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On some criteria for p-valence

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1. Preliminaries.

W. Kaplan [1] defined a close-to-convex function for $|z|<1$ as follows.

Let $f(z)$ be analytic for $|z|<1$. Then $f(z)$ is close-to-convex for $|z|<1$, if there exists a convex and schlicht function $\phi(z)$ for $|z|<1$, such that $f'(z)/\phi'(z)$ has positive real part for $|z|<1$. Furthermore he showed that $f(z)$ is close-to-convex if and only if

$$\int_{\theta_1}^{\theta_2} \Re\left(1+z\frac{f''(z)}{f'(z)}\right)d\theta > -\pi, \tag{1}$$

where $\theta_1<\theta_2$, $z=re^{i\theta}$ and $r<1$.

Lately T. Umezawa [2] obtained some criteria for univalence as follows.

Let $w=f(z)$ be regular in a simply connected closed region D_z whose boundary Γ_z consists of a regular curve and suppose $f'(z)\neq 0$ on Γ_z. If there holds one of the following conditions:

(i) For arbitrary arcs C_z on Γ_z

$$\int_{C_z} d\arg df(z) > -\pi \quad \text{and} \quad \int_{\Gamma_z} d\arg df(z) = 2\pi, \tag{2}$$

(ii) For arbitrary arcs C_z on Γ_z

$$\int_{C_z} d\arg df(z) < 3\pi,$$

then $f(z)$ is univalent in D_z.

As M. O. Reade [3] specified, above two criteria (1) and (2) are essentially equivalent to each other. In this paper we show some generalization of these criteria and some extension concerning p-valent functions.

2. Main criterion.

We can generalize above criteria for univalence as follows [4].

MAIN CRITERION. *Let $w=f(z)$ be regular on a simply connected closed domain D_z, whose boundary C_z consists of a regular curve and suppose $f'(z)\neq 0$ on C_z. If there holds the following condition for a suitable real function $\varphi(w)$, which is a single-valued and differentiable function of w, and for a suitable real* *constant k,*

(3) $$\int_{C_z} d\arg df(z) = 2\pi \qquad z \in C_z$$

and

(4) $$\int_{C_z'} [d\arg df(z) + kd\varphi(f(z))] > -\pi \qquad z \in C_z',$$

where C_z' is an arbitrary arc on C_z and the integration is taken in the positive direction with respect to D_z, then $f(z)$ is univalent in D_z.

The inequality (4) has the following geometrical meaning. Let L_t be the level curve of $\varphi(w)$ for $\varphi(w)=t$ on w-plane, and w_1 and w_2 be the intersections of L_t and C_w, where C_w means the map of C_z. Then the argument of the tangent of C_w at w_2 never drops to a value π radians below the previous value at w_1.

3. The case in which $\varphi(f(z)) = \arg f(z)$.

For $\varphi(f(z))$ we may use various real functions. For example we can put $\varphi(f(z)) = \Re f(z)$ or $\varphi(f(z)) = \Im f(z)$ or more generally $\varphi(f(z)) = \Im(e^{i\omega} f(z))$ [4]. In this paper we show some results obtained in the case in which $\varphi(f(z)) = \arg f(z)$.

LEMMA 1. *Let us denote by D_z a simply connected closed domain including $z=0$ in it and by C_z the boundaary of D_z. Let $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be regular on D_z and $f(z) \neq 0$ $(z \neq 0)$, $f'(z) \neq 0$ on D_z. If $w=f(z)$ is multivalent for D_z, then C_z has at least one arc C_z' such that both*

(5) $$\int_{C_z'} d\arg df(z) \leqq -\pi$$

and

(6) $$\int_{C_z'} d\arg f(z) = 0 \qquad z \in C_z'$$

hold.

PROOF. Let $z=\phi(\zeta)$ be the univalent regular function which maps the unit circle $|\zeta| \leqq 1$ onto D_z and suppose $\phi(0)=0$. Let us denote by $L_z(\rho)$ the map of $|\zeta|=\rho$ under $z=\phi(\zeta)$, by $L_w(\rho)$ the map of $L_z(\rho)$ under $w=f(z)$ and by $D_w(\rho)$ the region bounded by $L_w(\rho)$. We remark that, since $f'(z)$ never vanishes, for arbitrary ρ_1 and ρ_2 $(\rho_1 < \rho_2)$ $D_w(\rho_2)$ contains $D_w(\rho_1)$ entirely in it.

