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Perturbation of continuous spectra by unbounded operators, II.

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§ 1. Introduction.

In Part I of the present work[1] we proved that the absolutely continuous part of the spectrum of a self-adjoint (abbr. s. a.) operator is stable under the addition of a not necessarily bounded symmetric perturbation (see Theorem 1 of (I)). The main purpose of this Part II is to prove a similar theorem for perturbations of different kind which are not necessarily expressed as the addition of some symmetric operator. The formulation and the proof of the results are based on the theory of *closed Hermitian forms* in a Hilbert space.

Let $\mathfrak{H}$ be a Hilbert space with the inner product and the norm denoted by $(\ , \ )$ and $\| \ \|$. We follow mostly the terminology and notations of (I); for the notations $\boldsymbol{B}, \boldsymbol{S}, \boldsymbol{T}$ etc., see § 1 of (I). The deviation from the usage in (I) is that the Schmidt and the trace norm of $A \in \boldsymbol{B}$ are now denoted by $s(A)$ and $t(A)$[2], respectively, for typographical reasons. We also use the following new notations: $\boldsymbol{A}$ is the set of all s. a. operators in $\mathfrak{H}$, and $\boldsymbol{A}_{sb} \subset \boldsymbol{A}$ the set of all $H \in \boldsymbol{A}$ which is bounded below; γ_H, $H \in \boldsymbol{A}_{sb}$, is the *lower bound* of H, that is, the maximum of the number γ such that $(Hu, u) \geqq \gamma \|u\|^2$ for every $u \in \mathfrak{D}(H)$; $\boldsymbol{F}$ is the set of all operators $A \in \boldsymbol{T} \subset \boldsymbol{B}$ of finite rank. Furthermore, we use $\mathfrak{M}_0$, $\mathfrak{M}$, P_0 and P in the same sense as in (I) (see footnote 2) of (I)).

We are mainly concerned with the asymptotic properties of the family of unitary operators defined by

$$U_t(H, H_0) = \exp(itH)\exp(-itH_0), \quad -\infty < t < +\infty, \tag{1.1}$$

where $H_0, H \in \boldsymbol{A}$. As is stated in § 1 of (I), the existence of the *generalized wave operators*

$$W_\pm(H, H_0) = \operatorname*{s-lim}_{t \to \pm\infty} U_t(H, H_0)P_0 \text{ and } W_\pm(H_0, H) \tag{1.2}$$

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1) Kuroda [4]. This will be quoted as (I). The reference given in (I) will be quoted as e. g. von Neumann [15] of (I).

2) These are denoted in (I) by $\|A\|_2$ and $\|A\|_1$, respectively.

implies the unitary equivalence of the absolutely continuous parts of H_0 and H. In this Part II we consider the case in which $H_0, H \in \boldsymbol{A}_{sb}$. If one is only concerned with this unitary equivalence, it may be simpler to examine the existence of $W_\pm((H-\gamma)^{-1}, (H_0-\gamma)^{-1})$ and $W_\pm((H_0-\gamma)^{-1}, (H-\gamma)^{-1})$. (In fact, Putnam [**5**] treated some problems of the ordinary differential equation from this point of view.) In connection with the scattering theory in quantum mechanics, however, it seems worthwhile to examine the existence of $W_\pm(H, H_0)$ themselves and we shall treat the problem by considering the existence of $W_\pm(H, H_0)$ and $W_\pm(H_0, H)$

Before formulating the results, we shall examine some properties of Hermitian forms in § 2.

§ 2. Trace class of closed forms.

Let $J[u, v]$ be a *Hermitian bilinear form* (linear in u and conjugate linear in v) with the domain $\mathfrak{D}(J)$ which is a linear manifold dense in $\mathfrak{H}$.[3] J is *bounded below* if there exists a real number γ such that $J[u] \geqq \gamma \|u\|^2$ for every $u \in \mathfrak{D}(J)$. The maximum of such γ is called the *lower bound* of J and denoted by γ_J. J is *closed* if $u_n \in \mathfrak{D}(J)$, $u_n \to u$ and $J[u_n - u_m] \to 0$, $n, m \to \infty$, imply $u \in \mathfrak{D}(J)$ and $J[u_n - u] \to 0$. We denote by $\mathfrak{J}$ the set of all forms in $\mathfrak{H}$ and by $\mathfrak{J}_{sb} \subset \mathfrak{J}$ the set of all closed forms bounded below. For any linear manifold $\mathfrak{D}$ dense in $\mathfrak{H}$, we denote by $\mathfrak{J}_{sb}[\mathfrak{D}] \subset \mathfrak{J}_{sb}$ the set of all $J \in \mathfrak{J}_{sb}$ with the domain $\mathfrak{D}$[4]. In particular, $\mathfrak{J}_b \equiv \mathfrak{J}_{sb}[\mathfrak{H}]$ is the set of all *bounded* forms on $\mathfrak{H}$. A linear subset $\mathfrak{D}$ of the domain of a closed form J is called a *core* of J if the closure of the restriction of J to $\mathfrak{D}$ coincides with J, that is, if for any $u \in \mathfrak{D}(J)$ there exists a sequence $\{u_n\}$, $u_n \in \mathfrak{D}$, such that $u_n \to u$ and $J[u_n - u] \to 0$, $n \to \infty$.

According to a theorem of Friedrichs[3], there exists a uniquely determined one-to-one mapping $J \to \phi(J) = H$ from $\mathfrak{J}_{sb}$ onto $\boldsymbol{A}_{sb}$ such that 1) $\mathfrak{D}(H) \subset \mathfrak{D}(J)$ and $J[u, v] = (Hu, v)$ for every $u \in \mathfrak{D}(H)$ and $v \in \mathfrak{D}(J)$; 2) $\mathfrak{D}(H)$ is a core of J; and 3) $\gamma_H = \gamma_J$. H is called the *s. a. operator associated with* J. We denote by ψ the inverse mapping of $\phi : \psi(H) = J$. J is determined by the relations

3) For Hermitian bilinear forms, see Friendrichs [**1**]. More detailed exposition of the theory of closed forms and its connection with the perturbation theory of eigenvalues are given in Kato [**3**]. Throughout the present work we agree for brevity that 1) "form" means "Hermitian bilinear form with a dense domain" and 2) $J[u] = J[u, u]$.

4) $\mathfrak{J}_{sb}[\mathfrak{D}]$ may be empty. If not empty, however, it consists of an infinite number of elements. We agree throughout the present paper that, when we write $\mathfrak{J}_{sb}[\mathfrak{D}]$ without any comment on the nature of $\mathfrak{D}$, $\mathfrak{J}_{sb}[\mathfrak{D}]$ is assumed not to be empty.

$$\begin{cases} \mathfrak{D}(J) = \mathfrak{D}((H-\gamma)^{1/2}), \\ J[u, v] = ((H-\gamma)^{1/2}u, (H-\gamma)^{1/2}v) + \gamma(u, v), \end{cases} \tag{2.1}$$

where γ is an arbitrary number such that $\gamma \leqq \gamma_J$.[5)]

