}^{-1})$ which is homologous to a_2 we may assume that $a_{20}=a_{02}=1$. Then (3.1) will yield

$$1=(\varDelta_K a_{20})(\varDelta_F a_{11})=\varDelta_F a_{11}\,; \qquad 1=(\varDelta_F a_{02})(\varDelta_K a_{11})=\varDelta_K a_{11}\,.$$

Now $\varDelta_F=\varepsilon_1^F-\varepsilon_2^F$ (additively written); hence $\varepsilon_1^F(a_{11})=\varepsilon_2^F(a_{11})$. Since $\varDelta_K=-\varepsilon_1^K+\varepsilon_2^K$ we get $\varepsilon_1^K(a_{11})=\varepsilon_2^K(a_{11})$.

Now K is C-free, so let $\{k_i\}$ be a C-base and let $a_{11}=\sum f_i\otimes k_i$. Thus $\varepsilon_1^F(a_{11})=\sum(1\otimes f_i)\otimes k_i=\sum(f_i\otimes 1)\otimes k_i=\varepsilon_2^F(a_{11})$. Consequently, $f_i\in C$ and, therefore, $a_{11}\in(1\otimes K)^*$. From the second relation we now obtain that actually $a_{11}\in(F\otimes 1)^*$. Hence, $a_{11}=c\in C^*$.

But in this case, one verifies that $a_1=1\oplus c\oplus 1=\varDelta(c\oplus 1)$. i.e. $a_1\sim 1$. q.e.d.

We turn now to $H^2(F; K/C)$ and choose a representative a_3 of a cocycle and let:

$$a_3=a_{30}+a_{21}+a_{12}+a_{03}\,. \tag{3.2}$$

From the relation (3.1) in view of Remark 3.3 we get that a_{30} is a representative of a cocycle in $H^2(F/C)$ and a_{03} of $H^2(K/C)$.

Now to compute the first cohomology group of the quotient complex $[C^*(F/C)\otimes K]/C^*(F/C)$ we have to consider the sequence of quotient groups.

$$\cdots\longrightarrow(F\otimes K)^*/F^*\longrightarrow(F^2\otimes K)^*/(F^2)^*\xrightarrow{\varDelta_F}(F^3\otimes K)^*/(F^3)^*\longrightarrow\cdots.$$

First we obtain the map $\alpha: H^2(F, K/C)\to H^1[(C^*(F/C)\otimes K)/C^*(F/C)]$ as follows:

To the a_3 given above in (3.2), $\alpha(a_3)$ will be cocycle generated by a_{21}, and it is trivial to show that $\alpha(a_3b_3)=\alpha(a_3)\alpha(b_3)$. α is well defined: since $a_{21}\in(F^2\otimes K)^*$ and from (3.1) it follows that $(\varDelta_K a_{30})(\varDelta_F a_{21})=1$. Noticing that $\varDelta_K a_{30}\in(F^3)^*=(F^3\otimes 1)^*$ we get $\varDelta_F a_{21}\in(F^3)^*$ which means that a_{21} represents a cocycle in $H^1[C^*(F/C)\otimes K)/C^*(F/C)]$.

Furthermore, let $b_3=a_3(\varDelta a_2)$, $a_2=a_{20}+a_{11}+a_{02}$; then

$$b_3=[a_{30}+a_{21}+a_{12}+a_{03}][\varDelta_F a_{20}+(\varDelta_K a_{20}\cdot\varDelta_F a_{11})+\cdots]$$

so that $b_{21}=a_{21}(\varDelta_F a_{11})(\varDelta_K a_{20})$. Note that $\varDelta_K a_{20}\in(F^2)^*$ which means that b_{21} and a_{21} represent the same cocycle in the cohomology group of the quotient complex.

To prove that it is an isomorphism we define an inverse map β:

Let $x_{21}\in(F^2\otimes K)^*$ represent a cocycle in cohomology group of the quotient complex. This implies that $\varDelta_F x_{21}=y_{30}$ for some $y_{30}\in(F^{\cdot})^*$. Put $y_{30}^{-1}=x_{30}$

and we get the first relation: (*) $(\Delta_K x_{30})(\Delta_F x_{21})=1$ since $\Delta_K|F^3$ is actually the injection $(F^3)^*\to(F^3\otimes 1)^*\subseteqq(F^3\otimes K)^*$. Furthermore, applying Δ_F we get $1=(\Delta_F\Delta_K x_{30})(\Delta_F^2 x_{21})=(\Delta_K\Delta_F)x_{30}^{-1}$ since $\Delta_K\Delta_F+\Delta_F\Delta_K=0$. But as Δ_K is an injection we get the second relation $\Delta_F x_{30}=1$.

Next applying Δ_K on (*) we obtain: $(\Delta_K^2 x_{30})(\Delta_K\Delta_F x_{21})=1$.[5]

Consequently, $\Delta_F(\Delta_K x_{21})=1$. This means that $\Delta_K x_{21}\in(F^2\otimes K^2)^*$ represent a cocycle in $(F^2\otimes K^2)^*$. But $H^1(F\otimes K^2/K^2)=0$ hence $\Delta_K x_{21}=\Delta_F y_{12}$ with $y_{12}\in(F\otimes K^2)^*$ or equivalently we obtain the third relation: (***) $(\Delta_K x_{21})(\Delta_F x_{12})=1$, where $x_{12}=y_{12}^{-1}$.

To determine x_{03}, we apply Δ_K on the last relation. The fact that $\Delta_K^2=0$ implies that $\Delta_K\Delta_F x_{12}=\Delta_F(\Delta_K x_{12})=1$. Consequently, $\Delta_K x_{12}\in(F\otimes K^3)^*$ represents a cocycle in $H^0(F\otimes K^3/K^3)$, but the latter is zero so that $\Delta_K x_{12}\Delta_F x_{03}=1$ for some $x_{03}\in(F^3)^*$. Applying Δ_K we get $\Delta_K\Delta_F x_{03}=\Delta_F\Delta_K x_{03}=1$. Hence again $\Delta_F(K^3)^*$ is an injection so the last relation yields $\Delta_K x_{03}=1$ as required.

We now define $\beta(x_{21})$ to be the cocycle of the double complex represented by $x_3=x_{30}+x_{21}+x_{12}+x_{03}$ which were chosen above.

β is a well defined homomorphism: since one readily observes that $\beta(x_{21}y_{21})=\beta(x_{21})\beta(y_{21})$; and it suffices, therefore, to show that if x_{21} is a coboundary then $\beta(x_{21})$ is the zero cocycle.

Indeed, x_{21} a coboundary in the quotient complex is equivalent to $x_{21}=z_{20}(\Delta_F z_{11})=\Delta_K z_{20}\Delta_F z_{11}$. Let x_3 be any representative obtain from x_{21}, then another representative of the same cocycle will be:

$$y_3=x_3\Delta(z_{20}^{-1}+z_{11}^{-1})=y_{30}+1+y_{12}+y_{30}\,.$$

And it remains to show that a cocycle represented by y_3 is the zero cocycle. To this end we observe that the relation (3.1) yields first that $(\Delta_K y_{30})(\Delta_F 1)=1$ but since $\Delta_K|F^2\otimes K$ is an injection, it follows that $y_{30}=1$.

Next $(\Delta_K 1)(\Delta_F y_{12})=1$, so y_{12} represents a cocycle in $H^0(F\otimes K^2/K^2)$[6], but the latter is zero so $y_{12}=\Delta_F z_{02}$. Thus $y_3\Delta(1+1+z_{02}^{-1})=1+1+1+c_{03}$ is another representative of the same cocycle. The rest of the proof follows by showing as avove that $c_{03}=1$ which is a simple consequence of the fact that $\Delta_F c_{03}=1$ and Δ_F is an injection on c_{03}.

The conclusion of the proof will be obtained by establishing the isomorphism: $H^1[C^*(F/C)\otimes K/C^*(F/C)]\cong B(F)\cap B(K)$. To this end consider the exact sequence:

$$1\longrightarrow C^*(F/C)\xrightarrow{\rho}C^*(F/C)\otimes K\longrightarrow C^*(F/C)\otimes K/C^*(F/C)\longrightarrow 1$$

which yields the exact sequence

5) Since $\Delta_K^2=0$.

6) Note that since F is C-free, it follows that $H^0(F\otimes R/R)=0$ for arbitrary $R\supseteqq C$.

$$0=H^1[C^*(F/C),\otimes K]\longrightarrow H^1[C^*(F/C)\otimes K/C^*(F/C)]\longrightarrow$$
$$\longrightarrow H^2(F/C)\xrightarrow{\rho} H^2[C^*(F/C)\otimes K].$$

Thus, our cohomology group is isomorphic with $\mathrm{Ker}(\rho^*)$. From the remark given prior to Theorem 2.3 it follows readily that $\mathrm{Ker}(\rho^*)$ is isomorphic with the subgroup of $B(F/C)$ of all algebras which are split by K; and this completes the proof of the theorem. To prove the second isomorphism one has to assume that $H^1(F\otimes K/F)=H^1(F^2\otimes K/F^2)=0$.