Now we observe that ρ increases monotonously from 0 to 1. When ρ is sufficiently small, it is clear that $D_w(\rho)$ is univalent containing $w=0$ in it. Let ρ_0 be such a radius that $D_w(\rho)$ is univalent for $0<\rho<\rho_0$ and $D_w(\rho)$ is not univalent for $\rho_0<\rho$. For such ρ_0, $D_w(\rho_0)$ has at least one self-touching point. Let $w_0=f(z_1)=f(z_2)$ be such a point and for any z' and z'' $(\arg\phi^{-1}(z_1) < \arg\phi^{-1}(z') < \arg\phi^{-1}(z'') < \arg\phi^{-1}(z_2))$ suppose $f(z') \neq f(z'')$ on $L_w(\rho_0)$. On the loop $L_w'(\rho_0)$, the partial arc of $L_w(\rho_0)$ bounded by $f(z_1)$ and $f(z_2)$, w moves from $f(z_1)$ to $f(z_2)$ with a clock-wise direction when z moves from z_1 to z_2 on $L_z(\rho_0)$. Here we remark that the loop $L_w'(\rho_0)$ is simple and the inner region $\Delta_w'(\rho_0)$, bounded by $L_w'(\rho_0)$, does not contain $w=0$ in it. From this we have

$$\int_{L_z'(\rho_0)} d\arg df(z) = -\pi$$

and

$$\int_{L_z'(\rho_0)} d\arg f(z) = 0 \qquad z \in L_z'(\rho_0),$$

where $L_z'(\rho_0)$ is the map of $L_w'(\rho_0)$.

When ρ tends to 1 exceeding ρ_0, the region $\Delta_w'(\rho)$ may reduce with the direction of the positive normal of $L_w(\rho)$ and some parts of $L_w'(\rho)$ may self-overlap, but $\Delta_w'(\rho)$ cannot reduce to a point in accordance with $f'(z) \neq 0$ for $z \in D_z$. Hence, there should finally remain at least one simple loop $C'(w)$ in $\Delta_w'(\rho_0)$, which has a clock-wise encircling direction and clearly does not contain $w=0$ inside it. Denoting by $C'(z)$ the map of $C'(w)$, we have (5) and (6) for such $C'(z)$.

From Main Criterion and Lemma 1 we have following theorem immediately.

THEOREM 1. *Let us denote by D_z a simply connected closed domain including $z=0$ in it and by C_z the boundary of D_z. Let $w=f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ be regular on D_z and $f(z) \neq 0$ $(z \neq 0)$, $f'(z) \neq 0$ on D_z. If there holds for a suitable real constant k*

$$\int_{C_z'} [d\arg df(z) + k d\arg f(z)] > -\pi \qquad z \in C_z',$$

where C_z' is an arbitrary arc on C_z, then $f(z)$ is univalent on D_z.

REMARK 1. In this theorem k has to satisfy $k > -\frac{3}{2}$, because for C_z we have

$$\int_{C_z} [d\arg df(z) + k d\arg f(z)] = 2\pi(1+k) > -\pi .$$

REMARK 2. In this theorem we may modify above inequality as follows;

$$\int_{C_z'} [d\arg df(z) + k d\arg f(z)] > -\alpha\pi \qquad \left(1 \geqq \alpha \geqq 0,\ k > -\frac{\alpha}{2} - 1\right).$$

In this inequality, for $k=0$, $\alpha=0$, it coincides with the condition that $f(z)$ should be convex. For $k=0$, $\alpha=1$, $f(z)$ should be close-to-convex, and for $k=+\infty$, $f(z)$ should be star-like.

4. Extension to p-valence.

Theorem 1 may be extended to the case of p-valence. For this purpose we shall generalize Lemma 1.

LEMMA 2. *Let us denote by D_z a simply connectted closed domain including $z=0$ in it and by C_z the boundary of D_z. Let $w=f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n$ be regular on D_z and $f(z)\neq 0$, $f'(z)\neq 0$ except at $z=0$ on D_z. If $f(z)$ is at least $(p+1)$-valent, then C_z has at least one arc C_z' such that both*

$$\int_{C_z'} d\arg df(z) \leqq -\pi \tag{7}$$

and

$$\int_{C_z'} d\arg f(z) = 0 \qquad z\in C_z' \tag{8}$$

holds.

