Journal of the Mathematical Society of Japan Vol. 12, No. 2, April, 1960

On the conformal mapping of nearly circular domains.

By Hirosaku YOSHIKAWA

(Received Dec. 10, 1958)
(Revised May 18, 1959)

1. Let us denote by C a closed Jordan curve on w-plane, contained in $1-\varepsilon\leqq|w|\leqq 1+\varepsilon$ for $0<\varepsilon<1$ and surrounding the origin, and denote by D the interior of C. When ε is sufficiently small, D is a so-called nearly circular domain. Let $w=F(z)$ be the function mapping the interior of the unit circle $|z|<1$ conformally onto D such that $F(0)=0$, $F'(0)>0$. The estimates of various quantities related to D or $F(z)$ in terms of ε have been given by various authors, recently by S. E. Warschawski [8], E. Specht [5], and Z. Nehari and V. Singh [4]. In [8] and [5], $d\arg F(e^{i\theta})/d\theta$ is estimated under some additional conditions for C. We treat, in this paper, the similar problems under somewhat different conditions, where C is not necessarily starlike with respect to the origin and there may be several angular points on it. Further we derive the inequalities concerning $|F'(e^{i\theta})|$, $\arg F(e^{i\theta})-\theta$, etc. We consider next about the expansion of $F(z)$ by ε. The results obtained there are possibly helpful to the numerical computation of $F(z)$.

2. We begin with several lemmas.

LEMMA 1. *Let Δ be the sum of two open circular discs $|w|<1$ and $|w-a|<r$, where $0<r\leqq 1$ and $1-r<a<1+r$, and $e^{i\alpha}, e^{-i\alpha}$ $(0<\alpha<\pi/2)$ the intersections of those circumferences. Further we denote by $w=f(z)$ the function mapping $|z|<1$ conformally onto Δ such that $f(0)=0$, $f'(0)>0$, and put $f(e^{i\beta})=e^{i\alpha}$, $f(e^{-i\beta})=e^{-i\alpha}$. Then $d\arg f(e^{i\theta})/d\theta$ for $-\beta<\theta<\beta$ attains its maximum at $\theta=0$.*

PROOF. The function $w=f(z)$ is represented explicitly by the composition of the functions

$$z=\frac{1+i\zeta\tan\beta/2}{1-i\zeta\tan\beta/2}, \tag{1}$$

$$w=\frac{\cos\frac{\alpha-\delta}{2}}{\cos\frac{\alpha+\delta}{2}}\,\frac{1+i\omega\tan\frac{\alpha-\delta}{2}}{1-i\omega\tan\frac{\alpha+\delta}{2}} \tag{2}$$

and

$$\frac{1+\omega}{1-\omega}=\left(\frac{1+\zeta}{1-\zeta}\right)^{1+\delta/\pi}, \tag{3}$$

where δ $(0<\delta<\pi,\ 2\alpha+\delta\leqq\pi)$ is the angle between two circular arcs $(e^{-i\alpha}, 1, e^{i\alpha})$ and $(e^{-i\alpha}, a+r, e^{i\alpha})$, and $\beta=\alpha/(1+\delta/\pi)$. The relations (1) and (2) show that the arcs $(e^{-i\beta}, 1, e^{i\beta})$ on z-plane and $(e^{-i\alpha}, a+r, e^{i\alpha})$ on w-plane correspond respectively to the segments $-1<\zeta<1$ and $-1<\omega<1$. Further we obtain in virtue of (3) the inequality

$$|\zeta|\leqq|\omega| \tag{4}$$

on those segments. Now, with the notation

$$\psi(\theta)=\arg f(e^{i\theta}) \qquad (-\beta<\theta<\beta),$$

we have

$$\psi'(\theta)=\mathrm{Re}\left(\frac{z}{w}\frac{dw}{dz}\right) \qquad (z=e^{i\theta},\ w=f(e^{i\theta}))$$

$$=\mathrm{Re}\left[\frac{1}{2}\left(1+\frac{\delta}{\pi}\right)\cot\frac{\beta}{2}\cdot\left(\tan\frac{\alpha-\delta}{2}+\tan\frac{\alpha+\delta}{2}\right)\right.$$

$$\left.\times\frac{1-\omega^2}{1-\zeta^2}\frac{1+\zeta^2\tan^2\frac{\beta}{2}}{\left(1-i\,\omega\tan\frac{\alpha+\delta}{2}\right)\left(1+i\,\omega\tan\frac{\alpha-\delta}{2}\right)}\right]$$

$$=\frac{1}{2}\left(1+\frac{\delta}{\pi}\right)\cot\frac{\beta}{2}\cdot\frac{1-\omega^2}{1-\zeta^2}\left(1+\zeta^2\tan^2\frac{\beta}{2}\right)$$

$$\times\left(\frac{\tan\frac{\alpha+\delta}{2}}{1+\omega^2\tan^2\frac{\alpha+\delta}{2}}+\frac{\tan\frac{\alpha-\delta}{2}}{1+\omega^2\tan^2\frac{\alpha-\delta}{2}}\right).$$