The set of all $J \in \mathfrak{J}_{sb}$ such that $\phi(J) \in \boldsymbol{T}$ is called the *trace class of closed forms* and denoted by $\mathfrak{T}$. The trace norm $t(J)$ of $J \in \mathfrak{T}$ is defined by $t(J) = t(\phi(J))$. If in particular $\phi(J) \in \boldsymbol{F}$, J is said to be of finite rank. $t(J)$ is given by

$$\begin{aligned} t(J) = t(\phi(J)) &= \max_{\{\varphi_\nu\}} \sum_\nu |(\phi(J)\varphi_\nu, \varphi_\nu)| \\ &= \max_{\{\varphi_\nu\}} \sum_\nu |J[\varphi_\nu]|, \end{aligned} \tag{2.2}$$

where $\{\varphi_\nu\}$ ranges over all complete orthonormal sets (abbr. c. o. n. s.) of $\mathfrak{H}$ (cf. (1.7), (1.8) and (1.10) of (I)). The maximum is attained for a c. o. n. s. consisting of the eigenvectors of $\phi(J)$. Since $\|\phi(J)\| \leqq t(\phi(J))$, we obtain

$$|J[u]| = |(\phi(J)u, u)| \leqq t(J)\|u\|^2. \tag{2.3}$$

So far we have been considering various classes of operators and forms in a fixed Hilbert space $\mathfrak{H}$. For later use, however, it is necessary to investigate the relations between these classes in different Hilbert spaces which are identical with each other as vector spaces. Let $\mathfrak{X}$ be a vector space and let $(\ , \ )_1$ and $(\ , \ )_2$ denote two inner products (strictly positive definite bilinear forms) on $\mathfrak{X}$. We assume that $(\ , \ )_1$ and $(\ , \ )_2$ are equivalent to each other, that is, there exist positive constants M_1 and M_2 such that

$$\|u\|_1 \leqq M_1 \|u\|_2, \quad \|u\|_2 \leqq M_2 \|u\|_1, \qquad \text{where } \|u\|_i = (u, u)_i^{1/2},\ i = 1, 2 \tag{2.4}$$

for every $u \in \mathfrak{X}$. Then, the topologies generated in $\mathfrak{X}$ by $\| \ \|_1$ and $\| \ \|_2$ are identical with each other. We further assume that $\mathfrak{X}$ becomes a (complete) Hilbert space with the inner product $(\ , \ )_1$ (by (2.4) this is equivalent to assuming the same fact with $(\ , \ )_2$). When we consider $\mathfrak{X}$ as a Hilbert space with the inner product $(\ , \ )_i$, $i = 1, 2$, we write $\mathfrak{H}_i$ instead of $\mathfrak{X}$. Furthermore, the notations $\boldsymbol{B}, \mathfrak{T}, \mathfrak{J}_b, t(J), \phi$ etc. introduced above are used with index i, that is, $\boldsymbol{B}_i, \mathfrak{T}_i, \mathfrak{J}_{bi}, t_i(J), \phi_i$ etc., when they refer to the Hilbert space $\mathfrak{H}_i$.

(2.4) implies that the form $J_0[u, v] = (u, v)_2$ on $\mathfrak{H}_1$ belongs to $\mathfrak{J}_{b1}$. Moreover, on putting $A = \phi_1(J_0)$, we have $M_2^2 \geqq A \geqq M_1^{-2} > 0$. Let now $\{\varphi_\nu\}$ range over all c. o. n. s. of $\mathfrak{H}_1$. Then $\{\psi_\nu\}$ with $\psi_\nu = A^{-1/2}\varphi_\nu$ ranges over all c. o. n. s. of $\mathfrak{H}_2$. Hence, we have for any $J \in \mathfrak{T}_1$,

$$t_2(J) = \max_{\{\psi_\nu\}} \sum_\nu |J[\psi_\nu]| = \max_{\{\varphi_\nu\}} \sum_\nu |J[A^{-1/2}\varphi_\nu]| \tag{2.5}$$

5) Kato [**3**, Theorem 4.2].

$$= \max_{\{\varphi_\nu\}} \sum_\nu |(A^{-1/2}\phi_1(J)A^{-1/2}\varphi_\nu, \varphi_\nu)_1|$$

$$= t_1(A^{-1/2}\phi_1(J)A^{-1/2}) < \infty,$$

because $\phi_1(J) \in \boldsymbol{T}_1$ and $A^{-1/2} \in \boldsymbol{B}_1$ imply $A^{-1/2}\phi_1(J)A^{-1/2} \in \boldsymbol{T}_1$. This shows that $\mathfrak{T}_1 \subset \mathfrak{T}_2$. Hence, we obtain $\mathfrak{T}_1 = \mathfrak{T}_2$ by symmetry. Moreover, it follows from (2.5) that

$$t_2(J) \leqq \| A^{-1/2} \|_1^2 t_1(J) \leqq M_1^2 t_1(J), \quad t_1(J) \leqq M_2^2 t_2(J). \tag{2.6}$$

It is also easily seen that $\boldsymbol{S}_1 = \boldsymbol{S}_2$ and $\boldsymbol{T}_1 = \boldsymbol{T}_2$. Thus we have the following

LEMMA 2.1. *The classes of operators $\boldsymbol{B}, \boldsymbol{S}$ and $\boldsymbol{T}$ and the classes of forms $\mathfrak{J}_{sb}$ and $\mathfrak{T}$ do not depend on the particular inner products in $\mathfrak{X}$ so far as they are equivalent to each other.*

Next let $\mathfrak{D}$ be a dense linear manifold for which $\mathfrak{J}_{sb}[\mathfrak{D}]$ is not empty and let $J \in \mathfrak{J}_{sb}[\mathfrak{D}]$. Then the form $(J-\gamma)[u, v]$, $\gamma < \gamma_J$, defines an inner product in $\mathfrak{D}$ with which $\mathfrak{D}$ becomes a (complete) Hilbert space.

LEMMA 2.2. *Let $J_1, J_2 \in \mathfrak{J}_{sb}[\mathfrak{D}]$, $\gamma_1 < \gamma_{J_1}$ and $\gamma_2 < \gamma_{J_2}$. Then the two inner products $(J_1-\gamma_1)[\ ,\ ]$ and $(J_2-\gamma_2)[\ ,\ ]$ in $\mathfrak{D}$ are equivalent to each other.*

PROOF. Let $H_1 = \phi(J_1)$ and $H_2 = \phi(J_2)$. The first relation of (2.1) implies $\mathfrak{D}((H_1-\gamma_1)^{1/2}) = \mathfrak{D}((H_2-\gamma_2)^{1/2}) = \mathfrak{D}$. Hence, we have $B = (H_1-\gamma_1)^{1/2}(H_2-\gamma_2)^{-1/2} \in \boldsymbol{B}$ by the same argument as in the proof of Proposition 2.1 of (I). Using the second relation of (2.1), we then have $(J_1-\gamma_1)[u] = \| (H_1-\gamma_1)^{1/2}u \|^2 \leqq \| B \|^2 \| (H_2 -\gamma_2)^{1/2}u \|^2 = \| B \|^2 (J_2-\gamma_2)[u]$, $\| B \| > 0$. Hence, considering symmetry, we see that $(J_1-\gamma_1)[\ ,\ ]$ and $(J_2-\gamma_2)[\ ,\ ]$ are equivalent to each other. q. e. d.

$\mathfrak{D}$ will be denoted by $\mathfrak{H}(J-\gamma)$ when it is considered as a Hilbert space with the inner product $(J-\gamma)[\ ,\ ]$. By the preceeding lemma the set of all bounded operators on $\mathfrak{H}(J-\gamma)$ does not depend on the choice of J and γ. We denote this set by $\boldsymbol{B}(\mathfrak{D})$. $\boldsymbol{S}(\mathfrak{D})$, $\boldsymbol{T}(\mathfrak{D})$, $\mathfrak{J}_{sb}(\mathfrak{D})$ etc. are defined similarly. ($\mathfrak{J}_{sb}(\mathfrak{D})$ should not be confused with $\mathfrak{J}_{sb}[\mathfrak{D}]$.) In particular, $\mathfrak{T}(\mathfrak{D})$ is the set of all $J_1 \in \mathfrak{J}_{sb}(\mathfrak{D})$ which belongs to the trace class of closed forms on $\mathfrak{H}(J-\gamma)$ for *some* (or equivalently, for *each*) $J \in \mathfrak{J}_{sb}[\mathfrak{D}]$ and $\gamma < \gamma_J$. The ordinary, the Schmidt and the trace norm of $A \in \boldsymbol{B}(\mathfrak{D})$ in $\mathfrak{H}(J-\gamma)$ are denoted by $\| A \|_{J-\gamma}$, $s(A; J-\gamma)$ and $t(A; J-\gamma)$, respectively; the trace norm of $J_1 \in \mathfrak{T}(\mathfrak{D})$ as a form on $\mathfrak{H}(J-\gamma)$ by $t(J_1; J-\gamma)$; and the s. a. operator associated with $J' \in \mathfrak{J}_{sb}(\mathfrak{D})$ in $\mathfrak{H}(J-\gamma)$ by $H = \phi(J'; J-\gamma)$. We also write $J' = \psi(H; J-\gamma)$.