PROOF. Let us apply the same notations as in Lemma 1. When ρ is sufficiently small, $D_w(\rho)$ is p-valent with the branch point of $(p-1)$-th order at $w=0$. Thus we suppose a p-sheeted Riemann surface as a "basic surface" and we denote by Σ this surface and by $D_w^*(\rho)$ the domain $D_w(\rho)$ developed on Σ. We remark that since $f'(z)$ never vanishes except for $z=0$, for arbitrary ρ_1 and ρ_2 $(\rho_1<\rho_2)$ $D_w(\rho_2)$ contains $D_w(\rho_1)$ entirely in it.

When ρ is sufficiently small, $D_w^*(\rho)$ is univalent on each sheet of Σ. Let us suppose that $D_w^*(\rho)$ becomes two-valent on some sheet of Σ. Then as in Lemma 1 there is such a radius ρ_0 that $D_w^*(\rho)$ is univalent for ρ smaller than ρ_0 and $D_w^*(\rho)$ is no longer univalent for ρ greater than ρ_0. Thus we have some loop of $L_w^*(\rho_0)$, the boundary of $D_w^*(\rho_0)$, which is simple and does not contain $w=0$. When ρ exceeds ρ_0 and tends to 1, there remains at least one loop C_w' which is simple and does not contain $w=0$ and has a clock-wise encircling direction. For this C_w' hold (7) and (8).

On the other hand if D_w has a part which is at least $(p+1)$-valent, then D_w^* is at least two-valent on some sheet of Σ. Thus we have this lemma.

From this lemma we have next theorem immediately.

THEOREM 2. *Let us denote by D_z a simply connected closed domain including $z=0$ in it and by C_z the boundary of D_z. Let $w=f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n$ be regular on D_z and $f(z)\neq 0$, $f'(z)\neq 0$ except at $z=0$ on D_z. If there holds for a suitable real constant k,*

$$\int_{C_z'}[d\arg df(z)+k\,d\arg f(z)] > -\pi \qquad z\in C_z', \tag{9}$$

where C_z' is an arbitrary arc on C_z, then $f(z)$ is p-valent on D_z.

REMARK. In this theorem k has to satisfy $k>-\dfrac{1}{2p}-1$, because for C_z we have

$$\int_{C_z}[d\arg df(z)+kd\arg f(z)] = 2\pi p(1+k) > -\pi\,.$$

5. Some applications of Theorem 2.

THEOREM 3. *Let D_z and $w=f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n$ satisfy the hypothesis of Theorem 2. If there holds for a suitable convex function (may be multivalent) $\phi(z)$ and for real constants α and k,*

$$\Re e^{i\alpha}\frac{f'(z)f(z)^k}{\phi'(z)}>0 \qquad z\in D_z\,, \tag{10}$$

then $f(z)$ is p-valent for D_z.

PROOF. Let C_z' be an arbitrary arc on C_z and z_1, z_2 be the initial and end point respectively. Then from (10) we have

$$\arg\frac{f'(z_2)f(z_2)^k}{\phi'(z_2)}-\arg\frac{f'(z_1)f(z_1)^k}{\phi'(z_1)}>-\pi\,.$$

Thus we have

$$[\arg df(z_2)+k\arg f(z_2)]-[\arg df(z_1)+k\arg f(z_1)] \\ -[\arg d\phi(z_2)-\arg d\phi(z_1)]>-\pi\,. \tag{11}$$

Since $\phi(z)$ is convex, we have

$$\arg d\phi(z_2)>\arg d\phi(z_1)\,. \tag{12}$$

By (11) and (12) we have

$$[\arg df(z_2)+k\arg f(z_2)]-[\arg df(z_1)+k\arg f(z_1)]>-\pi\,,$$

$$\int_{C_z'}[d\arg df(z)+kd\arg f(z)]>-\pi\,.$$

Thus by Theorem 2 we see that $f(z)$ is p-valent.