Hence, by the relation $\beta<\alpha$ and (4), we get

$$\psi'(\theta)\leqq\frac{1}{2}\left(1+\frac{\delta}{\pi}\right)\cot\frac{\beta}{2}$$

$$\times\left(\tan\frac{\alpha+\delta}{2}\frac{1+\omega^2\tan^2\frac{\alpha}{2}}{1+\omega^2\tan^2\frac{\alpha+\delta}{2}}+\tan\frac{\alpha-\delta}{2}\frac{1+\omega^2\tan^2\frac{\alpha}{2}}{1+\omega^2\tan^2\frac{\alpha-\delta}{2}}\right),$$

and so

$$\psi'(0)-\psi'(\theta)\geqq\frac{1}{2}\left(1+\frac{\delta}{\pi}\right)\cot\frac{\beta}{2}\frac{\sin\delta/2}{\cos^2\alpha/2}\,\omega^2$$

$$\times\left(\frac{\tan\frac{\alpha+\delta}{2}\sin\left(\alpha+\frac{\delta}{2}\right)}{1-(1-\omega^2)\sin^2\frac{\alpha+\delta}{2}}-\frac{\tan\frac{\alpha-\delta}{2}\sin\left(\alpha-\frac{\delta}{2}\right)}{1-(1-\omega^2)\sin^2\frac{\alpha-\delta}{2}}\right)\geqq 0,$$

since $0<\alpha<\pi/2$, $0<\delta<\pi$ and $2\alpha+\delta\leqq\pi$. Thus we have

(5) $$\psi'(\theta) \leqq \psi'(0) = \left(1 + \frac{\delta}{\pi}\right) \frac{\sin\alpha \cot\beta/2}{\cos\alpha + \cos\delta},$$

the desired result of Lemma 1.

LEMMA 2. *Let $f(z)$ be the function defined in Lemma 1. Then $|f'(e^{i\theta})|$ for $-\beta < \theta < \beta$ attains its maximum at $\theta = 0$.*

PROOF. With the same notations as in the proof of Lemma 1, the relation

$$|f'(e^{i\theta})| = \left(1 + \frac{\delta}{\pi}\right) \frac{\sin\alpha \cot\beta/2}{1+\cos(\alpha+\delta)} \frac{1-\omega^2}{1-\zeta^2} \frac{1+\zeta^2 \tan^2 \dfrac{\beta}{2}}{1+\omega^2 \tan^2 \dfrac{\alpha+\delta}{2}}$$

holds. Hence we have, regarding (4) and the relation $\beta/2 < (\alpha+\delta)/2 < \pi/2$,

$$|f'(e^{i\theta})| \leqq |f'(1)| = \left(1 + \frac{\delta}{\pi}\right) \frac{\sin\alpha \cot\beta/2}{1+\cos(\alpha+\delta)}.$$

Thus Lemma 2 is proved.

Fixing r, the functions $f(z)$ and $\psi(\theta)$ in Lemma 1 depend on a for $1-r < a < 1+r$, and so we denote them again by $f_a(z)$ and $\psi_a(\theta)$ respectively. Then we have

LEMMA 3. *$\psi_a'(0)$ is a strictly increasing function of a.*

PROOF. For every $a \leqq b$ the function

$$g(z) = f_a^{-1}\left(\frac{a+r}{b+r} f_b(z)\right)$$

is clearly holomorphic in $|z| < 1$, besides in a neighbourhood of $z = 1$, and satisfies the conditions $|g(z)| < 1$ and $g(0) = 0$. Hence we have $|g(z)| \leqq |z|$ and so, regarding $g(1) = 1$,

$$\frac{a+r}{b+r} \frac{f_b'(1)}{f_a'(1)} = g'(1) \geqq 1.$$

Thus we find

(6) $$\psi_a'(0) \leqq \psi_b'(0),$$

where the equality occurs only for $a = b$ [**1**].

It is also proved easily that, fixing $a+r$, $\psi'(0)$ is a strictly decreasing function of r.

LEMMA 4. *Let $f(\theta)$ be a piecewise smooth function with period 2π, satisfying the relations*

(7) $$\int_0^{2\pi} f(\theta) d\theta = 0,$$

(8) $$-q(\theta) \leqq f'(\theta) \leqq p(\theta),$$

where $p(\theta)$ and $q(\theta)$ are piecewise continuous and periodic functions with period 2π. Then, putting

$$P(\theta, t) = \min\left[\int_0^t q(\theta+s)ds,\ \int_t^{2\pi} p(\theta+s)ds\right], \tag{9}$$

$$Q(\theta, t) = \min\left[\int_0^t p(\theta+s)ds,\ \int_t^{2\pi} q(\theta+s)ds\right], \tag{10}$$

the inequality

$$-\frac{1}{2\pi}\int_0^{2\pi} Q(\theta, t)dt \leqq f(\theta) \leqq \frac{1}{2\pi}\int_0^{2\pi} P(\theta, t)dt \tag{11}$$

holds.