In dealing with the continuity properties of $W_\pm$, we further need the following notions. Let $J, J' \in \mathfrak{J}_{sb}[\mathfrak{D}]$. If in addition $J-J' \in \mathfrak{T}(\mathfrak{D})$, we write $J \sim J'$. The relation $\sim$ is an equivalence relation on $\mathfrak{J}_{sb}[\mathfrak{D}]$. Let $\tilde{\mathfrak{J}}$ be one of the equivalence classes of $\mathfrak{J}_{sb}[\mathfrak{D}]$ with respect to the relation $\sim$. On fixing a $J_0 \in \mathfrak{J}_{sb}[\mathfrak{D}]$ and $\gamma < \gamma_{J_0}$, we introduce a metric in $\tilde{\mathfrak{J}}$ in which the distance $d(J, J')$ is equal to $t(J-J'; J_0-\gamma)$. By virtue of (2.6), however, we see that the topology generated in $\tilde{\mathfrak{J}}$ by this metric does not depend on the choice of J_0 and γ. From now on we regard $\tilde{\mathfrak{J}}$ as a topological space with the topology thus defined.

With this we stated all necessary tools for the formulation of our theorems.

§ 3. Theorems and their applications.

THEOREM 1. *Let* $J_0 \in \mathfrak{J}_{sb}$ *and* $\mathfrak{D} = \mathfrak{D}(J_0)$. *Let* $J_1 \in \mathfrak{J}$ *be a form such that* $\mathfrak{D}(J_1) \supset \mathfrak{D}$ *and* $J_1|_{\mathfrak{D}} \in \mathfrak{T}(\mathfrak{D})$, *where* $J_1|_{\mathfrak{D}}$ *is the restriction of* J_1 *to* $\mathfrak{D}$. *Then:* i) $J = J_0 + J_1 \in \mathfrak{J}_{sb}[\mathfrak{D}]$; ii) *if we put* $H_0 = \phi(J_0)$ *and* $H = \phi(J)$, $W_\pm(H, H_0)$ *and* $W_\pm(H_0, H)$ *exist.*

For applications of Theorem 1 it is sometimes convenient to state the assumptions in a somewhat stronger form. We first note that each densely defined, closed linear operator V in $\mathfrak{H}$ admits a unique decomposition $V = W|V|$, where $|V| = (V^*V)^{1/2}$ and W is a partially isometric operator such that its initial set is identical with the closure of the range of $|V|$ (von Neumann [**15**] of (I)). If in particular V is symmetric, i.e. $V^* \supset V$, we have

$$\text{(3.1)} \qquad \begin{cases} \mathfrak{D}(|V^*|^{1/2}) \supset \mathfrak{D}(|V|^{1/2}) \quad \text{and} \\ \| |V^*|^{1/2}u \| \leqq \| |V|^{1/2}u \| \quad \text{for each } u \in \mathfrak{D}(|V|^{1/2}) \end{cases}$$

(see the proof of Proposition 4.1 of (I)).

COROLLARY. *Let* $J_0, \mathfrak{D}$ *and* H_0 *be as in Theorem 1. Let* V *be a closed symmetric operator such that* $\mathfrak{D}(|V|^{1/2}) \supset \mathfrak{D}$ *and*

$$\text{(3.2)} \qquad A = |V|^{1/2}(H_0 - \gamma)^{-1/2} \in \boldsymbol{S} \ (= \textit{Schmidt class})$$

for some $\gamma < \gamma_{J_0}$. *Let* J_1 *be a form defined by*

$$\text{(3.3)} \qquad \begin{cases} \mathfrak{D}(J_1) = \mathfrak{D}(|V|^{1/2}) \supset \mathfrak{D}, \\ J_1[u, v] = (W|V|^{1/2}u, |V^*|^{1/2}v), \ u, v \in \mathfrak{D}(J_1) \end{cases}$$

(*W being defined as above*). *Then* $J_1|_{\mathfrak{D}} \in \mathfrak{T}(\mathfrak{D})$ *and hence all the assertions of Theorem* 1 *hold.*

REMARK. When $V \in \boldsymbol{T}_s$, V satisfies the assumptions of Corollary for any s.a. operator H_0 which is bounded below. Thus Theorem 1 is a partial generalization of a theorem of Kato referred to in § 1 of (I) (Kato [**5**] of (I)). If we confine ourselves to the case treated in this Corollary, the condition (3.2) seems to be stronger than the corresponding one in Theorem 1 of (I). In the present theorem, however, no relations between the domains of H_0 and V are required. Thus the above theorem neither implies nor is implied by Theorem 1 of (I).

The next theorem concerns the continuity properties of the mappings $(J, J') \rightarrow W_{\pm}(\phi(J'), \phi(J))$.

THEOREM 2. *Let* $\tilde{\mathfrak{J}}$ *be one of the equivalence classes of* $\mathfrak{J}_{sb}[\mathfrak{D}]$ *with respect to the relation* $\sim$ *with the topology introduced above (see* § 2) *and let* $J, J' \in \tilde{\mathfrak{J}}$, $H = \phi(J)$ *and* $H' = \phi(J')$. *Then the mappings* $(J, J') \rightarrow W_{\pm}(H', H)$ *from* $\tilde{\mathfrak{J}} \times \tilde{\mathfrak{J}}$ *into the set* $\boldsymbol{U}$ *of all partially isometric operators are strongly continuous in* J' *for fixed* J *and weakly continuous in* J *for fixed* J'. *For a fixed* $J \in \tilde{\mathfrak{J}}$, *the mapping* $J' \rightarrow S(H') = W_{\pm}(H', H)^* W_{-}(H', H)$[6] *from* $\tilde{\mathfrak{J}}$ *into* $\boldsymbol{U}$ *is strongly continuous.*

Applications. EXAMPLE 1. Consider an ordinary differential operator given *formally* by

$$Hu = -d^2u/dx^2 + q(x)u, \quad -\infty < x < +\infty,$$

where $q(x)$, a real-valued measurable function, is only assumed to belong to $L^1(-\infty, +\infty)$. To define H properly in $\mathfrak{H} = L^2(-\infty, +\infty)$ we shall use Theorem 1 and Corollary to it. Let H_0 and V be s. a. operators defined as follows. $\mathfrak{D}(H_0)$ comprises all functions $u \in L^2(-\infty, +\infty)$ such that $u' = du/dx$ exists and is absolutely continuous in $(-\infty, +\infty)$ and that $u'' \in L^2$; $H_0 u = -u''$ for each $u \in \mathfrak{D}(H_0)$. V is a multiplicative operator $(Vu)(x) = q(x)u(x)$ with the maximal domain. Then $H_0 \geqq 0$. We shall prove (3.2) for $\gamma = -1$[7]. A simple calculation using Fourier transforms gives

$$((H_0 + 1)^{-1/2}u)(x) = \int_{-\infty}^{\infty} k(x-y)u(y)dy,$$

where

$$k(z) = (2\pi)^{-1}\int_{-\infty}^{\infty} \exp(ipz)(p^2+1)^{-1/2}dp.$$

This shows that $|q(x)|^{1/4}k(x-y)$ is a kernel of Hilbert-Schmidt type, from which (3.2) follows immediately. Hence we can define H according to Theorem 1 and Corollary to it.[8] Theorem 1 and Lemma 1.1 of (I) then imply that the absolutely continuous parts of H and $H_0 = -d^2/dx^2$ are unitarily equivalent.