THEOREM 4. *Let D_z and $w=f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n$ satisfy the hypothesis of Theorem 2 and furthermore let D_z be convex. Then the following condition is sufficient for p-valence of $f(z)$ for D_z:*

$$\Re e^{i\alpha}\frac{f'(z)f(z)^k}{\psi(z)}>0\,, \tag{13}$$

where $\psi(z)$ is a suitable star-like function with respect to $z=0$ (may be multivalent).

PROOF. As in Theorem 3 following inequalities hold:

$$\arg\frac{f'(z_2)f(z_2)^k}{\psi(z_2)}-\arg\frac{f'(z_1)f(z_1)^k}{\psi(z_1)}>-\pi\,,$$

$$[\arg df(z_2)+k\arg f(z_2)]-[\arg df(z_1)+k\arg f(z_1)] \\ -[\arg\psi(z_2)-\arg\psi(z_1)]-[\arg dz_2-\arg dz_1]>-\pi\,.$$

Observing that $\psi(z)$ is star-like and D_z is convex, we have

$$\arg\psi(z_2)>\arg\psi(z_1)\,, \qquad \arg dz_2>\arg dz_1\,.$$

This implies

$$[\arg df(z_2)+k\arg f(z_2)]-[\arg df(z_1)+k\arg f(z_1)]>-\pi\,,$$

and this yields (9).

COROLLARY 1. *Let D_z in Theorem 4 be the closed disc $|z|\leqq r$. If there holds any one of the following inequalities for a suitable convex function $\phi(z)$ or star-like function $\psi(z)$ (both may be multivalent) and for real constants α and k,*

$$\text{(10)}\qquad \Re e^{i\alpha}\frac{f'(z)f(z)^k}{\phi'(z)}>0\,,\qquad\qquad \text{(10}'\text{)}\qquad \Re e^{i\alpha}\frac{zf'(z)f(z)^k}{\psi(z)}>0\,,$$

$$\text{(13)}\qquad \Re e^{i\alpha}\frac{f'(z)f(z)^k}{\psi(z)}>0\,,\qquad\qquad \text{(13}'\text{)}\qquad \Re e^{i\alpha}\frac{f'(z)f(z)^k}{z\phi'(z)}>0\,,$$

then $f(z)$ is p-valent for D_z.

PROOF. The well-known relation:

$$F(z) \text{ is convex} \rightleftarrows zF'(z) \text{ is star-like}$$

yields (10′) from (10) and (13′) from (13) immediately.

COROLLARY 2. *Let $w=f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n$ be regular for $|z|\leqq r$ and $f(z)\neq 0$, $f'(z)\neq 0$ except for $z=0$. If for some positive integer n $f(z)^n$ be close-to-convex (multivalent except for $p=1$, $n=1$), then $f(z)$ is p-valent for $|z|\leqq r$.*

PROOF. Putting $k=n-1$ we have

$$\Re e^{i\alpha}\frac{f'(z)f(z)^{n-1}}{\phi'(z)}=\frac{1}{n}\Re e^{i\alpha}\frac{[f(z)^n]'}{\phi'(z)}\,.$$

This means that $f(z)^n$ is close-to-convex (may be multivalent).

COROLLARY 3. *Let $f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n$ be regular for $|z|\leqq r$. If there holds*

$$\text{(14)}\qquad\qquad \Re e^{i\alpha}\frac{zf'(z)}{f(z)}>0\,,$$

then $f(z)$ is p-valent (p-valent spiral-like).

PROOF. As easily seen from the assumption, $f(z)$ and $f'(z)$ can not vanish except for $z=0$ for $|z|\leqq r$. Thus $f(z)$ satisfies the assumption of Corollary 1. Thus putting $\psi(z)=1$, $k=-1$ in (10′) we have (14).

We may obtain various sufficient conditions for p-valence substituting appropriate concrete star-like or convex function into ψ or ϕ respectively, for example $\Re e^{i\alpha}\frac{f'(z)}{z^{p-1}}>0$ [**2**, p. 226], but the details are omitted here.

6. Some sufficient conditions for p-valence.

In this section we show some sufficient conditions for p-valence, following the idea introduced by S. Ozaki [**5**] and T. Umezawa [**6**]. For this purpose we prepare the following lemmas.