In one case we can determine the group $H(F; K/C)$:

THEOREM 3.3. *Let* $F\supseteqq K$, *then* $H^n(F,K/C)\cong H^n(K/C)$, $n\geqq 0$.

PROOF. First we observe that Theorem 2.9 yields $H^n(F^\nu\otimes K/F^\nu)=0$ for $\nu\geqq 1$.

Let $a_n=a_{0n}+a_{1,n-1}+\cdots+a_{n0}$, $a_{ik}\in(F^i\otimes K^k)^*$, represent a cocycle in $H^n(F, K/C)$ then it follows by (3.1) that $a_{0n}\in(K^n)^*$ represents a cocycle $\bar{a}_{0n}\in H^n(K/C)$. We obtain the isomorphism required by mapping: $\bar{a}_n\rightarrow\bar{a}_{0n}$.

Indeed, the map is onto. For, let $\bar{a}_{0n}\in H^n(K/C)$, which means $\Delta_K a_{0n}=1$. Define $a_{i,n-i}$ stepwise to obtain an element $a_n=\sum a_{i,n-i}$ for which $\Delta a_n=1$. This is carried out as follows: $\Delta_K^{0n}: K^n\rightarrow F\otimes K^n$ is an injection. Since $1=\Delta_F\Delta_K a_{0n}$ and the latter $=\Delta_K\Delta_F a_{0n}$ it follows that $\Delta_K a_{0n}\in F\otimes K^n$ is a cocycle in $C^*(F\otimes K/C)$. But as remarked above $H^n(F\otimes K/F)=H^n(C^*(K/C)\otimes F)$ consequently, $\Delta_F a_{0n}=\Delta_K a_{1,n-1}^{-1}$ for some $a_{1,n-1}\in F\otimes K^{n-1}$. Now $1=\Delta_F^2 a_{0n}=\Delta_F\Delta_K a_{1,n-1}^{-1}$ $=\Delta_K\Delta_F a_{1,n-1}$, which implies that $\Delta_F a_{1,n-1}\in F^2\otimes K^{n-1}$ is a cocycle. Again $H^{n-1}(F^2\otimes K/F^2)=0$ yields $\Delta_F a_{1,n-1}=\Delta_K a_{2,n-2}^{-1}$ and clearly this procedure can be continued to yield the required a_n. Furthermore, the method of choosing the $a_{i,n-i}$ proves the validity of the relations (3.1) and hence $\Delta a_n=1$, as required.

Now suppose $a_{0n}=\Delta_K b_{0,n-1}$ i.e. a_{0n} is a coboundary; we wish to show the corresponding a_n is also a coboundary. Indeed, $a_n\Delta(b_{0n+1}^{-1}+1+\cdots+1)=$ $1+a'_{1,n-1}+\cdots+a'_{n,0}=a'_n$ is homologous to a_n. Our proof will be obtained by showing that if we can find an element $b_n=1+\cdots+1+b_{i,n-i}+\cdots+b_{n0}$ homologous to a_n then we can find $b'_n=1+\cdots+1+b'_{i+1,n-i-1}+\cdots+b'_{n0}$ of the same class. Indeed, (3.1) yields that $\Delta_K b_{i,n-i}=\Delta_F 1=1$ so $b_{i,n-i}$ is a cocycle in $C^*(F\otimes K/F)$ but the latter is zero, hence $b_{i,n-i}=\Delta_F c_{i+1,n-i-1}$ and the element $b'_n=b_n\Delta(1+\cdots+c_{i+1,n-i-1}^{-1}+1+\cdots+1)$ will satisfy our requirements. The rest follows now easily.

Actually the preceding proof yields more:

COROLLARY 3.4. *Let F, K be two free C-algebras with the property that* $H^{n+1-\nu}(F^\nu\otimes K/F^\nu)=0$ *for* $n\geqq\nu\geqq 1$ *and all* n *then* $H^n(F,K/C)\cong H^n(K/C)$.

In our case where $K\subseteqq F$ we can actually write down the explicit map which yields the isomorphism. First we obtain a more general result:

THEOREM 3.5. *Let* $\varphi: R\rightarrow F$, $\psi: R\rightarrow K$ *be two C-homomorphisms then the* *map*: $\xi:(R^n)^* \to \sum(F^\nu \otimes K^{n-\nu})^*$ *given by*

$$\xi(r_1\otimes\cdots\otimes r_n)=\sum(\varphi^\nu\otimes\psi^{n-\nu})(r_1\otimes\cdots\otimes r_n)$$
$$=\sum\varphi(r_1)\otimes\cdots\otimes\varphi(r_\nu)\otimes\psi(r_{\nu+1})\otimes\cdots\otimes\psi(r_n)$$

induces a homomorphism $\xi^*:H^*(R/C)\to H^*(F,K/C)$.

PROOF. Clearly ξ is also a ring homomorphism on each component R^n and writing $\xi=\sum\varphi^\nu\otimes\psi^{n-\nu}$ one obtains the relations:

$$\xi\varepsilon_i=\sum(\varphi^\nu\otimes\psi^{n-\nu})\varepsilon_i=\sum_{i<\nu}\varepsilon_i^F(\varphi^{\nu-1}\otimes\psi^{n-\nu})+\sum_{\nu\leqq i}\varepsilon_i^K(\varphi^\nu\otimes\psi^{n-1-\nu}).$$

So that (additively written): for $\xi=\xi^n$

$$\xi\Delta=\xi\sum_{i=1}^n(-1)^{i-1}\varepsilon_i=\sum_{i=1}^n\sum_{i<\nu}(-1)^{i-1}\varepsilon_i^F(\varphi^{\nu-1}\otimes\psi^{n-\nu})+\sum_{i=1}^n\sum_{i\geqq\nu}(-1)^{i-1}\varepsilon_i^K(\varphi^\nu\otimes\psi^{n-1-\nu})$$
$$=\sum_{\lambda=0}^{n-1}(\sum_{i\leqq\lambda}(-1)^{i-1}\varepsilon_i^F+\sum_{i\leqq\lambda}(-1)^{i-1}\varepsilon_i^K)(\varphi^\lambda\otimes\psi^{n-1-\lambda})=\sum(\Delta_F+\Delta_K(\varphi^\lambda\otimes\psi^{n-1-\lambda})=\Delta\xi\,.$$

That is ξ is a complex homomorphism. q. e. d.

REMARK 3.6. In the particular case $R=K$, $K\subseteqq F$. Then $\varphi=$ identity and $\psi:K\to F$ is the injection of F in K, the above ξ is given by $\xi(a_1\otimes\cdots\otimes a_n)=\sum a_1\otimes\cdots\otimes a_n$, but each term is considered in a different ring $F^\nu\otimes K^{n-\nu}$. In this case ξ^* is an isomorphism as follows readily from the construction of the isomorphism of the proof of Theorem 3.3. Indeed, let $\xi(a_{0n})=a_{0n}+\cdots=a_n$, $a_{0n}\in(K^n)^*$ and the isomorphism of Theorem 3.3 was achieved by mapping: $a_n\to a_{0n}$.

For further application we recall the simple result of (3.1):

THEOREM 3.6. *There are always homomorphisms*:

$$\lambda_F^*: H^n(F,K/C)\longrightarrow H^n(F/C)\,;\qquad \lambda_K^*: H^n(F,K/C)\longrightarrow H^n(K,C)$$

given by: $\lambda_F(a_n)=a_{n0}$, $\lambda_K(a_n)=a_{0n}$ *where* $a_n=a_{0n}+a_{1,n-1}+\cdots+a_{n0}$.

We conclude with the simple observation, that since $\xi^*=\lambda_K^{*-1}$.

COROLLARY 3.7. *If* $F\supseteqq K$ *then the composite*:

$\lambda_F^*\lambda_K^{*-1}: H^*(K/C)\to H^*(F,K/C)\to H^*(F/C)$ *is exactly the lift map of Definition* 2.3.