Note that Theorem 1 of (I) can not be applied to the present problem in general. It is even impossible in general to define H as the sum $H_0 + V$ under the only assumption that $q \in L^1$. In fact, there is an example in which $\mathfrak{D}(H_0) \cap \mathfrak{D}(V) = \{0\}$ holds: put $q(x) = \sum_{k=1}^{\infty} a_k(x-x_k)^{-1/2}\eta(x-x_k)$, where $\{x_k\}$ is the sequence of all rational numbers in an arbitrary order, $\{a_k\}$ is a sequence

6) $S(H')$ is the *scattering operator* defined in § 1 of (I).

7) The proof is analogous to that of Theorem 5.1 of Kuroda [**10**] of (I).

8) Another way to define H is given in Stone [**14**] of (I), Chapt. X. on the basis of the theory of differential equations.

of real numbers with $\sum_{k=1}^{\infty}|a_k|<\infty$ and $\eta(x)$, $-\infty<x<+\infty$, is equal to 1 for $|x|<1$ and vanishes elsewhere.

EXAMPLE 2. Let $\mathfrak{H}=L^2(1,+\infty)$ and H_0 be a multiplicative operator $(H_0u)(x)=xu(x)$ with the maximal domain. Let $J_0=\psi(H_0)$ and J_ε, $\varepsilon>0$, a form defined by

$$\left\{\begin{array}{l}\mathfrak{D}(J_\varepsilon)=\mathfrak{D}(H_0{}^{1/2})\equiv\mathfrak{D}\,,\\[2mm] J_\varepsilon[u,v]=2\varepsilon\int_1^\infty u(x)x^{-\varepsilon}dx\int_1^\infty v(x)x^{-\varepsilon}dx,\ u,v\in\mathfrak{D}\,.\end{array}\right.\tag{3.4}$$

(These integrals are convergent because $x^{1/2}u\in L^2(1,+\infty)$ for each $u\in\mathfrak{D}$.) Then J_ε is expressible in the form

$$J_\varepsilon[u,v]=J_0[u,\varphi_\varepsilon]J_0[\varphi_\varepsilon,v]\,,$$

where we put $\varphi_\varepsilon(x)=(2\varepsilon)^{1/2}x^{-(1+\varepsilon)}$ $(J_0[\varphi_\varepsilon]=1)$. Hence, $J_\varepsilon\in\mathfrak{T}(\mathfrak{D})$ and $\phi(J_\varepsilon;J_0)$ is of rank 1. Theorem 1 then shows that $J_0+J_\varepsilon\in\mathfrak{Z}_{sb}[\mathfrak{D}]$, and $W_\pm(H_\varepsilon,H_0)$ and $W_\pm(H_0,H_\varepsilon)$ exist, where $H_\varepsilon=\phi(J_0+J_\varepsilon)$. Since φ_ε depends continuously on ε in the strong topology of $\mathfrak{H}(J_0)$, J_ε depends continuously on ε in the sense of $t(\ ;J_0)$. Hence, Theorem 2 can be applied, too.

If $\varepsilon>1/2$, we have $\varphi_\varepsilon\in\mathfrak{D}(H_0)$. Then J_ε can be expressed as $J_\varepsilon[u,v]=c_\varepsilon(u,\psi_\varepsilon)(\psi_\varepsilon,v)$, where $\psi_\varepsilon(x)=(2\varepsilon-1)^{1/2}x^{-\varepsilon}$ and $c_\varepsilon=2\varepsilon(2\varepsilon-1)^{-1}$ $(\|\psi_\varepsilon\|=1)$. Hence $H_\varepsilon=H_0+c_\varepsilon(\cdot,\psi_\varepsilon)\psi_\varepsilon$ and the problem is reduced to that of one-dimensional perturbation (cf. also Lemma 4.2). If $0<\varepsilon\leqq1/2$, however, we can find no such simple expression for H_ε and the use of the theory of closed forms is essential. In fact, simple consideration gives $\mathfrak{D}(H_0)\cap\mathfrak{D}(H_\varepsilon)=\{0\}$, if $0<\varepsilon\leqq1/2$.

§ 4. Lemmas on trace class of closed forms.

In this section we prove three lemmas for later use.

LEMMA 4.1. *Let $J\in\mathfrak{Z}_{sb}[\mathfrak{D}]$ and $J_1\in\mathfrak{T}(\mathfrak{D})$. Then*

$$\lim_{\gamma\to-\infty}t(J_1;J-\gamma)=0\,.\tag{4.1}$$

PROOF. Let $\beta<\gamma_J$ be fixed and $\gamma\leqq\beta$. Put $H=\phi(J)$ and $A=(H-\gamma)^{-1/2}\cdot(H-\beta)^{1/2}$. A is an operator on $\mathfrak{D}$ to $\mathfrak{D}$ and we have $(J-\beta)[Au]\leqq\|(H-\beta)^{1/2}\cdot(H-\gamma)^{-1/2}\|^2(J-\beta)[u]\leqq(J-\beta)[u]$. Hence $A\in\boldsymbol{B}(\mathfrak{D})$ and $\|A\|_{J-\beta}\leqq1$. Furthermore, simple calculation shows $A\dagger=A$, where $A\dagger$ is the adjoint of A in $\mathfrak{H}(J-\beta)$. Since $V_\beta\equiv\phi(J_1;J-\beta)\in\boldsymbol{T}(\mathfrak{D})$, we have $|V_\beta|^{1/2}A\in\boldsymbol{S}(\mathfrak{D})$.[9] Let now $\{\varphi_\nu\}$ range over all c.o.n.s. of $\mathfrak{H}(J-\gamma)$. Then $\{\psi_\nu\}$ with $\psi_\nu=A^{-1}\varphi_\nu$ ranges

9) $|V_\beta|$ and $|V_\beta|^{1/2}$ are the absolute value and its positive square root of $V_\beta\in\boldsymbol{B}(\mathfrak{D})$ as an operator in $\mathfrak{H}(J-\beta)$. When no misunderstanding occur, we shall use these notations without any comment.

over all c. o. n. s. of $\mathfrak{H}(J-\beta)$. Hence, using (2.2), we have the estimate:

$$(4.2)\qquad \begin{aligned} t(J_1;J-\gamma) &= \max_{\{\varphi_\nu\}} \sum_\nu |J_1[\varphi_\nu]| = \max_{\{\varphi_\nu\}} \sum_\nu |(J-\beta)[V_\beta\varphi_\nu, \varphi_\nu]| \\ &\leqq \max_{\{\varphi_\nu\}} \sum_\nu (J-\beta)[|V_\beta|^{1/2}\varphi_\nu] = \max_{\{\psi_\nu\}} \sum_\nu (J-\beta)[|V_\beta|^{1/2}A\psi_\nu] \\ &= s(|V_\beta|^{1/2}A;J-\beta)^2 = s(A|V_\beta|^{1/2};J-\beta)^2 \\ &\leqq \sum_{k=1}^N (J-\beta)[A|V_\beta|^{1/2}\psi_k] + \sum_{k=N+1}^\infty (J-\beta)[|V_\beta|^{1/2}\psi_k], \end{aligned}$$

where $\{\psi_k\}$ is a countable subset of a fixed $\{\psi_\nu\}$ and N is an arbitrary positive integer. Let $\varepsilon>0$ be fixed arbitrarily. By choosing sufficiently large N, we can make the second term on the right-hand side of (4.2) smaller than ε, irrespective of γ. On fixing such an N, let γ tend to $-\infty$. Since $(J-\beta)[Au]\to 0$, $\gamma\to-\infty$, for any $u\in\mathfrak{D}$, the first term becomes smaller than ε if γ is sufficiently small. This proves (4.1). q. e. d.