LEMMA 3 [4]. *Let* $h(re^{i\theta})$ *be a real function continuous for* $0 \leqq \theta \leqq 2\pi$ *satisfying the following for some positive number* $m\left(m > \frac{1}{2}\right)$,

$$(16) \qquad -m < h(re^{i\theta}) < \frac{(2h_0+1)m}{2m-1} \qquad (0 \leqq \theta \leqq 2\pi),$$

where $h_0 = \frac{1}{2\pi}\int_0^{2\pi} h(re^{i\theta})d\theta$ *and* $h_0 > -\frac{1}{2}$, *then for arbitrary interval* C *of* θ (*or the sum of these intervals*) *on* $[0, 2\pi]$ *there holds*

$$(17) \qquad \int_C h(re^{i\theta})d\theta > -\pi .$$

LEMMA 4. *Let* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ (p: *positive integer*) *be meromorphic for* $|z| \leqq r$ *and satisfy*

$$(18) \qquad \Re\left[e^{i\alpha}\left(1+z\frac{f''(z)}{f'(z)}+kz\frac{f'(z)}{f(z)}\right)\right] > K$$

for suitable real constants α, K *and* k, *where* $k \neq -\frac{q-1}{q}$ *for any integer* q, *then* $f(z)$ *is regular for* $|z| \leqq r$ *and* $f(z) \neq 0$, $f'(z) \neq 0$ *for* $0 < |z| \leqq r$.

PROOF. Let us assume that $f(z)$ has zero or pole of $|q|$-th order at $z = z_0$ ($z_0 \neq 0$). Then we can put

$$f(z) = z^p(z-z_0)^q g(z) \qquad g(0) \neq 0, \infty, \qquad g(z_0) \neq 0, \infty,$$
$$F(z) \equiv zf'(z) = z^p(z-z_0)^{q-1}G(z),$$

where $G(z) = p(z-z_0)g(z) + qzg(z) + z(z-z_0)g'(z)$, $G(0) \neq 0, \infty$, $G(z_0) \neq 0, \infty$. An elementary calculation shows

$$h(z) \equiv 1+z\frac{f''(z)}{f'(z)}+kz\frac{f'(z)}{f(z)} = z\frac{F'(z)}{F(z)}+kz\frac{f'(z)}{f(z)}$$
$$= p(1+k)+(q-1+kq)\frac{z}{z-z_0}+z\frac{G'(z)}{G(z)}+z\frac{g'(z)}{g(z)} .$$

Since $q-1+kq$ never vanishes, we see that $h(z)$ has a pole at $z = z_0$ and $h(z) \to \infty$ for $z \to z_0$. This contradicts (18).

THEOREM 6. *Let* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ (p: *positive integer*) *be meromorphic for* $|z| \leqq r$ *and satisfy*

$$(19) \qquad -m < \Re\left[1+z\frac{f''(z)}{f'(z)}+kz\frac{f'(z)}{f(z)}\right] < \frac{[2(k+1)p+1]m}{2m-1}$$

for real constants m $\left(m > \frac{1}{2}\right)$ *and* k $\left(k > -\left(\frac{1}{2p}+1\right)\right.$, $k \neq -\frac{q-1}{q}$ *for any integer* q), *then* $f(z)$ *is regular and* p-*valent for* $|z| \leqq r$.

PROOF. From Lemma 4, we see that $f(z)$ is regular for $|z|\leqq r$ and $f(z)\neq 0$, $f'(z)\neq 0$ for $0<|z|\leqq r$. Then $f(z)$ satisfies the assumption of Theorem 2. As is well-known [5, p. 49], we have

$$d\arg df(z)=\Re\Big(1+z\frac{f''(z)}{f'(z)}\Big)d\theta\,,$$

$$d\arg f(z)=\Re\Big(z\frac{f'(z)}{f(z)}\Big)d\theta$$

for $z=re^{i\theta}$. Thus we have

$$\int_{|z|=r}[d\arg df(z)+kd\arg f(z)]=\int\Re\Big[1+z\frac{f''(z)}{f'(z)}+kz\frac{f'(z)}{f(z)}\Big]d\theta\,,\qquad (z=re^{i\theta})\,.$$

Since $h(z)\equiv\Re\Big[1+z\frac{f''(z)}{f'(z)}+kz\frac{f'(z)}{f(z)}\Big]$ is harmonic for $|z|\leqq r$ by above statement,

$$h_0=\frac{1}{2\pi}\int_0^{2\pi}h(re^{i\theta})d\theta=h(0)=(k+1)p\,.$$

Thus we see that (19) is equivalent to (16) and so (16) yields (17). (17) means that $f(z)$ satisfies (9) in Theorem 2. Hence $f(z)$ is p-valent for $|z|\leqq r$.