PROOF. We have from (8)

$$f(\theta+t) \geqq f(\theta) - \int_0^t q(\theta+s)ds\,,$$

$$f(\theta+t) \geqq f(\theta) - \int_t^{2\pi} p(\theta+s)ds$$

at the same time for $0 \leqq t \leqq 2\pi$. Hence

$$f(\theta+t) \geqq f(\theta) - P(\theta, t)\,,$$

and it follows, noticing (7), that

$$2\pi f(\theta) - \int_0^{2\pi} P(\theta, t)dt \leqq \int_0^{2\pi} f(\theta+t)dt = 0\,.$$

We have therefore

$$f(\theta) \leqq \frac{1}{2\pi}\int_0^{2\pi} P(\theta, t)dt\,,$$

and similarly

$$f(\theta) \geqq -\frac{1}{2\pi}\int_0^{2\pi} Q(\theta, t)dt\,.$$

Thus (11) is proved.

When we put $q(\theta) \equiv \infty$, the inequality (11) becomes

$$\begin{aligned} -\frac{1}{2\pi}\int_0^{2\pi}(2\pi - t)p(\theta+t)dt \leqq f(\theta) \\ \leqq \frac{1}{2\pi}\int_0^{2\pi} t\,p(\theta+t)dt\,. \end{aligned} \tag{12}$$

3. We now consider a nearly circular domain D as defined in § 1. Let $w = F(z)$ be the function mapping $|z| < 1$ onto D conformally such that $F(0) = 0$, $F'(0) > 0$. We suppose that this domain satisfies the following additional conditions.

(i) *Boundary C is piecewise smooth and*

$$w'(s) \in H_s^{\alpha} \qquad (0 < \alpha \leqq 1) \tag{13}$$

on each divided closed arc, where $w = w(s)$ is the representation of C by its arc length.

Expression (13) implies that $w'(s)$ satisfies, as the function of s, Hölder's condition of order α.

(ii) *Through each point* $w=F(e^{i\theta})$ *on* C *there exists at least one circle of radius* $\rho(\theta)$, *contained in the closed domain* $\bar{D}$, *where* $\rho(\theta)$ *is a piecewise continuous function and* $\varepsilon<\rho(\theta)\leqq 1-\varepsilon$ $(0<\varepsilon<1/2)$.

Hence there may be a finite number of angular points on C, but the interior angles at them must be greater than π. Because of the condition (i) $F'(z)$ is continuous in $|z|\leqq 1$ [**2**], [**6**], and vanishes at the points on $|z|=1$ corresponding to such angular points.

Let w_0 be an arbitrary point on C, different from the angular points, and Γ the circle through w_0 such as mentioned in the condition (ii). Further let Δ be the sum of $|w|<1-\varepsilon$ with the interior of Γ. Next we denote by $\zeta=f(z)$ the function mapping $|z|<1$ onto Δ conformally such that $f(0)=0$, $f'(0)>0$, and by $w=g(\zeta)$ the function mapping Δ onto D such that $g(0)=0$, $g(w_0)=w_0$. Then the relation

$$F(z)=g(f(e^{i\gamma}z))$$

holds for some real γ. Hence we have

$$|F'(z_0)|=|g'(w_0)|\cdot|f'(e^{i\gamma}z_0)|, \tag{14}$$

where $F(z_0)=w_0$. Now the function

$$p(z)=F^{-1}(g^{-1}(F(z)))$$

is holomorphic in $|z|<1$, besides in a neighbourhood of $z=z_0$, and satisfies the conditions $|p(z)|<1$, $p(0)=0$ and $p(z_0)=z_0$, and so we have

$$|p'(z_0)|=|g'(w_0)|^{-1}\geqq 1. \tag{15}$$

Hence it follows from (14) and (15) that

$$|F'(z_0)|\leqq|f'(e^{i\gamma}z_0)|. \tag{16}$$

Putting

$$\varphi(\theta)=\arg F(e^{i\theta}),\quad \psi(\theta)=\arg f(e^{i\theta}),\quad z_0=e^{i\theta_0},$$

and regarding that C and Γ touch each other at w_0, it becomes

$$\varphi'(\theta_0)\leqq\psi'(\gamma+\theta_0). \tag{17}$$

On the other hand, using Lemma 1 and 3, we find

$$\psi'(\gamma+\theta_0)\leqq\frac{d}{d\theta}\arg f_0(e^{i\theta})\Big|_{\theta=0}, \tag{18}$$

where $f_0(z)$ $(f_0(0)=0,\ f_0'(0)>0)$ is the function mapping $|z|<1$ onto Δ_0, the sum of $|w|<1-\varepsilon$ and $|w-(1-\rho+\varepsilon)|<\rho$, $\rho=\rho(\theta_0)$. Then, denoting the right-hand side of (18) by $A(\rho)$, the relation