We conclude with a general case where $H^0(F/C)$ and $H^1(F/C)$ are zero:

THEOREM 3.8. *If F is C-free then $H^0(F/C)=0$, and if C be a commutative ring with a unit satisfying the minimum condition for ideals and F be a finitely generated free C-algebra with the generators $k_1=1$, $k_2,\cdots,k_n$. Then $H^1(F/C)=0$.*

PROOF. Let $a\in F^2$ be a cocycle, i. e. $\Delta a=(\varepsilon_1a)(\varepsilon_2a)^{-1}\varepsilon_3a=1$. Thus $\varepsilon_2a=\varepsilon_1a\varepsilon_3a$ and if $a=\sum a_i\otimes k_i$ then:

$$\sum(1\otimes a_i\otimes k_i)(a\otimes 1)=\sum a_i\otimes 1\otimes k_i$$

yields $(1\otimes a_i)a=a_i\otimes 1$. Furthermore, we note that for the homomorphism $\mu:F^2\to F$ given by $\mu(x\otimes y)=xy$, we get that $\Delta a=1$ yields $\mu a=\sum a_ik_i=1$.

Consider the set $\{u \mid u \in F, (1\otimes u)a = u\otimes 1\} = K$. Then K is a C-module containing all a_i and we have to show that K contains an invertible element u which will give $a=(1\otimes u^{-1})$ as required. To this end we consider the radical N of C and observe that $\bar{C}=C/N$ is semi simple.

Now if $u, v \in K$ then $u\otimes v=(u\otimes 1)(1\otimes v)=(1\otimes u)a(v\otimes 1)a^{-1}=v\otimes u$ hence if $u=\sum \alpha_i k_i$, $v=\sum \beta_j k_j$ we get that $\alpha v=\beta u$ for any $\alpha=\alpha_i$, $\beta=\beta_i$; and for $u\neq 0$ and $v\neq 0$ some α, β are non zero. The module K/NK is a $\bar{C}$ module and since it must be free and one dimensional with respect to any field of $\bar{C}$, one readily verifies that there exists $x\in K$ such that all $a_i=\lambda_i x+n_i$ for $n_i\in NK$, and $\lambda_i\in C$. Consequently, $1=\sum a_i k_i=(\sum \lambda_i k_i)x+n$ for some $n\in NK$. Clearly, $1-n$ is invertible hence x has also an inverse. We also obtain that $a=\sum a_i\otimes k_i=x\otimes y \pmod{NK}$ with $y=\sum \lambda_i k_i$ and note that y has also an inverse. Multiply by $1\otimes y^{-1}$; we get

$$x\otimes 1\equiv a(1\otimes y)^{-1}\equiv \sum a_i\otimes k_i y^{-1}\equiv \sum \mu_{ij} a_i\otimes k_j \pmod{NK},$$

from which we deduce that $x\equiv \sum \mu_i a_i \bmod (NK)$. Set $u=\sum \mu_i a_i\in K$ and $u=x+n=x(1+nx^{-1})$ and $(1+nx^{-1})^{-1}x^{-1}$ exists since $n\in NK$. q. e. d.

REMARK. In the proof we actually used only the fact that if $N(C)$, $N(F)$ are the Jacobson radicals of C and F respectively and 1) every module over $C/N(C)$ is free or $C/N(C)$ is a direct sum of fields; and 2) $FN(C)\subseteqq N(F)$.

4. The fundamental exact sequence.

The aim of the present section is to obtain some exact sequences which in the classical case are known as the fundamental exact sequences. Our first result in this direction is

THEOREM 4.1. *Let F, K be two C-free algebras and such that $H^i(F^j\otimes K/F^j)=H^i(F\otimes K^j)=0$ for $i+j\leqq 3$, $i=0,1$ and $H^2(F\otimes K/K)=0$ then there exists an exact sequence:*

$$0\longrightarrow H^2(K/C)\xrightarrow{\lambda^*} H^2(F/C)\xrightarrow{\rho^*} H^2(F\otimes K/K). \tag{4.1}$$

In particular, the condition holds for $K\subseteqq F$; and then λ is the "lift" homomorphism and ρ is the "restriction".

PROOF. It follows from the requirement of the theorem and from the fact that $H^1(F^\nu\otimes K/F^\nu)=0$, in view of Corollary 3.4, that the procedure of the proof of Theorem 3.3 can be applied to our case. Namely, for $a_{03}\in(K^3)^*$ we can find $\sigma(a_{03})=a_3=a_{30}+a_{21}+a_{12}+a_{03}\in C^2(F, K/C)$ so that σ induces a homomorphism $\sigma^*: H^2(F/C)\to H^2(F, K/C)$.

The map λ^* of the theorem is the composite $\lambda^*=\lambda_F^*$, where $\lambda_F(a_3)=a_{30}$ is given in Theorem 3.6. The homomorphism ρ^* is the restriction i.e. $\rho(a_{30})=a_{30}\otimes 1\in F^3\otimes K$ and clearly $\sigma=\Delta_K$ of the double complex.

From the proof of Theorem 3.3 it follows immediately that σ^* is an injection (in fact an isomorphism !), so is also λ_F^*. Indeed, let $\lambda_F a_3 = a_{30} = \Delta_F b_{20}$ for $b_{20} \in F^2$, then $b_3 = a_3 \Delta(1+1+b_{20}^{-1}) = b_{03}+b_{12}+b_{21}+1$ represents the same element of $H^2(F, K/C)$. Since $\Delta b_3 = 1$, it follows from the relation (3.1) that $\Delta_F b_{21} \Delta_K 1 = 1$. i.e. b_{12} is a cocycle in $H^1(F \otimes K/K)$. The latter is zero, hence $b_{21} = \Delta_F c_{11}$. Consider $c_3 = b_3 \Delta(1+c_{11}^{-1}+1) = c_{03}+c_{12}+1+1$, and similarly one obtains that the original a_3 is a coboundary. This proves that λ_K^* is an injection and the exactness of the first part of (4.1) is shown.

To prove the second part of (4.1), let $a_{30} \in (F^3)^*$ be such that $\rho^*(a_{30}) = 1$, i.e. $\rho(a_{30}) = \Delta_K a_{30} = \Delta_F a_{21}^{-1}$ [7] for some $a_{21} \in F^2 \otimes K$. We follow now a procedure similar to the proof of Theorem 3.3 to show that there exists a_3 such that $a_3 = a_{30}+a_{21}+a_{12}+a_{03}$ is a cocycle in $C^*(F, K/C)$ which shows that $a_{30} = \lambda(a_3)$ as required. Indeed, from the way a_{21} was chosen we get that $1 = \Delta_K^2 a_{30} = \Delta_K \Delta_F a_{21} = \Delta_F \Delta_K a_{21}$. Hence $\Delta_K a_{21}$ is a cocycle in $H^1(F \otimes K/K)$ and again the latter is zero. Hence, $\Delta_K \Delta_{21} = \Delta_F a_{12}^{-1}$, etc....

If $K \subseteq F$, then Corollary 3.7 means that λ^* is actually the lift map.

To obtain the complete fundamental sequence including the transgression we note that the two homomorphisms $\varepsilon_1 = \varepsilon_1^K: F^n \otimes K \to F^n \otimes K^2$ given by $\varepsilon_1(a \otimes k) = a \otimes 1 \otimes k$ and $\varepsilon_2 = \varepsilon_2^K: F^n \otimes K \to F^n \otimes K^2$ defined by $\varepsilon_2(a \otimes k) = a \otimes k \otimes 1$ are in fact complex homomorphism $C^*(F/C) \otimes K \to C^*(F/C) \otimes K^2$. Hence, they induce $\varepsilon_i^*: H^n(F \otimes K/K) \to H^n(F \otimes K^2/K^2)$ and we shall denote $H^n(F \otimes K/K)^0 = \{\bar{c} \mid \bar{c} \in H^n(F \otimes K/K)$ for which $\varepsilon_1^*(\bar{c}) = \varepsilon_2^*(\bar{c})\}$.

Thus the complete fundamental exact sequence is:

THEOREM 4.2. *Let F, K be two free C-modules such that $H^2(F \otimes K/F) = 0$ and $0 = H^1(F^i \otimes K/F^i)$; $i = 3, 2, 1$ (which is always valid if $K \subseteq F$) then there exists an exact sequence:*

$$0 \longrightarrow H^2(K/C) \xrightarrow{\lambda^*} H^2(F/C) \xrightarrow{\rho^*} H^2(F \otimes K/K)^0 \xrightarrow{t^*} H^3(K/C) \xrightarrow{\lambda^*} H^3(F/C)$$

where λ^* is the lift map and ρ^* is the restriction.

PROOF. The proof will be carried in steps:

a) Consider first the set of all elements of $H^3(F, K/C)$ which have a representation of the form $a_4 = a_{04}+a_{13}+a_{22}+a_{31}+1$.