A form $J_1\in\mathfrak{T}(\mathfrak{D})$ is not necessarily bounded as a form in $\mathfrak{H}$, even if $\phi(J_1;J-\gamma)$ is of rank 1. Nevertheless, we have the following

LEMMA 4.2. *Let $J_1\in\mathfrak{T}(\mathfrak{D})$ and for some $J\in\mathfrak{J}_{sb}[\mathfrak{D}]$ and $\gamma<\gamma_J$ let $\hat{V}\equiv\phi(J_1;J-\gamma)$ be of finite rank and hence be expressible in the form $\hat{V}=\sum_{k=1}^r c_k(J-\gamma)[\cdot,\varphi_k]\varphi_k$ with $\{\varphi_k\}$ having the property $(J-\gamma)[\varphi_j,\varphi_k]=\delta_{jk}$ and with real numbers c_k. Put $H=\phi(J)$. Then, if $\varphi_k\in\mathfrak{D}(H)$, $k=1,2,\cdots,r$, J_1 is uniquely extended to a bounded form J_1' on $\mathfrak{H}$ and J_1' is of finite rank. If in particular $J_1\geqq 0$, then $V\equiv\phi(J_1')\geqq 0$ and we have*

$$(4.3)\qquad t(J_1;J-\gamma)=s(V^{1/2}(H-\gamma)^{-1/2})^2.\,^{10)}$$

PROOF. Without loss of generality we may assume that $J\geqq c>0$ and $\gamma=0$. Since $\varphi_k\in\mathfrak{D}(H)$, we have for any $u,v\in\mathfrak{D}$

$$J_1[u,v]=J[\hat{V}u,v]=\sum_{k=1}^r c_k(u,H\varphi_k)(H\varphi_k,v)=(Vu,v),$$

where we put $V=\sum_{k=1}^r c_k(\cdot,H\varphi_k)H\varphi_k\in\boldsymbol{F}\cap\boldsymbol{A}$. This proves the first statement, if we put $J_1'=\psi(V)$. $J_1\geqq 0$ implies $c_k\geqq 0$, $k=1,2,\cdots,r$, from which $V\geqq 0$ follows. Let $\{\varphi_\nu\}$ be a c. o. n. s. of $\mathfrak{H}(J)$ containing all φ_k. Then $\{\psi_\nu\}$ with $\psi_\nu=H^{1/2}\varphi_\nu$ is a c. o. n. s. of $\mathfrak{H}$ and we have

$$\begin{aligned} s(V^{1/2}H^{-1/2})^2 &= \sum_\nu \|V^{1/2}H^{-1/2}\psi_\nu\|^2 = \sum_\nu (VH^{-1/2}\psi_\nu, H^{-1/2}\psi_\nu) \\ &= \sum_\nu J_1[\varphi_\nu] = t(J_1;J). \qquad \text{q. e. d.} \end{aligned}$$

LEMMA 4.3. *Let $J\in\mathfrak{J}_{sb}[\mathfrak{D}]$ and $H=\phi(J)=\int\lambda dE(\lambda)$. Let $\hat{V}_t$ be the restric-*

10) It can be shown that the validity of the assumptions on V and φ_k does not depend on the choice of J and γ, but this fact is not needed in the sequel.

tion of $V_t=\exp(itH)$, $-\infty<t<+\infty$, *to* $\mathfrak{D}$. *Then* $\hat{V}_t$ *is a unitary operator in* $\mathfrak{H}(J-\gamma)$, $\gamma<\gamma_J$, *and there exists a uniquely determined s.a. operator* $\hat{H}=\int \lambda d\hat{E}(\lambda)$ *in* $\mathfrak{H}(J-\gamma)$ *such that* $\hat{V}_t=\exp(it\hat{H})$. *Furthermore, for any* $u, v\in\mathfrak{D}$ *we have*

$$(E(\lambda)(H-\gamma)^{1/2}u,\ (H-\gamma)^{1/2}v)=(J-\gamma)[\hat{E}(\lambda)u, v]\,. \tag{4.4}$$

PROOF. Without loss of generality we may assume that $J\geqq c>0$ and $\gamma=0$. After a simple consideration using the relation $J[\hat{V}_t u]=\|H^{1/2}\exp(itH)u\|=\|H^{1/2}u\|=J[u]$ we see that $\{\hat{V}_t\}$ forms a strongly continuous one-parameter group of unitary operators in $\mathfrak{H}(J)$. Hence, $\hat{H}$ mentioned in the lemma exists by virtue of Stone's theorem. Since $(\exp(itH)H^{1/2}u, H^{1/2}v)=J[\exp(it\hat{H})u, v]$, $u, v\in\mathfrak{D}$, we get

$$\int_{-\infty}^{\infty}\exp(it\lambda)d(E(\lambda)H^{1/2}u,\ H^{1/2}v)=\int_{-\infty}^{\infty}\exp(it\lambda)dJ[\hat{E}(\lambda)u, v]\,.$$

By the uniqueness of the Fourier-Stieltjes transforms, we therefore obtain (4.4).

§ 5. Proof of Theorems.

1. We first show that Theorem 1 can be reduced to the following

PROPOSITION 5.1. *Let* J_0, $\mathfrak{D}$, H_0 *and* J_1 *be as in Theorem* 1 *and let* $J_0\geqq c>0$ *for some positive c. Furthermore, let*

$$K=t(J_1|_{\mathfrak{D}}\,;J_0)<1\,. \tag{5.1}$$

Then: i) $J=J_0+J_1\in\mathfrak{J}_{sb}[\mathfrak{D}]$; ii) *if* $H=\phi(J)$, $W_\pm(H, H_0)$ *exist.*

Assume that Proposition 5.1 holds true and let J_0, J_1, J, H_0 and H be as in Theorem 1. By virtue of Lemma 4.1, there exists $\gamma<\gamma_{J_0}$ such that $t(J_1|_{\mathfrak{D}}; J_0-\gamma)<1$. Hence, it follows from i) of Proposition 5.1 that $J_0-\gamma+J_1=J-\gamma\in\mathfrak{J}_{sb}[\mathfrak{D}]$. This implies $J\in\mathfrak{J}_{sb}[\mathfrak{D}]$. Noting the relations $\phi(J_0-\gamma)=H_0-\gamma$ and $\phi(J-\gamma)=H-\gamma$, we then see by ii) of Proposition 5.1 that $W_\pm(H, H_0)=W_\pm(H-\gamma, H_0-\gamma)$ exist. The relations $J\in\mathfrak{J}_{sb}[\mathfrak{D}]$ and $J_1|_{\mathfrak{D}}\in\mathfrak{T}(\mathfrak{D})$ imply that the assumptions of Theorem 1 are satisfied also for J and J_1 in place of J_0 and J_1. Accordingly, the existence of $W_\pm(H_0, H)$ follows in the same way as above.

2. We now prove Proposition 5.1 in several steps. To simplify the description we assume that J_0, $\mathfrak{D}$, H_0, J_1, and J are as in Proposition 5.1, unless otherwise explicitly stated. Furthermore, we put for brevity $\hat{\mathfrak{H}}=\mathfrak{H}(J_0)$, $\hat{\mathrm{B}}=\mathrm{B}(\mathfrak{D})$, $\hat{\mathfrak{T}}=\mathfrak{T}(\mathfrak{D})$, $[u, v]=J_0[u, v]$, $|||u|||=[u, u]^{1/2}$, $|||A|||=\|A\|_{J_0}$ for $A\in\hat{\boldsymbol{B}}$, $\hat{t}(J')=t(J'\,;J_0)$ for $J'\in\hat{\mathfrak{T}}$ etc. and $\hat{V}=\phi(J_1|_{\mathfrak{D}}\,;J_0)$. Then $\hat{V}\in\hat{\boldsymbol{T}}$ and $\hat{t}(\hat{V})=K$.