COROLLARY 4. *In Theorem 6 we may replace (19) with any one of the following conditions,*

$$\Re\Big[1+z\frac{f''(z)}{f'(z)}+kz\frac{f'(z)}{f(z)}\Big]<(k+1)p+\frac{1}{2}\,,\tag{20}$$

$$\Re\Big[1+z\frac{f''(z)}{f'(z)}+kz\frac{f'(z)}{f(z)}\Big]>-\frac{1}{2}\,,\tag{21}$$

$$\Big|\Re\Big[1+z\frac{f''(z)}{f'(z)}+kz\frac{f'(z)}{f(z)}\Big]\Big|<(k+1)p+1\,,\tag{22}$$

$$\Big|\Re\Big[z\frac{f''(z)}{f'(z)}+kz\frac{f'(z)}{f(z)}\Big]\Big|<\frac{(k+1)p+1+\sqrt{\{(k+1)p-1\}^2+4}}{2}\,.\tag{23}$$

PROOF. The following special cases of (19) give (20)~(23) respectively:

$$m\to+\infty \qquad (20)\,,$$

$$m\to\frac{1}{2} \qquad (21)\,,$$

$$m=\frac{[2(k+1)p+1]m}{2m-1} \qquad (22)\,,$$

$$m+1=\frac{[2(k+1)p+1]m}{2m-1}-1 \qquad (23)\,.$$

REMARK. Putting $p=1$ and $k=0$ in Corollary 4, we have Ozaki's criteria for univalence [5, p. 56]. Putting $k=0$, we have Ozaki's criteria for p-valence as the special case $k=p$ in his Theorem 3 [5, p. 57].

7. Some extension of radius of convexity.

In this section we consider a function $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$, which is regular and univalent for $|z|<1$. As a sufficient condition that $f(z)$ should satisfy

$$\int [d\arg df(z)-kd\arg f(z)] > -\alpha\pi \qquad (|z|=r<1)$$

or

$$\int_{\theta_1}^{\theta_2} \Re\Big[1+z\frac{f''(z)}{f'(z)}-kz\frac{f'(z)}{f(z)}\Big]d\theta > -\alpha\pi \qquad (z=re^{i\theta}, \theta_1\leqq\theta\leqq\theta_2),$$

we have

$$\Re\Big[1+z\frac{f''(z)}{f'(z)}-kz\frac{f'(z)}{f(z)}\Big] > -\frac{\alpha}{2}. \tag{24}$$

Now we seek such a radius that (24) should hold. For this purpose we employ the following lemma due to Golusin [7].

LEMMA 5. *Let $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ be regular and univalent for $|z|<1$, then we have*

$$p\Re\Big[1+z\frac{f''(z)}{f'(z)}-\Big(1-\frac{1}{p}\Big)\frac{zf'(z)}{f(z)}\Big] \geqq \frac{1-2(p+1)|z|+|z|^2}{1-|z|^2} \qquad (p\geqq 1). \tag{25}$$

THEOREM 7. *Let $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ be regular and univalent for $|z|<1$. Then $f(z)$ satisfies*

$$\Re\Big[1+z\frac{f''(z)}{f'(z)}-kz\frac{f'(z)}{f(z)}\Big] > -\frac{\alpha}{2} \tag{26}$$

for

(i) $$|z| < \frac{2(2-k)-\sqrt{12-8k+\alpha^2}}{2(1-k)-\alpha} \qquad (2(1-k)-\alpha\neq 0),$$

(ii) $$|z| < \frac{\alpha}{\alpha+2} \qquad (2(1-k)-\alpha=0),$$

where constant k and α satisfy $1\geqq k\geqq 0$, $1\geqq\alpha\geqq 0$.