Now $\Delta a_4 = 1$, yields by (3.1) that (1) $\Delta_F a_{31} = 1$ and (2) $\Delta_K a_{31} = \Delta_F a_{22}^{-1}$. The first relation is equivalent to the fact that a_{31} is a cocycle in $H^2(F \otimes K/K)$. Noting that this $\Delta_K = \varepsilon_1^K(\varepsilon_2^K)^{-1}$, it follows that (2) is equivalent to the fact that under the induced homomorphism $\Delta_K^*: H^2(F \otimes K/K) \to H^2(F \otimes K^2/K^2)$ the class of a_{31} is mapped onto the zero.

a′) Conversely, let $a_{31} \in (F^3 \otimes K)^*$ an element with the properties that its class $\bar{a}_{31} \in H^2(F \otimes K/K)$ and $\varepsilon_1^* \bar{a}_{31} = \varepsilon_2^* \bar{a}_{31}$ then clearly the first condition implies

7) Note that ρ is exactly the same map as Δ_K.

(1) and the second condition yields (2) for some $a_{22} \in (F^2 \otimes K^2)^*$. Then we can continue as in the proof of the previous theorems: from (2) it follows that $1 = \Delta_K^2 a_{31} = \Delta_K \Delta_F a_{22}^{-1} = \Delta_F(\Delta_K a_{22})$. Hence $\Delta_K a_{22}$ is a cocycle in $H^1(C^*(F \otimes K^3/K^3))$ which is zero. Thus $\Delta_K a_{22} = \Delta_F a_{13}^{-1}$ for some $a_{13} \in F \otimes K^3$ and so on...

b) Next we observe that if a_{31} represents the zero cocycle in $H^2(F \otimes K/K)$ then the corresponding a_4 is homologous to zero in $H^3(F, K/C)$. Indeed, if $a_{31} = \Delta_F a_{21}$ then a_4 is homologous to $a_4 \Delta(1+1+a_{21}^{-1}+1) = b_{04}+b_{13}+b_{22}+1+1 = b_4$. Since $\Delta b_4 = 1$, conditions (3.1) imply that $\Delta_F b_{22} = 0$ which means that b_{22} is a cocycle in $H^1(F \otimes K^2/K^2) = 0$. So $b_{22} = \Delta_F c_{12}^{-1}$ and we continue as in the proof of Theorem 3.3.

c) Let $a_4 = \Delta(a_{03}+a_{12}+a_{21}+a_{30}) = a_{04}+ \cdots +a_{31}+1$, then it follows by definition of Δ that $\Delta_F a_{30} = 1$ and $a_{31} = (\Delta_F a_{21}) \cdot (\Delta_K a_{30})$ which means that $\bar{a}_{30} \in H^2(F/C)$ and that in $H^2(F \otimes K/K)$, a_{31} is homologous to $\Delta_K a_{30}$; but in the present case $\Delta_K = \rho_K \colon F^3 \to F^3 \otimes K$ is the restriction ρ of Definition 2.1. Conversely, for a given a_{31} which is homologous in $H^2(F \otimes K/K)$ with an image $\rho(a_{30})$ of a cocycle of $H^2(F/C)$ the corresponding a_4 is coboundary Δa_3 in the double complex $C(F, K/C)$. The proof is similar to the proof of (b), starting by $b_4 = a_4 \Delta(1+1+a_{21}^{-1}+a_{30}^{-1})$ where $H^0(F \otimes K^3/K^3) = 0$, $a_{31} = (\Delta_F a_{21})(\Delta_K a_{30})$, which exists by assumption on a_{31}. Here again $H^0(F \otimes K^4/K^4) = 0$, $b_4 = b_{04}+b_{13}+b_{22}+1+1$ and one continues as in (b).

d) Consider the map $\sigma(a_{31}) = a_4 = a_{04}+a_{13}+a_{22}+a_{31}+1$ where a_4 is chosen as in (a′). Though σ is not unique, it induces by (b) a homomorphism $\sigma^* \colon H^2(F \otimes K/K)^0 \to H^2(F, K/C)$ and the image $\sigma^*[H^2(F \otimes K/K)^0]$ is exactly the set of all cocycles of the form $a_4 = a_{40}+ \cdots +a_{31}+1$. But this group is clearly the kernel of $\lambda_F^* \colon H^3(F, K/C) \to H^3(F/C)$. Indeed $\lambda_F(x_4) = \lambda(x_{04}+ \cdots +x_{31}+x_{40}) = x_{40}$ and if $\bar{x}_{40} = 1$ in $H^3(F/C)$ it follows that $x_{40} = \Delta_F y_{30}$ and consequently $x_4 \Delta(1+ \cdots +1+y_{30}^{-1}) = x'_{04}+ \cdots +x'_{31}+1$ is of the preceding form.

Furthermore, Kernel (σ^*) is by (c) $\rho^*[H^2(F/C)]$. Hence we obtain the exactness of the sequence:

$$H^2(F/C) \xrightarrow{\rho^*} H^2(F \otimes K/K)^0 \xrightarrow{\sigma^*} H^3(F, K/C) \xrightarrow{\lambda_K^*} H^3(F/C)\,.$$

To conclude the proof, we define $t^* = \lambda_K^* \sigma^*$ where $\lambda^* \colon H(F, K/C) \to H(K/C)$ given by $\lambda_K(\sum a_{i,n-i}) = a_{0n}$ which is shown to be an isomorphism in the proof of Theorem 3.3. Then $\lambda_F^* \lambda_K^{*-1} = \lambda^*$ by Corollary 3.7. This together with Theorem 4.2 complete the proof.

5. The normal separable case.

It is our purpose in the preceding section to show that the notions introduced above coincide with the respective classical notions. Let F be a normal separable extension of an infinite field C, and let $\mathcal{G}$ be its group of automorphisms.

Let $\Phi_n=\{(\varphi_1,\cdots,\varphi_n), \varphi_i\in\mathcal{G}\}$ be the set of n-tuples of elements of $\mathcal{G}$. For $\alpha\in\mathcal{G}$, $\varphi\in\Phi_n$ we put $\alpha\varphi=(\alpha\varphi_1,\cdots,\alpha\varphi_n)$.

Let $C_n=C_n(\mathcal{G})$ be the free abelian group generated by the n-tuples of Φ_n, then clearly $C_n(\mathcal{G})$ is also a free $\mathcal{G}$-module generated by the elements $\{(\varphi_1,\cdots,\varphi_{i-1},1,\varphi_i,\cdots,\varphi_{n-1})\}$ for any fixed i.

Put $\Phi_0=\{(\cdot)\}$, and $C_0=Z$ and set $\alpha(\cdot)=(\cdot)$. We shall use also the notations:

$$\sigma_i(\varphi_1,\cdots,\varphi_n)=(\varphi_1,\cdots,\hat{\varphi}_i,\cdots,\varphi_n); \qquad \sigma_1(\varphi)=(\cdot) \tag{5.1}$$

$$\tau_i(\varphi_1,\cdots,\varphi_n)=(\varphi_1,\cdots,\varphi_{i-1},1,\varphi_i,\cdots,\varphi_n).$$

The set of all groups $C_n(\mathcal{G})$ from a complex $C(\mathcal{G})$ with respect to the derivation,

$$d=d_n=\sum_{i=1}^{n}(-1)^{i-1}\sigma_i. \tag{5.2}$$

Consider F^* as a right $\mathcal{G}$-module by setting $a\alpha^{-1}=\alpha(a)$, then the homology groups of the complex $F^*\otimes_{\mathcal{G}}C(\mathcal{G})$ with respect to the derivation $1\otimes d$.

Our aim is first to prove:

THEOREM 5.1. $H_n(\mathcal{G};F^*)\cong H_n(F/C)$.

We start with some preliminary result in order to obtain an isomorphism between the complex $F^*\otimes_{\mathcal{G}}C(\mathcal{G})$ and $\mathcal{G}_*(F/C)$.

Following [**5**] we consider the pairing of F^n with Φ_n into F. That is, consider the function (a,φ) for $a\in F^n$, $\varphi\in\Phi_n$ which is linear in the first variable and given by:

$$(a,\varphi)=(a_1\otimes\cdots\otimes a_n,(\varphi_1,\cdots,\varphi_n))=\varphi_1(a_1)\cdot\varphi_2(a_2)\cdots\varphi_n(a_n). \tag{5.3}$$

We quote some properties of (a,φ):

$$\begin{gathered}(a+b,\varphi)=(a,\varphi)+(b,\varphi); \qquad (ab,\varphi)=(a,\varphi)(b,\varphi)\\ (a,\alpha\varphi)=\alpha(a,\varphi); \qquad (\psi a,\varphi)=(a,\varphi\psi)\\ (\varepsilon_i a,\varphi)=(a,\sigma_i\varphi)\\ (\nu_i a,\varphi)=\prod_{\alpha\in\mathcal{G}}(a,\tau_i\varphi\alpha^{(i)})=\prod_{\alpha\in\mathcal{G}}[a,(\varphi_1,\cdots,\varphi_{i-1},\alpha,\varphi_i,\cdots,\varphi_n)]\end{gathered} \tag{5.4}$$

where $\varphi\psi=(\varphi_1,\cdots,\varphi_n)(\psi_1,\cdots,\psi_n)=(\varphi_1\psi_1,\cdots,\varphi_n\psi_n)$ and $\alpha^{(i)}=(1,\cdots,1,\alpha,1,\cdots,1)$ with α standing at the i-th place.