PROPOSITION 5.2. i) $J\in\mathfrak{J}_{sb}[\mathfrak{D}]$. ii) *If* $H=\phi(J)$, *then*

(5.2) $$H \geqq c(1-K) > 0 \textit{ and } \| H_0^{1/2} H^{-1/2} \|^2 \leqq (1-K)^{-1}.$$

PROOF. For any $u \in \mathfrak{D}$ we have by (2.3) and (5.1) $|J_1[u]| \leqq \hat{t}(\hat{V}) |\!|\!| u |\!|\!|^2 = KJ_0[u]$. Hence,

(5.3) $$(1+K)J_0[u] \geqq J[u] \geqq (1-K)J_0[u] \geqq c(1-K) \| u \|^2.$$

Since $J_0 \in \mathfrak{J}_{sb}[\mathfrak{D}]$ and $\mathfrak{D}(J) = \mathfrak{D}$, it follows easily from (5.3) and $0 \leqq K < 1$ that $J \in \mathfrak{J}_{sb}[\mathfrak{D}]$ (cf. Theorem 3.4 of Kato [**3**]). (5.2) is a consequence of (5.3): the first inequality is clear and the second is proved as $\| H_0^{1/2} H^{-1/2} u \|^2 = J_0[H^{-1/2} u] \leqq (1-K)^{-1} J[H^{-1/2} u] = (1-K)^{-1} \| u \|^2$. q. e. d.

We next determine the explicit form of H^{-1}, when $\hat{V}$ is of finite rank.[11] $\hat{V}$ is then expressible in the form

(5.4) $$\hat{V} = \sum_{k=1}^{r} c_k [\cdot, \varphi_k] \varphi_k,$$

where c_k are real and $\{\varphi_k\}$ has the property $[\varphi_j, \varphi_k] = \delta_{jk}$. By (5.1) we have $|c_k| \leqq \sum |c_k| < 1$.

PROPOSITION 5.3. *Let $\hat{V}$ be of finite rank and hence be expressible in the form* (5.4). *Then we have for any $u \in \mathfrak{H}$*

(5.5) $$H^{-1} u = H_0^{-1} u - \sum_{k=1}^{r} \frac{c_k [H_0^{-1} u, \varphi_k]}{1 + c_k} \varphi_k.$$

PROOF. Let $w \in \mathfrak{H}$, $w' = H_0^{-1} w$ and $v = H^{-1} u$ (note that $H^{-1}, H_0^{-1} \in \boldsymbol{B}$). Then we have

$$\begin{aligned} (H_0^{-1} u, w) = (u, w') = (Hv, w') &= J[v, w'] = (J_0 + J_1)[v, w'] \\ &= (H_0^{1/2} v, H_0^{1/2} w') + \sum_{k=1}^{r} c_k [v, \varphi_k][\varphi_k, w'] \\ &= (v, w) + \sum_{k=1}^{r} c_k [v, \varphi_k](\varphi_k, w). \end{aligned}$$

Since $w \in \mathfrak{H}$ is arbitrary, we obtain

(5.6) $$H^{-1} u = v = H_0^{-1} u - \sum_{k=1}^{r} c_k [v, \varphi_k] \varphi_k.$$

Taking the inner product with φ_k in $\hat{\mathfrak{H}}$, we get $[v, \varphi_k] = [H_0^{-1} u, \varphi_k] - c_k [v, \varphi_k]$. Whence $[v, \varphi_k] = [H_0^{-1} u, \varphi_k](1 + c_k)^{-1}$. Inserting this into (5.6) we get (5.5). q. e. d.

3. We now prove Proposition 5.1 under the additional assumption that $J_1 \geqq 0$ or $J_1 \leqq 0$. We first observe that by Lemma 4.3 there exists a s.a. operator $\hat{H}_0 = \int \lambda d\hat{E}_0(\lambda)$ in $\hat{\mathfrak{H}}$ such that

11) Similar result was given in Friedrichs [**2**] for one-dimensional perturbation.

(5.7) $$\exp(itH_0)u = \exp(it\hat{H}_0)u, \text{ and } \| E_0(\lambda)H_0^{1/2}u \| = ||| \hat{E}_0(\lambda)u |||$$

for every $u \in \hat{\mathfrak{H}}$. Let $\mathfrak{M}_0$ and $\hat{\mathfrak{M}}_0$ be absolutely continuous subspaces of $\mathfrak{H}$ and $\hat{\mathfrak{H}}$ with respect to H_0 and $\hat{H}_0$, respectively, and $\mathfrak{L}$ the set of all $u \in \mathfrak{M}_0 \cap \mathfrak{D}(H_0^{1/2})$ such that

(5.8) $$d \| E_0(\lambda)H_0^{1/2}u \|^2/d\lambda \leqq m^2, \text{ a. e.,}$$

for some positive m (depending on u). By (5.7) and (5.8) we then obtain $\hat{\mathfrak{M}}_0 = \mathfrak{M}_0 \cap \mathfrak{D}$ and

(5.9) $$d ||| \hat{E}_0(\lambda)u |||^2/d\lambda \leqq m^2, \text{ a. e.,}$$

for each $u \in \mathfrak{L}$.

Proposition 5.4. *Let $J_1 \geqq 0$ or $J_1 \leqq 0$. Then we have for any $u \in \mathfrak{L}$ and for any real s and t the inequality*

(5.10) $$\| (U_t - U_s)u \| \leqq C\{\eta(t; u) + \eta(s; u)\},$$

where $U_t = U_t(H, H_0)$,

(5.11) $$C = \{8\pi m^2 K(1-K)^{-1}\}^{1/4}$$

and

(5.12) $$\eta(t; u) = \left(\int_t^\infty ||| \, |\hat{V}|^{1/2} \exp(-it\hat{H}_0)u |||^2 dt\right)^{1/4}.$$

(*Note that $\eta(t; u)$ is finite by virtue of Lemma* 2.1 *of* (I) *and* (5.9).) (5.10) *also holds if $\int_t^\infty$ in* (5.12) *is replaced by $\int_{-\infty}^t$. Furthermore, $W_\pm(H, H_0)$ exist.*

Proof. For brevity we assume $J_1 \geqq 0$. The other case can be dealt with similarly. For the moment we further assume that $\hat{V}$ is of finite rank and hence is expressible in the form (5.4) with $0 \leqq c_k < 1$. Since $\mathfrak{D}(H_0)$ is a core of J_0, $\mathfrak{D}(H_0)$ is dense in $\hat{\mathfrak{H}}$. Hence, there exists a sequence $\{\{\varphi_k^{(n)} | k = 1, 2, \cdots, r\} | n = 1, 2, \cdots\}$ of orthonormal set in $\hat{\mathfrak{H}}$ such that $\varphi_k^{(n)} \in \mathfrak{D}(H_0)$ for each k and n, and $||| \varphi_k^{(n)} - \varphi_k ||| \to 0$, as $n \to \infty$. Put $\hat{V}_n = \sum_{k=1}^r c_k[\cdot, \varphi_k^{(n)}]\varphi_k^{(n)}$ and $J_1^{(n)}[u, v] = [\hat{V}_n u, v]$, $u, v \in \mathfrak{D}$. Then $J_1^{(n)} \in \hat{\mathfrak{T}}$, $J_1^{(n)} \geqq 0$ and $\hat{t}(J_1^{(n)}) = \sum_{k=1}^r c_k = \hat{t}(\hat{V}) = K$. Since $\varphi_k^{(n)} \in \mathfrak{D}(H_0)$, we see by Lemma 4.2 that there exists a $V_n \in \boldsymbol{F} \cap \boldsymbol{A}$ such that $J_1^{(n)}[u, v] = (V_n u, v)$. This implies that $H_n \equiv \phi(J_0 + J_1^{(n)}) = H_0 + V_n$. In such a case the existence of $W_\pm^{(n)} = W_\pm(H_n, H_0)$ was proved by Kato (see [4] of (I) or Corollary to Lemma 3.1 of (I)). Moreover, we have for any $u \in \mathfrak{L}$ and for any real s and t an inequality similar to (5.10) with U_t, U_s, C and η replaced by $U_t^{(n)}$, $U_s^{(n)}$, C_n and η_n, where $U_t^{(n)} = \exp(itH_n)\exp(-itH_0)$,

(5.11′) $$C_n = \{8\pi m^2 s(V_n^{1/2}H_0^{-1/2})^2 \| H_0^{1/2}H_n^{-1/2} \|^2\}^{1/4}$$

and η_n is defined by (5.12) with $\hat{V}$ replaced by $\hat{V}_n$. The proof of this inequality is similar to that of (2.6) of (I) and we shall only sketch its outline.