PROOF. Putting $1-\frac{1}{p}=k$ in (25), we have

$$\Re\Big[1+z\frac{f''(z)}{f'(z)}-kz\frac{f'(z)}{f(z)}\Big] \geqq \frac{1-k-2(2-k)|z|+(1-k)|z|^2}{1-|z|^2}.$$

Thus as a sufficient condition for (26), we have

$$\frac{1-k-2(2-k)|z|+(1-k)|z|^2}{1-|z|^2} > -\frac{\alpha}{2}.$$

This yields

$$F(|z|, k, \alpha) \equiv (2(1-k)-\alpha)|z|^2-4(2-k)|z|+2(1-k)+\alpha > 0.$$

Noticing $F(0, k, \alpha)=2(1-k)+\alpha>0$, $F(1, k, \alpha)=-4$, we have (i) or (ii) in each case.

COROLLARY 5. *Under the same assumption as Theorem 7, we have for* $|z|<\frac{4-\sqrt{12+\alpha^2}}{2-\alpha}$ *the inequalily*

$$\Re\left[1+z\frac{f''(z)}{f'(z)}\right]>-\frac{\alpha}{2}$$

and so

$$\int d\arg df(z)>-\alpha\pi \qquad (|z|=r).$$

This corollary means that for such r, the argument of any tangent on the arc $f(re^{i\theta})$ never drops to a value $\alpha\pi$ radians below the previous value. For example, putting $\alpha=\frac{1}{2}$ we have $|z|<\frac{1}{3}$. Thus we see that for $r<\frac{1}{3}$ the argument of any tangent on the arc $f(re^{i\theta})$ never drops to a value $\frac{\pi}{2}$ radians below the previous value.

COROLLARY 6. *Under the same assumption as Theorem 7, $f(z)$ is convex for* $|z|<2-\sqrt{3}$.

This case corresponds to $\alpha=0$ in Corollary 5.

COROLLARY 7. *Under the same assumption as Theorem 7, let* $|z|<4-\sqrt{13}$. *Then $f(z)$ is close-to-convex or, more precisely, $f(z)$ is convex in one direction* [**6**].

This case corresponds to $\alpha=1$ in Corollary 5. It is known that if $f(z)$ satisfies $\Re\left(1+z\frac{f''(z)}{f'(z)}\right)>-\frac{1}{2}$, then $f(z)$ is not merely close-to-convex but also convex in one direction [**6**].

8. The case for meromorphic functions.

In Theorem 1, let $f(z)$ be $F(z)^{-1}$. Then $F(z)$ has an expansion $F(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_nz^n$ at $z=0$. Furthermore we have by an elementary culculation

$$d\arg df+kd\arg f=d\arg dF-(k+2)d\arg F. \tag{27}$$

Since $F(z)$ is univalent if and only if $f(z)$ is univalent, we have the following theorem.

THEOREM 8. *Let us denote by D_z a simply connected closed domain including $z=0$ in it and C_z the boundary of D_z. Let $f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_nz^n$ be regular on D_z except at $z=0$ and suppose $f(z)\neq 0$, $f'(z)\neq 0$ on D_z. If there holds for a suitable real constant k*

$$\int_{C_z'}[d\arg df(z)-kd\arg f(z)]>-\pi \qquad (z\in C_z'), \tag{28}$$

where C_z' is an arbitrary arc on C_z, then $f(z)$ is univalent on D_z.

REMARK 1. In this theorem k has to satisfy $k > \frac{1}{2}$.

REMARK 2. For $k \to +\infty$, $f(z)$ should be star-like.

REMARK 3. Though we have (28) immediately from (27), we may prove this theorem as follows. Suppose that $f(z)$ be multivalent, then D_w has some overlapping parts and accordingly, C_w has two loops separated by these parts. One of these loops should encircle $w=0$, so that the other loops C_w' can not encircle $w=0$. Thus for C_z' we have

$$\int_{C_z'} [d \arg df(z) - kd \arg f(z)] = \int_{C_z'} d \arg df(z) \leqq -\pi .$$

Just as we deduced theorems of §5 from Theorem 2, we could deduce many results concerning $f(z) = z^{-p} + \sum_{n=-p+1}^{\infty} a_n z^n$ from theorem 8. We omit them as these results are easily obtained in the same manner.

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