With the exception of the last property, all proofs are straightforward. For the last property we need:

LEMMA 5.2. *Let R be an arbitrary commutative C-algebra and $K\subseteqq F$. Consider $K\otimes R$ as a subalgebra of $F\otimes R$ then* $\mathrm{Norm}(K\otimes R/R;x)=\prod(\varphi\otimes 1)(x)$ *where φ ranges over the isomorphisms of K into the normal field F.*

PROOF. Let $u_1, \cdots, u_m$ be a C-base of F and let $x=\sum \xi_i u_i$, $\xi_i \in R$. The characteristic polynomial $Pc(\lambda; x)$ of x is by definition det $|\lambda 1 - g_{ik}(\xi)|$ where $xu=\sum g_{ik}(\xi)u_k$ and its last term is $(-1)^n \operatorname{Norm}(F \otimes R/R; x)$.

On the other hand, consider the polynomial

$$H(x;\lambda)=\prod_{\varphi}[\lambda-(\varphi\otimes 1)(x)]=\prod[(\lambda-\sum\xi_j\varphi(u_j)]=\prod(\lambda-l_\varphi(\xi))$$

where $l_\varphi(\xi)$ is a linear polynomial in ξ. The lemma is now an immediate consequence of the fact that $H(x;\lambda)\equiv Pc(x;\lambda)$.

Indeed, if all $\xi_i \in C$ the result is well known (e.g. [**2**, p. 137]). Now C was assumed to be infinite and both $H(x;\lambda)$ and $Pc(x;\lambda)$ are polynomials in ξ; hence, they are identical.

The last property of (5.4) follows now easily since:

First, we have for $a=\varepsilon_i^{-1}b$ that $(\varepsilon_i\varepsilon_i^{-1}b, \varphi)=(b,\varphi)=(\varepsilon_i^{-1}b, \sigma_i\varphi)$. Hence,

$$\begin{aligned}(\nu_i a, \varphi) &= (\varepsilon_i^{-1}\operatorname{Norm}(F^n/\varepsilon_i F^{n-1}; a), \varphi)=(\varepsilon_i^{-1}\prod_{\alpha}(\alpha^{(i)}a), \sigma_i\tau_i\varphi)\\ &=\prod_{\alpha}(\alpha^{(i)}a, \tau_i\varphi)=\prod_{\alpha}(a, \tau_i\varphi\alpha^{(i)}).\end{aligned}$$

To the function of (5.4) we add the definition $(a,(\cdot))=a$ for $a\in C=F^0$.

The following result of [**5**] (Lemma 2.2) will be used here extensively:

LEMMA 5.3. *For $x\in F_n$, let $p_x \in \operatorname{Hom}^{\mathcal{G}}(\Phi_n, L)$ defined as $p_x(\varphi)=(x,\varphi)$. Then the mapping $\tau: x\to p_x$ determines an isomorphism: $(F^n)^*\cong \operatorname{Hom}^{\mathcal{G}}(\Phi_n, F^*)$.*

We turn now to the proof of Theorem 5.1:

Consider the mapping $f: C_*(F/C)\to F^*\otimes_{\mathcal{G}} C(G)$ defined by:

(5.5) $$f(x)=\sum(x,\tau_1\varphi)\otimes\tau_1\varphi \qquad \text{for} \quad x\in F^n$$

and where φ ranges over all Φ_{n-1}.

Note first that in the definition of f we could have chosen any $2\leqq i\leqq n$ instead of 1.

Indeed, setting $\psi=(\varphi_1^{-1}\varphi_2, \cdots, \varphi_1^{-1}\varphi_{i-1}, \varphi_1^{-1}, \varphi_1^{-1}\varphi_i, \cdots)$ where $\varphi=(\varphi_1,\cdots,\varphi_{n-1})$, we get $\tau_i\varphi=\varphi_1(\tau_1\psi)$. Now since $(x,\alpha\varphi)=\alpha(x,\varphi)=(x,\varphi)\alpha^{-1}$ by definition, it follows that:

$$(x,\tau_i\varphi)\otimes\tau_i\varphi=(x,\tau_1\psi)\varphi_1^{-1}\otimes\varphi_1\tau_1\psi=(x,\tau_1\psi)\otimes\tau_1\psi.$$

Clearly ψ will also range over all Φ_{n-1} if φ does so.

It follows now by (5.4) that

$$\begin{aligned}f(xy)&=\sum(xy,\tau_1\rho)\otimes\tau_1\varphi=\sum(x,\tau_1\rho)(y,\tau_1\varphi)\otimes\tau_1\varphi\\&=\sum(x,\tau_1\varphi)\otimes\tau_1\varphi+\sum(y,\tau_1\varphi)\otimes\tau_1\varphi^{8)}.\end{aligned}$$

It is evident that the elements of $F^*\otimes C_n(\mathcal{G})$ can be written uniquely in the form $\xi=\sum a_\varphi\otimes\tau_i\varphi$ with φ ranging over Φ_{n-1}. The functions $\mathcal{G}$ of

8) Note that the elements of F^* are written multiplicatively, whereas the groups $F^*\otimes C_n(\mathcal{G})$ are written additively.

$\mathrm{Hom}^{\mathcal{G}}(\Phi_n, F^*)$ are uniquely determined by their values $\mathcal{G}(\tau_i\varphi)$ since every $\varphi\in\Phi_n$ can be expressed uniquely in the form $\varphi=\alpha\tau_i\psi$, $\psi\in\Phi_{n-1}$ so that $\mathcal{G}(\varphi)=\alpha\mathcal{G}(\tau_i\psi)$. It follows, therefore, by Lemma 5.3 that the mapping f is an isomorphism. It remains now to show that f is also a complex isomorphism.

The last property of (5.4) yields: for $x\in F^n$, $\nu_i x\in F^{n-1}$ hence, for $i>1$:

$$\begin{aligned} f(\nu_i x) &= \sum_{\varphi\in\Phi_{n-2}} (\nu_i x, \tau_1\varphi)\otimes\tau_1\varphi = \sum_{\alpha}\prod (x, \tau_i(\tau_1\varphi)\alpha^{(i)})\otimes\tau_1\varphi \\ &= \sum_{\varphi\in\Phi_{n-1}} (x, \tau_1\psi)\otimes\sigma_i(\tau_1\psi)\,. \end{aligned}$$

For, $\tau_1\psi=(\tau_i\tau_1\varphi)\alpha^{(i)}=(1, \varphi_1, \cdots, \varphi_{i-2}\alpha, \varphi_{i-1}, \cdots, \varphi_{n-2})$ and ψ will range over all Φ_{n-1} when α and φ range over all $\mathcal{G}$ and Φ_{n-2} respectively. The same result holds also for $i=1$. Indeed,

$$\begin{aligned} (x, (\tau_1\tau_1\varphi)\alpha^{(1)})\otimes\tau_1\varphi &= (x, (\alpha, 1, \varphi_1, \cdots))\otimes\tau_1\varphi = (x, \alpha(1, \alpha^{-1}, \alpha^{-1}\varphi_1, \cdots)\otimes\tau_1\varphi \\ &= \alpha(x, \tau_1\psi)\otimes\tau_1\varphi = (x, \tau_1\psi)\alpha^{-1}\otimes\tau_1\varphi \\ &= (x, \tau_1\psi)\otimes\alpha^{-1}\tau_1\varphi = (x, \tau_1\psi)\otimes\psi = (x, \tau_1\psi)\otimes\sigma_1\tau_1\psi\,, \end{aligned}$$

and here $\psi=\alpha^{-1}\tau_1\varphi=(\alpha^{-1}, \alpha^{-1}\varphi_1, \cdots, \alpha^{-1}\varphi_{n-2})$, and $\sigma_1\cdot\tau_1$ is the identity. Consequently,

$$\begin{aligned} f\mathfrak{N}(x) &= f(\prod\nu_i(x)^{(-1)^{i-1}}) = \sum(-1)^{i-1}f(\nu_i x) \\ &= \sum(x, \tau_1\psi)\otimes\psi + \sum(-1)^{i-1}\sum(x, \tau_1\psi)\otimes\sigma_i\tau_1\psi \\ &= \sum(-1)^{i-1}(1\otimes\sigma_i)(\sum(x, \tau_1\psi)\otimes\tau_1\psi) = (1\otimes d)f(x) \end{aligned}$$

which concludes the proof of the theorem.