We start from the inequality (2.11) of (I) with W_+ etc. replaced by $W_+^{(n)}$ etc. ($W_+^{(n)} = \text{s-lim}\, U_t^{(n)} P_0$). Considering the relation $V_n^{1/2} W_+^{(n)} \exp(-itH_0)u = V_n^{1/2} H_0^{-1/2} H_0^{1/2} H_n^{-1/2} W_+^{(n)} \exp(-itH_0) H_0^{1/2} u$, we obtain as in (I) the inequality similar to (2.12) of (I). Since its right-hand side is equal to

$$C_n \int_s^\infty \| V_n^{1/2} \exp(-itH_0)u \|^2 dt = C_n \int_s^\infty J_1^{(n)}[\exp(-it\hat{H}_0)u] dt = C_n \eta_n(s\,;u)\,,$$

we obtain the desired inequality.

As in the proof of Proposition 5.2, we see from $\hat{t}(\hat{V}_n) = K$ that $\| H_0^{1/2} H_n^{-1/2} \|^2 \leqq (1-K)^{-1}$. Hence, by considering (4.3), we can replace C_n in the above inequality by C as given in (5.11). Now let n tend to infinity. We first show that $\eta_n(t\,;u) \to \eta(t\,;u)$, $n \to \infty$. By virtue of Lemma 2.2 of (I) it suffices to prove that $\hat{s}(\hat{V}_n^{1/2} - \hat{V}^{1/2}) \to 0$. As is shown in the proof of (3.3) of (I), $|\!|\!| \varphi_k^{(n)} - \varphi_k |\!|\!| \to 0$ implies that $\hat{s}(c_k^{1/2}[\cdot, \varphi_k^{(n)}]\varphi_k^{(n)} - c_k^{1/2}[\cdot, \varphi_k]\varphi_k) \to 0$. Hence, noting the expression $\hat{V}^{1/2} = \sum_{k=1}^{r} c_k^{1/2}[\cdot, \varphi_k]\varphi_k$ and similar one for $\hat{V}_n^{1/2}$ and making use of the triangle inequalities, we obtain $\hat{s}(\hat{V}_n^{1/2} - \hat{V}^{1/2}) \to 0$. Next we show that $\text{s-lim}\, U_t^{(n)} = U_t$. Since $p_n \equiv |\!|\!| \varphi_k^{(n)} - \varphi_k |\!|\!| \to 0$ implies $\| \varphi_k^{(n)} - \varphi_k \| \leqq p_n \| H_0^{-1/2} \| \to 0$, (5.5) implies that $\text{s-lim}\, H_n^{-1} = H^{-1}$. From this, we have $\text{s-lim}(H_n - \zeta)^{-1} = (H-\zeta)^{-1}$ for each non-real ζ (cf. Theorem 12.2 of Kato [**3**]). According to the general theory of semi-groups of operators referred to in the proof of Proposition 4.3 of (I)[12], the last relation implies $\text{s-lim} \exp(itH_n) = \exp(itH)$. $\text{s-lim}\, U_t^{(n)} = U_t$ follows directly from this. By taking the limit $n \to \infty$, we thus obtain (5.10) if $\hat{V}$ is of finite rank.

We next consider the general case. Since $\hat{V} \in \hat{\boldsymbol{T}}$, $\hat{V}$ is expressible in the form (5.4) with r replaced by ∞. Put

$$\hat{V}_n = \sum_{k=1}^{n} c_k[\cdot, \varphi_k]\varphi_k, \qquad n = 1, 2, \cdots, \tag{5.13}$$

and, using this $\hat{V}_n$, define $J_1^{(n)}$, H_n and $U_t^{(n)}$ as above. Then, by the part of the proposition already proved we have the inequality similar to (5.10) with U_t etc. replaced by $U_t^{(n)}$, $U_s^{(n)}$, C_n and η_n, where C_n and η_n are defined by (5.11) and (5.12) with K and V replaced by $K_n = \hat{t}(\hat{V}_n)$ and $\hat{V}_n$, respectively. Since (5.13) implies $K_n \leqq K$ and $|\!|\!| \hat{V}_n^{1/2} u |\!|\!| \leqq |\!|\!| \hat{V}^{1/2} u |\!|\!|$, we can again replace K_n and $\hat{V}_n$ by K and $\hat{V}$, respectively. Hence, if we prove $\text{s-lim}\, U_t^{(n)} = U_t$, then taking the limit $n \to \infty$ will give the inequality (5.10) itself. It is easily verified that J and $J^{(n)} = J_0 + J_1^{(n)}$, $n = 1, 2, \cdots$, have the properties: 1) $\mathfrak{D}(J^{(n)}) = \mathfrak{D}(J) = \mathfrak{D}$; 2) $J_0[u] \leqq J^{(m)}[u] \leqq J^{(n)}[u] \leqq J[u]$, $u \in \mathfrak{D}$, $m \leqq n$; 3) $\lim J^{(n)}[u] = J[u]$, $n \to \infty$, $u \in \mathfrak{D}$. According to the theory of closed forms[13], 1), 2) and 3)

12) See also Theorem 5.1 of Trotter's work added at the end of (I).

13) Theorem 10.1 or 10.2 of Kato [3], according to $\hat{V} \geqq 0$ or $\hat{V} \leqq 0$.

imply s-lim$(H_n-\zeta)^{-1}=(H-\zeta)^{-1}$ for every non-real ζ, whence we get s-lim $U_t^{(n)} = U_t$ as required.

Since $\eta(t\,;u)$ and $\eta(s\,;u)$ are convergent, the right-hand side of (5.10) tends to zero as $s,t\to\infty$. This implies that $U_t u$ has a limit as $t\to\infty$, if $u\in\mathfrak{L}$. The existence of $W_+(H,H_0)$ follows from this by a standard argument.

The inequality (5.10) with $\int_t^\infty$ replaced by $\int_{-\infty}^t$ and the existence of $W_-(H,H_0)$ can be proved similarly. q. e. d.

4. Completion of the proof of Proposition 5.1. Let $\{F(\lambda)\}$ be the resolution of the identity corresponding to $\hat{V}$ and put $\hat{V}_\pm=\int_0^{\pm\infty}\lambda dF(\lambda)$ and $J_1^{(\pm)}=\hat{\psi}(\hat{V}_\pm)$. Then, $J_1|_{\mathfrak{D}}=J_1^{(+)}-J_1^{(-)}$, $J_1^{(\pm)}\geqq 0$ and $\hat{t}(J_1^{(\pm)})<\hat{t}(J_1|_{\mathfrak{D}})<1$. Hence, Propositions 5.2 and 5.4 show that $J'=J_0+J_1^{(+)}\in\mathfrak{J}_{sb}[\mathfrak{D}]$ and $W_\pm(H',H_0)$ exist, where $H'=\phi(J')$. On the other hand, by taking sufficiently small real number γ and applying Proposition 5.4 to $J'-\gamma\in J_{sb}[\mathfrak{D}]$ and $J_1^{(-)}\in\mathfrak{T}(\mathfrak{D})$, we can conclude as in § 5.1 that $W_\pm(H,H')=W_\pm(H-\gamma,H'-\gamma)$ exist. Since the existence of $W_\pm(H',H_0)$ and $W_\pm(H,H')$ have been proved, that of $W_\pm(H,H_0)$ follows directly from (1.6) of (I). q. e. d.

This completes the proof of Theorem 1.