Next we show that the restriction and transfer defined in Section 2 coincide with the classical definitions for the normal case. But we shall carry it through only for the cohomology groups. To this end we need the following generalization of [**5**, Lemma 2.2]:

Let $F_1, \cdots, F_n$ be finite algebraic extensions of C and let F be a normal extension of C containing all F_i. Denote by $\mathcal{G}(F_i/C)$ the set of all isomorphisms of F_i into F. Thus $\mathcal{G}(F/C)$ will be the Galois group of F.

Let $\Phi=\{(\varphi_1, \cdots, \varphi_n);\ \varphi_i\in\mathcal{G}(F_i)\}$ and define $\alpha(\varphi_1, \cdots, \varphi_n)=(\alpha\varphi_1, \cdots, \alpha\varphi_n)$. Again (a, φ) will denote the pairing of $F_1\otimes\cdots\otimes F_n$ with Φ into F which is linear in the first variable and defined as in (5.3). Namely,

$$(a_1\otimes\cdots\otimes a_n, (\varphi_1, \cdots, \varphi_n)) = \varphi_1(a_1)\varphi_2(a_2)\cdots\varphi_n(a_n)\,.$$

The same proof of [**5**, Lemma 2.2] will yield the following generalization:

LEMMA 5.6. *For $x\in F_1\otimes\cdots\otimes F_n$, let $p_x\in\mathrm{Hom}^{\mathcal{G}}(\Phi, F)$ be defined as $p_x(\varphi)=(x, \varphi)$. If $F_2, \cdots, F_n$ are separable extensions of C then the mapping: $x\to p_x$ determines an isomorphism $(F_1\otimes\cdots\otimes F_n)^*\cong\mathrm{Hom}^{\mathcal{G}}(\Phi, F^*)$.*

The proof will not be reproduced here as it is the same as that of [**5**, Lemma 2.2], noticing that under the assumptions $F_1\otimes\cdots\otimes F_n$ is still semi simple.

Let K be a fixed subfield of F with a corresponding subgroup $\mathcal{H}$ of $\mathcal{G}=\mathcal{G}(F/C)$. The complex $C(\mathcal{G})$ given in the beginning of this section can be considered also as a free $\mathcal{H}$-complex. We shall use our lemma for the fields $F,\cdots,F,K$ and since $\mathcal{G}(K/C)$ can be identified with right cosets of $\mathcal{G}$ mod $\mathcal{H}$ we shall denote them by $\bar{\alpha}$ for $\alpha\in\mathcal{G}$.

LEMMA 5.5. *For* $x\in F^n\otimes K$, *let* $\bar{p}_x\in \mathrm{Hom}^{\mathcal{H}}(C_n(\mathcal{G}), F^*)$ *be given by* $\bar{p}_x(\varphi)=(x,(\varphi,\bar{1}))$, *where* (x,ψ) *is the pairing of* $F^n\otimes K$ *with* $\psi=(\mathcal{G}^n,\mathcal{G}/\mathcal{H})$ *into* F. *Then the mapping* $\eta: x\to p_x$ *defines an isomorphism* $C^*(F/C)\otimes K\cong \mathrm{Hom}^{\mathcal{H}}(C(\mathcal{G}),F^*)$.

PROOF. The groups of $C^*(F/C)\otimes K$ are $(F^n\otimes K)^*$ and by Lemma 5.4 it follows that $\tau: x\to p_x$ is an isomorphism of $(F^n\otimes K)^*$ with $\mathrm{Hom}^{\mathcal{G}}(\Psi_{n+1},F^*)$ where $\Psi_{n+1}=\{(\varphi_1,\cdots,\varphi_n,\bar{\varphi}_{n+1}),\varphi_i\in\mathcal{G}(F/C),\bar{\varphi}_{n+1}\in\mathcal{G}(K/C)\}$. To conclude that η is an isomorphism between $(F^n\otimes K)^*$ and $\mathrm{Hom}^{\mathcal{H}}(C_n(\mathcal{G}),F^*)$ it remains to show that the map $f_x\to\bar{f}$ where $\bar{f}(\varphi)=f(\varphi,\bar{1})$ is an isomorphism between $\mathrm{Hom}^{\mathcal{G}}(\Psi_{n+1},F^*)$ and $\mathrm{Hom}^{\mathcal{H}}(C_n(\mathcal{G}),F^*)$. This is clear from the observation that for an $\bar{f}\in\mathrm{Hom}(C_n(\mathcal{G}),F^*)$, $\bar{f}(\alpha\varphi)=\alpha\bar{f}(\varphi)$ will hold if and only if $f(\alpha\varphi,\bar{1})=\alpha f(\varphi,\bar{1})$ is valid only for $\alpha\in\mathcal{H}$; and from the fact that the functions f of $\mathrm{Hom}^{\mathcal{G}}(\Psi_{n+1},F^*)$ are uniquely determined by $f(\varphi,\bar{1})$ if $\alpha f(\varphi,\bar{1})=f(\alpha\varphi,\bar{1})$, for all $\alpha\in\mathcal{H}$, holds.

At this stage one can reproduce the proof of [1, Theorem 1] to show that $\eta: x\to\bar{p}_x$ is actually a complex isomorphism from which the theorem follows.

From the general theory of these complexes as developed in [1] and [5] we know that the mapping $\tau_F: x\to p_x$ of Lemma 5.3 induces a complex isomorphism: $C(F/C)\cong\mathrm{Hom}^{\mathcal{G}}(C(\mathcal{G}),F^*)$. We also recall that the restriction $\rho: C(F/C)\to C(F/C)\otimes K$ was given by $\rho(x)=x\otimes 1$ for $x\in F^n$. We now prove:

THEOREM 5.6. *The induced homomorphism of the composite:*

$$\eta\rho\tau_F^{-1}: \mathrm{Hom}^{\mathcal{G}}(C(\mathcal{G}),F^*)\longrightarrow C^*(F/C)\longrightarrow C^*(F/C)\otimes K\longrightarrow \mathrm{Hom}^{\mathcal{H}}(C(\mathcal{G}),F^*)$$

yields the restriction homomorphisms: $H^n(\mathcal{G};F^*)\to H^n(\mathcal{H};F^*)$.

PROOF. For $f\in\mathrm{Hom}^{\mathcal{G}}(C_n(\mathcal{G}),F^*)$ we have $f=\tau_F x$ for $x\in(F^n)^*$ where x is determined by the relation $f(\varphi)=(x,\varphi)$ i.e. $f=p_x$. Now $(\eta\rho)(x)=\eta(x\otimes 1)=g\in\mathrm{Hom}^{\mathcal{H}}(C_n(\mathcal{G}),F^*)$ and g is given by

$$g(\varphi)=p_{x\otimes 1}(\varphi)=(x\otimes\bar{1},(\varphi,1))=(x,\varphi)=f(\varphi).$$

That is $(\eta\rho\tau_F^{-1})f=f$ but here f is considered as invariant only under $\mathcal{H}$. Hence the induced map $(\eta\rho\tau_F^{-1})^*=\eta^*\rho^*\tau_F^{*-1}=i(\mathcal{H};\mathcal{G})$ is the restriction map given in [3, p. 254].

Next we show:

THEOREM 5.7. *The induced homomorphism of the composite map:*

$$\tau_F\tau\eta^{-1}: \mathrm{Hom}^{\mathcal{H}}(C(\mathcal{G});F^*)\longrightarrow C(F/C)\otimes K\longrightarrow C(F/C)\longrightarrow \mathrm{Hom}^{\mathcal{G}}(C(\mathcal{G}),F^*)$$

yields the transfer map: $H^n(\mathcal{H};F^*)\to H^n(\mathcal{G};F^*)$.

PROOF. For $\bar{f} \in \mathrm{Hom}^{\mathcal{H}}(C_n(G), F^*)$, $\eta^{-1}\bar{f} = x \in F^n \otimes K$ where (Lemma 5.5) $\bar{p}_x(\varphi) = (x, \varphi) = \bar{f}(\varphi)$. By definition of τ (Theorem 2.4) and by Lemma 5.2 it follows that: $\tau(x) = \mathrm{Norm}(F^n \otimes K/F^n; x) = \prod(1 \otimes \bar{\alpha})(x)$ with $\bar{\alpha}$ ranging over all cosets of $\mathcal{G} \bmod \mathcal{H}$. Hence, $\tau_F\tau(x) = g \in \mathrm{Hom}^{\mathcal{G}}(C_n(\mathcal{G}); F^*)$ satisfies, by (5.4),

$$g(\varphi) = p_{\tau(x)}(\varphi) = (\prod(1 \otimes \bar{\alpha})(x), \varphi) = \prod(x, (\varphi, \bar{\alpha}))^*$$
$$= \prod \alpha_i(x, (\alpha_i^{-1}\varphi, \bar{1}))^* \prod_{\alpha} \alpha_i f(\alpha_i^{-1}\varphi)$$

where $(a, \varphi)^*$ is the pairing of $(\mathcal{G}^n, \mathcal{G}/\mathcal{H})$ and $F^n \otimes K$ in F, and α_i range over a set of representatives of the cosets $\bar{\alpha}$. But, clearly, the relation between g and f is exactly the transfer map as given in [**3**, p. 254].