5. Proof of Corollary to Theorem 1. To show that $J_1[u,v]$ is a Hermitian form we have only to prove $J_1[u,v]=\overline{J_1[v,u]}$. Since $V=W|V|=|V^*|W$ and $W|V|^{1/2}=|V^*|^{1/2}W$ (see von Neumann [**15**] and Kato [8] of (I)), we have $V=|V^*|^{1/2}W|V|^{1/2}$. Hence, if $u,v\in\mathfrak{D}(V)$, it follows from (3.3) that $J_1[u,v]=(Vu,v)=(u,Vv)=\overline{J_1[v,u]}$. For every $u\in\mathfrak{D}(J_1)=\mathfrak{D}(|V|^{1/2})$, however, there exists a sequence $\{u_n\}$ such that $u_n\in\mathfrak{D}(|V|)=\mathfrak{D}(V)$ and $\||V|^{1/2}(u_n-u)\|\to 0$, $n\to\infty$. We therefore obtain $J_1[u,v]=\overline{J_1[v,u]}$, $u,v\in\mathfrak{D}(J_1)$, by a limiting procedure. Let now $\{\varphi_\nu\}$ range over all c. o. n. s. of $\mathfrak{H}(J_0-\gamma)$ and $\psi_\nu=(H_0-\gamma)^{1/2}\varphi_\nu$. Then $\{\psi_\nu\}$ ranges over all c. o. n. s. of $\mathfrak{H}$ and we have by (3.3) and (3.1) $t(J_1\,;J_0-\gamma)=\max\limits_{\{\varphi_\nu\}}\sum\limits_\nu J_1[\varphi_\nu]\leqq\sum\limits_\nu\||V|^{1/2}(H_0-\gamma)^{-1/2}\psi_\nu\|^2=s(A)^2$, which shows that $J_1\in\mathfrak{T}(\mathfrak{D})$. q. e. d.

6. Proof of Theorem 2. Since $\tilde{\mathfrak{J}}$ can be regarded as a metric space, the topology in $\tilde{\mathfrak{J}}$ satisfies the first countability axiom. Hence it suffices to prove the continuity by considering a sequence $\{J_n\}$ converging to J. In view of relations (1.5) and (1.6) of (I), however, it suffices for this purpose to show that, if $J_n\to J_0$ in $\tilde{\mathfrak{J}}$ and $H_n=\phi(J_n)$, we have

$$\text{s-lim } W_\pm^{(n)}=P_0,\ \text{where } W_\pm^{(n)}=W_\pm(H_n,H_0) \tag{5.14}$$

(for details, see the proof of Theorem 2 of (I)).

PROPOSITION 5.5. *Let J_0, J_1, K etc. be as in Proposition* 5.1 *and let* $K<1/2$. *Then we have for any* $u \in \mathfrak{L}$

$$\| W_\pm(H, H_0)u - u \| \leqq mK^{1/2}f(K)\,, \tag{5.15}$$

where m is given by (5.8) *and $f(K)$ is a continuous function of K defined in the interval* $[0, 1/2)$.

PROOF. Let $\hat{V}_\pm, J_1^{(\pm)}, J'$ and $H' = \int_{-\infty}^{\infty} \lambda dE'(\lambda)$ be as in § 5.4 and the notations such as $|||\ |||$, $\hat{\mathfrak{T}}$ etc. have the same meaning as above. Put $K_\pm = \hat{t}(J_1^{(\pm)})$. Then $0 \leqq K_\pm \leqq K$ and $K = K_+ + K_-$. Using (1.6) of (I), we have for any $u \in \mathfrak{H}$

$$\| W_\pm(H, H_0)u - u \| \leqq \| W_\pm(H', H_0)u - u \| \tag{5.16}$$
$$+ \| \{ W_\pm(H, H') - 1 \} W_\pm(H', H_0)u \|\,.$$

In order to estimate the first term on the right-hand side, we note that the inequality (5.10) with $\hat{V}, K$ etc. replaced respectively by $\hat{V}_+, K_+$ etc. holds true. On putting $s=0$ and letting $t \to \infty$ in this inequality, we obtain for any $u \in \mathfrak{L}$

$$\begin{aligned} &\| W_\pm(H', H_0)u - u \| \\ &\leqq \{8\pi m^2 K_+(1-K_+)^{-1}\}^{1/4} \left(\int_0^\infty ||| \hat{V}_+^{1/2} \exp(-it\hat{H}_0)u |||^2 dt \right)^{1/4} \\ &\leqq (4\pi m^2 K_+)^{1/2}(1-K_+)^{-1/4}\,, \end{aligned} \tag{5.17}$$

where we use Lemma 2.1 of (I) for the estimation of the integral in the second member of (5.17). To deal with the second term on the right-hand side of (5.16) we note that $H' \geqq c(1-K_+) > 0$ and $\| B \|^2 \leqq (1-K_+)^{-1}$, where $B = H_0^{1/2} H'^{-1/2}$. Since B^* ($=$ the adjoint of B in $\mathfrak{H}$) transforms each c. o. n. s. of $\hat{\mathfrak{H}}$ into a c. o. n. s. of $\mathfrak{H}(J')$, we obtain the inequality

$$t(J_1^{(-)}; J') \leqq \| B^* \|^2 \hat{t}(J_1^{(-)}) \leqq K_-(1-K_+)^{-1} \equiv L$$

in the same way as we got (2.6). Since $K_\pm \leqq K < 1/2$, we have $L<1$. On the other hand, it follows from Lemma 1.1 of (I) and (5.8) that, if $u \in \mathfrak{L}$, $v = W_\pm(H', H_0)u \in \mathfrak{M}' \cap \mathfrak{D}(H'^{1/2})$ and $d \| E'(\lambda) H'^{1/2} v \|^2 / d\lambda = d \| E_0(\lambda) H_0^{1/2} u \|^2 / d\lambda \leqq m^2$. Hence, we have for any $u \in \mathfrak{L}$

$$\| \{ W_\pm(H, H') - 1 \} W_\pm(H', H_0)u \| \leqq (4\pi m^2 L)^{1/2}(1-L)^{-1/4} \tag{5.18}$$

similarly as above. Substitute (5.17) and (5.18) into (5.16). On putting $L' = K(1-K)^{-1}$ and noting $0 \leqq K_+ \leqq K \leqq L' < 1$ and $L \leqq L'$, we then obtain

$$\begin{aligned} &\| W_\pm(H, H_0)u - u \| \\ &\leqq (4\pi m^2)^{1/2} \{ K_+^{1/2}(1-K_+)^{-1/4} + K_-^{1/2}(1-K_+)^{-1/2}(1-L)^{-1/4} \} \\ &\leqq (4\pi m^2)^{1/2}(1-K)^{-1/2}(1-L')^{-1/4}(K_+^{1/2} + K_-^{1/2}) \\ &\leqq mK^{1/2}f(K)\,, \end{aligned}$$

where we put $f(K)=2(2\pi)^{1/2}(1-K)^{-1/2}(1-L')^{-1/4}$. q. e. d.

Proof of (5.14). Without loss of generality we may assume that $J_0 \geqq c > 0$ and $J_n \geqq c > 0$ for some positive c. Put $J_1^{(n)} = J_n - J_0$. Then $J_n \to J_0$, $n \to \infty$, in $\tilde{\mathfrak{J}}$ implies $K_n = \hat{t}(J_1^{(n)}) \to 0$. Hence, if n is sufficiently large, we have

$$\| W_\pm^{(n)} u - u \| \leqq m K_n^{1/2} f(K_n), \quad u \in \mathfrak{L}.$$

Since $K_n \to 0$ and consequently $f(K_n)$ is bounded in n, the right-hand side tends to zero as $n \to \infty$. Hence $W_\pm^{(n)} u \to u$ if $u \in \mathfrak{L}$, (5.14) follows from this by a standard argument.

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Added in proof. Corrections to Part I ([**4**]).

In Theorems 1 and 2 of [**4**, p. 249], for "symmetric", read "closed symmetric".

In Theorem 2 of [**4**, pp. 249-250], for "$V_n' = V - V_n$", read "V_n' be the closure of $V - V_n$".