REMARK 5.8. The fundamental difference between the restriction and transfer maps for arbitrary fields and the respective notions for $H^n(\mathcal{G}, F^*)$ and $H^n(\mathcal{H}, F^*)$ is that the first are maps between $C^*(F/C) \otimes K$ and $C^*(F/C)$, whereas the latter are between $H^n(\mathcal{H}, F^*)$ and $H^n(\mathcal{G}, F^*)$. The corresponding groups for $H^n(\mathcal{H}, F^*)$ are the homology groups of $C^*(F/K)$ and not of $C^*(F/C) \otimes K$. These last groups, as we have seen in Lemma 5.5, are isomorphic with the homology groups of $\mathrm{Hom}^{\mathcal{H}}(C(\mathcal{G}), F^*)$ and the former with the homology groups of $\mathrm{Hom}^{\mathcal{H}}(C(\mathcal{H}), F^*)$. Now, for groups we know that both the complex $C(\mathcal{H})$ and $C(\mathcal{G})$ can be used to compute $H^n(\mathcal{H}, F^*)$ while for the fields the complexes $C^*(F/K)$ and $C^*(F/C \otimes K)$ yield the groups $H^*(F/K)$ and $H^*(F \otimes K/K)$ which we do not know if they are isomorphic for arbitrary extensions $F \supseteqq K \supseteqq C$. Though in some case it is known (Theorem 2.8 and Theorem 5.12 in the end of this section) to be true.

From Theorem 2.6 we have a homomorphism $\mu^*: H^*(F \otimes K/K) \to H^*(F/K)$ induced by the map $\mu(x \otimes k) = xk$ for $x \otimes k \in F^n \otimes K$. Using the same methods of the proof of Theorems 5.6 and 5.7, one readily verifies that: $\tau_F\mu\eta^{-1}$: $\mathrm{Hom}^{\mathcal{H}}(C(\mathcal{G}), F^*) \to C^*(F/C) \otimes K \to C^*(F/K) \to \mathrm{Hom}^{\mathcal{H}}(C(\mathcal{G}), F^*)$ is the restriction of $f \in \mathrm{Hom}^{\mathcal{H}}(C(\mathcal{G}), F^*)$ to $C(\mathcal{H})$. This map is known to induce an isomorphism $H^n(\mathrm{Hom}^{\mathcal{H}}(C(\mathcal{G}), F^*)) \cong \mathrm{Hom}^{\mathcal{H}}(C(\mathcal{H}), F^*)$ which clearly yields that in this case $\mu^*: H^n(F \otimes K/K) \to H^n(F/K)$ is also an isomorphism. We shall see that this is a special case of a more general result (Theorem 5.2).

To conclude the analogue with the classical groups we have to consider the "lift" map and the groups appearing in the fundamental exact sequence.

For the lift map we have:

THEOREM 5.9. *Let $\mathcal{H}$ be a normal subgroup of $\mathcal{G}$ then the composite map:*

$$\tau_F\lambda\tau_K^{-1}: \mathrm{Hom}^{\mathcal{G}/\mathcal{H}}(C(\mathcal{G}/\mathcal{H}), K^*) \longrightarrow C(K/C) \longrightarrow C(F/C) \longrightarrow \mathrm{Hom}^{\mathcal{G}}(C(\mathcal{G}), F^*)$$

induces the lift map: $H^n(\mathcal{G}/\mathcal{H}, K^*) \to H^n(\mathcal{G}, F^*)$.

PROOF. For $f \in \mathrm{Hom}^{\mathcal{G}/\mathcal{H}}(C(\mathcal{G}/\mathcal{H}), K^*)$, $\tau_K^{-1}f = x \in K^n$ where $f(\bar{\varphi}) = (x, \bar{\varphi})$ with $\bar{\varphi} = (\bar{\varphi}_1, \cdots, \bar{\varphi}_n)$, $\bar{\varphi}_i \in \mathcal{G}/\mathcal{H}$. Now $\lambda x = x \in F^n$, and thus $(\tau_F\lambda\tau_K^{-1})f = \tau_F x = g \in \mathrm{Hom}^{\mathcal{G}}(C(\mathcal{G}), F^*)$ is given by $g(\varphi) = (x, \varphi)$ for $\varphi \in \Phi_n = (\mathcal{G}, \cdots, \mathcal{G})$. But since

$x \in K^n$ we see that $g(\varphi)$ depends only on the classes $\bar{\varphi}$. i.e. $g(\varphi)=f(\bar{\varphi})$ thus, $f \rightarrow g$ is the known "lift" map. q.e.d.

The last result in this connection is:

THEOREM 5.10. *Let $\mathcal{H}$ be a normal subgroup of $\mathcal{G}$ then we have the isomorphism:* $H^*(F \otimes K/K)^0 \cong H^*(\mathcal{H}, F^*)^{\mathcal{G}}$.

In fact we shall prove more, that this isomorphism is induced by the isomorphism $H^*(F \otimes K/K) \rightarrow H^*(\mathcal{H}, F^*)$ discussed above.

Let $(\Phi_n, \bar{\mathcal{G}})$ and $(\Phi_n, \bar{\mathcal{G}}, \bar{\mathcal{G}})$ be the set of $\{(\varphi, \bar{\alpha})\}$ and $\{(\varphi, \bar{\alpha}, \bar{\beta})\}$, $\varphi \in \Phi_n$ and $\bar{\alpha}$, $\bar{\beta}$ cosets of $\mathcal{G}/\mathcal{H}$. Consider the commutative diagram:

$$\begin{array}{ccc} (F^n \otimes K)^* & \xrightarrow{\tau} & \mathrm{Hom}^{\mathcal{G}}((\Phi_n \bar{\mathcal{G}}), F^*) \\ {\scriptstyle \varepsilon_i^K} \downarrow & & \downarrow \\ (F^n \otimes K^2)^* & \xrightarrow{\tau} & \mathrm{Hom}\,((\Phi_n \bar{\mathcal{G}}, \bar{\mathcal{G}}), F^*)\,. \end{array}$$

The last vertical map is by definition $\tau\varepsilon_i^K\tau^{-1}$, and τ is the map $\tau(x)=p_x$ of Lemma 5.4.

Let $f \in \mathrm{Hom}^{\mathcal{G}}(\Phi_n, \bar{\mathcal{G}})$ then by definition τ and by (5.4) it follows that

$$\begin{aligned}[][(\tau\varepsilon_1^K\tau^{-1})f](\varphi, \bar{\alpha}_1, \bar{\alpha}_2) &= p\varepsilon_1^K\tau^{-1}f(\varphi, \bar{\alpha}_1, \bar{\alpha}_2) = (\varepsilon_1^K\tau^{-1}f, (\varphi, \bar{\alpha}_1, \bar{\alpha}_2)) \\ &= (\tau^{-1}f, \sigma_1^K(\varphi, \bar{\alpha}_1, \bar{\alpha}_2)) = (\tau^{-1}f, (\varphi_1, \bar{\alpha}_2)) \\ &= f(\varphi_1, \bar{\alpha}_2) \end{aligned}$$

and similarly $(\tau\varepsilon_2^K\tau^{-1})f(\varphi, \bar{\alpha}_1, \bar{\alpha}_2)=f(\varphi_1, \bar{\alpha}_1)$.

Now $\Delta_K: F^n \otimes K \rightarrow F^n \otimes K^2$ is $\varepsilon_1^K(\varepsilon_2^K)^{-1}$. Hence

$$(5.6) \qquad (\tau\Delta_K\tau^{-1}f)(\varphi, \bar{\alpha}_1, \bar{\alpha}_2)=f(\varphi, \bar{\alpha}_1)f(\varphi, \bar{\alpha}_2)^{-1}\,.$$

Next we consider the isomorphism $\zeta: \mathrm{Hom}^{\mathcal{G}}((\Phi_n, \bar{\mathcal{G}}), F^*) \rightarrow \mathrm{Hom}^{\mathcal{H}}(\Phi_n, F^*)$ given in the proof of Lemma 5.5. Namely $(\zeta f)(\varphi)=f(\varphi, \bar{1})$ and similarly $(\zeta f)(\varphi, \bar{\alpha})=f(\varphi, \bar{1}, \bar{\alpha})$ will be an isomorphism of $\mathrm{Hom}^{\mathcal{G}}((\Phi, \bar{\mathcal{G}}, \bar{\mathcal{G}}), F^*) \rightarrow \mathrm{Hom}^{\mathcal{H}}((\Phi_n, \mathcal{G}), F^*)$. Consider now $\zeta\tau\Delta_K(\zeta\tau)^{-1}f$ for $f \in \mathrm{Hom}^{\mathcal{H}}(\Phi, F^*)$, by (5.6) it follows that:

$$(7.8) \qquad \begin{aligned}[][(\zeta\tau\Delta_K\tau^{-1}\zeta^{-1})f](\varphi, \bar{\alpha}) &= (\tau\Delta_K\tau^{-1}\zeta^{-1}f)(\varphi, \bar{1}, \bar{\alpha}) \\ &= (\zeta^{-1}f)(\varphi, \bar{1})[(\zeta^{-1}f)(\varphi, \bar{\alpha})]^{-1} \\ &= f(\varphi)[\alpha f(\alpha^{-1}\varphi)]^{-1} = f(\varphi)(\alpha_c f(\varphi)^{-1} \end{aligned}$$

since $\zeta^{-1}f(\varphi, \bar{\alpha})=\alpha(\zeta^{-1}f)(\alpha^{-1}\varphi, \bar{1})=\alpha f(\alpha^{-1}\varphi)=\alpha_c f$.

Since $\zeta\tau$ induces isomorphism of the respective homology groups, hence $(\zeta\tau)^*$ induces an isomorphism between Ker Δ_K^* and Kernel $(\zeta\tau\Delta_K(\zeta\tau)^{-1})^*$! The first is by definition $H^*(F \otimes K/K)^0$ and the latter is, by (5.7) the set of all cocycles of $H^*(\mathcal{H}; F^*)$ for which $\alpha_c f \sim f$ i.e. $H^*(\mathcal{H}; F^*)^{\mathcal{G}}$ and the proof is thus concluded.

Summarizing the last three theorems, and noticing that all the isomorphisms involved commute, the existence of the fundamental exact sequence ([4]) is readily obtained from Theorem 4.2.

COROLLARY 5.11. *If $\mathcal{H}$ is normal in $\mathcal{G}$ then the following sequence is exact:*

$$0 \longrightarrow H^2(\mathcal{G}/\mathcal{H}, F^*) \longrightarrow H^2(\mathcal{G}, F^*) \longrightarrow H^2(\mathcal{H}, F^*)^{\mathcal{G}}$$
$$\longrightarrow H^3(\mathcal{G}/\mathcal{H}, F^*) \longrightarrow H^3(\mathcal{G}, F^*).$$

We close the paper with a case where we can prove the validity of the condition of Theorem 2.8; hence for which $H^*(F/K) \cong H^*(F \otimes K/K)$.

THEOREM 5.12. *Let $F \supseteq K \supseteq C$ be finite algebraic extension of a field C and such that K is separable over C*[9]*; and suppose that each of the fields $(\psi F)K$, which is generated by K and by conjugate ψF of F, contains F (e.g. if F is normal) then there exists a homomorphism $\vartheta : F \to F \otimes K$ such that $k = 1 \otimes k$ for all $k \in K$. Clearly, an application of Theorem 2.8, yields now the fact that if; F is normal then $H^*(F \otimes K/K) \cong H^*(F/K)$.*

PROOF. Let $\Phi_2 = \{(\varphi_1, \varphi_2), \varphi_1 \in G(F/C), \varphi_2 \in G(K/C)\}$ (in the notations of Lemma 5.4) and let $\mathcal{G} = \mathcal{G}(L/C)$ where L is any normal extension of C containing F. It follows from Lemma 5.5 that $\tau : F \otimes K \to \mathrm{Hom}^{\mathcal{G}}(\Phi_2, L)$ is an isomorphism. Let $\mathcal{H} \subseteq \mathcal{G}$ be the subgroup of $\mathcal{G}$ leaving the elements of K invariant, and let $\mathcal{G}(F/C) = \cup \mathcal{H}\varphi_i$.

First we observe that every element of Φ_2 can be expressed in the form $(\varphi, \psi) = \alpha(\varphi_i, 1)$. Indeed, let α_0 be any element of $\mathcal{G}$ such that $\alpha | K = \psi$ then $(\varphi, \psi) = \alpha_0(\alpha_0^{-1}\varphi, 1) = \alpha(\varphi_i, 1)$ with $\alpha_0^{-1}\varphi = h\varphi_i$ and $\alpha = \alpha_0 h$. Next, we prove that if $\alpha(\varphi_i, 1) = \beta(\varphi_i, 1)$ then $i = j$ and $\alpha(a) = \beta(a)$ for all $a \in F$. Indeed, the fact that $\alpha\varphi_i = \beta\varphi_j$ yields that $i = j$ by the definition of φ_i's, and that $\alpha | K = \beta | K$ i.e. $\beta = \alpha h$. This in turn yields $h\varphi_i = \varphi_i$ so that $h\varphi_i(a) = \varphi_i(a)$ for all $a \in F$. In other words, h leaves $\varphi_i(F)$ invariant, but it leaves also K invariant since $h \in \mathcal{H}$. Consequently, $\varphi_i(F)K$ is invariant under h and therefore $h(a) = a$ for all $a \in F \subseteq \varphi_i(F)K$ by assumption. Thus, we conclude $\beta(a) = \alpha h(a) = \alpha(a)$ for all $a \in F$ and our assertion is proved.

We recall that $\tau : F \otimes K \to \mathrm{Hom}^{\mathcal{G}}(\Phi_2, L^*)$ is given by $(\tau x)(\varphi, \psi) = p_x(\varphi, \psi) = (x, (\varphi, \psi))$. For $a \in F \otimes K$ we define $\vartheta(a)$ by the relation:

$$(5.8) \qquad (\tau\vartheta a)(\varphi, \psi) = \alpha a \quad \text{where} \quad (\varphi, \psi) = \alpha(\varphi_i, 1).$$

From the result obtain above, we can show that the function $f(\varphi, \psi) = \alpha(a)$ is a well defined element in $\mathrm{Hom}^{\mathcal{G}}(\Phi_2, L^*)$. Indeed, $f(\gamma\varphi, \gamma\psi) = \gamma f(\varphi, \psi) = (\gamma\alpha)a$ since $(\gamma\varphi, \gamma\psi) = \gamma\alpha(\varphi_i, 1)$ and if $(\varphi, \psi) = \alpha(\varphi_i, 1) = \beta(\varphi_j, 1)$ then $\alpha(a) = \beta(a)$ by the previous result.

Consequently, ϑ is uniquely determined, and it is readily seen that ϑ is a homomorphism.

---

9) F is not necessarily separable.

Now for $k \in K$,

$$[\tau(1\otimes k)](\varphi, \psi) = \tau(1\otimes k)[\alpha(\varphi_i, 1)] = \alpha[\tau(1\otimes k)(\varphi_i, 1)]$$
$$= \alpha(1\otimes k, (\varphi_i, 1)) = \alpha\varphi_i(1)k$$
$$= \alpha k = (\tau\vartheta k)(\varphi, \psi).$$

Hence $\vartheta k = 1\otimes k$, and the proof is completed.

Another evident case where such ϑ exists is that the field $F = H\otimes K$ is the tensor product of two fields since then $\vartheta : F \to F\otimes K$ is merely the map we denoted earlier as $\varepsilon_1^K : H\otimes K \to H\otimes K\otimes K$.

Hebrew University
Jerusalem, ISRAEL

References

[1] S. A. Amitsur, Simple algebras and cohomology groups of arbitrary fields, Trans. Amer. Math. Soc., **90** (1959), 73–112.
[2] N. Bourbaki, Elements de mathematique, Livre II Algebra ch. 8 Herman, 1958.
[3] H. Cartan and S. Eilenberg, Homological algebra, Princeton Univ. Press, 1956.
[4] G. Hochschild and J. P. Serre, Cohomology of group extensions, Trans. Amer. Math. Soc., **57** (1953), 591–603.
[5] A. Rosenberg and D. Zelinsky, On Amitsur's complex, Trans. Amer. Math. Soc., **97** (1960), 327–356.
[6] T. Nakayama, Note on the fundamental exact sequences in homology and cohomology for non normal subgroups, Proc. Japan Acad., **10** (1958), 661–663.
[7] T. Nakayama, Fundamental exact sequences in (co)-homology for for non normal subgroups, Nagoya Math. J., **18** (1961), 63–92.
[8] A. Hattori, On fundamental exact sequences, J. Math. Soc. Japan, **12** (1960), 65–80.

Added in proof: Rosenberg and Zelinsky have pointed out to the author that instead of the assumptions of the freeness of F over C and that the unit 1 is one of the generators of F over C, which are used in Sections 1–4, it suffices to assume that F is C-flat and that the unit map: $C \to F$ splits. The modifications of the proof are as in [**5**].