(19) $$A(\rho)=\left(1+\frac{\delta}{\pi}\right)\frac{\sin\alpha\cot\beta/2}{\cos\alpha+\cos\delta}$$

holds in virtue of (5), with

(20) $$\cos\delta=\frac{\rho-2\varepsilon+\rho\varepsilon}{\rho(1-\varepsilon)},$$

(21) $$\cos\alpha=\frac{1-\rho-\rho\varepsilon+\varepsilon^2}{(1-\varepsilon)(1-\rho+\varepsilon)},$$

(22) $$\beta=\alpha\Big/\left(1+\frac{\delta}{\pi}\right).$$

Fixing ε, $A(\rho)$ is strictly decreasing, as noticed after the proof of Lemma 3 We have thus from (17) the inequality $\varphi'(\theta_0)\leqq A(\rho(\theta_0))$. It follows similarly from (16) and Lemma 2 and 3 that $|F'(z_0)|\leqq(1+\varepsilon)A(\rho(\theta_0))$.

Considering furthermore the relations

$$\int_0^{2\pi}(\varphi(\theta)-\theta)d\theta=\int_0^{2\pi}\arg\frac{F(e^{i\theta})}{e^{i\theta}}\,d\theta=2\pi\arg F'(0)=0\,,$$

$$(\varphi(\theta)-\theta)'\leqq A(\rho(\theta))-1\,,$$

we obtain from (12) the estimate of $\varphi(\theta)-\theta$. Thus we have

THEOREM 1. *If a nearly circular domain D satisfies the conditions* (i) *and* (ii), *$F'(z)$ is continuous in $|z|\leqq 1$ and*

$$\frac{d}{d\theta}\arg F(e^{i\theta})\leqq A(\rho(\theta))\,,$$

$$|F'(e^{i\theta})|\leqq(1+\varepsilon)A(\rho(\theta))\,,$$

$$-\frac{1}{2\pi}\int_0^{2\pi}(2\pi-t)[A(\rho(\theta+t))-1]dt\leqq\arg F(e^{i\theta})-\theta$$

$$\leqq\frac{1}{2\pi}\int_0^{2\pi}t[A(\rho(\theta+t))-1]dt\,,$$

where $A(\rho)$ is given by (19), (20), (21) *and* (22).

When ε is sufficiently small and $\rho(\theta)=\text{const}=1-k\varepsilon$ for a fixed k $(1\leqq k<(1-\varepsilon)/\varepsilon)$, D is necessarily starlike with respect to the origin and then we can estimate A from above by the simple expression of ε and k, as follows. Since

(23) $$\delta\leqq 2\tan\frac{\delta}{2}=\frac{2\sqrt{k}\,\varepsilon}{\sqrt{1-(k+1)\varepsilon}},$$

$$\alpha\leqq\cos^{-1}\frac{k-1}{k+1}=2\cot^{-1}\sqrt{k},$$

we have

$$\tag{24} \begin{aligned} \cot\frac{\beta}{2} &= \cot\left(\frac{\alpha}{2}-\frac{\alpha}{2}\frac{\delta/\pi}{1+\delta/\pi}\right) \\ &\leqq \cot\frac{\alpha}{2}\cdot\left(1-\frac{\alpha}{\sin\alpha}\frac{\delta/\pi}{1+\delta/\pi}\right)^{-1} \\ &\leqq \sqrt{k(1-(k+1)\varepsilon)}\cdot\left[1-(k+1)\left(1+\frac{2}{\pi}\cot^{-1}\sqrt{k}\right)\varepsilon\right]^{-1}. \end{aligned}$$

Inserting (20), (21), (23) and (24) in (19) we obtain, after some computations

$$A \leqq \left[1-\frac{2}{\pi}(\sqrt{k}+(k+1)\cot^{-1}\sqrt{k})\varepsilon-3(k+1)^2\varepsilon^2\right]^{-1},$$

where the coefficient of ε is best possible, but that of ε^2 is somewhat rough. We have further from Lemma 4

$$\begin{aligned} |\arg F(e^{i\theta})-\theta| &\leqq \pi(1-1/A) \\ &\leqq 2(\sqrt{k}+(k+1)\cot^{-1}\sqrt{k})\varepsilon+3\pi(k+1)^2\varepsilon^2, \end{aligned}$$

since $-1 \leqq (\arg F(e^{i\theta})-\theta)' \leqq A-1$. However the inequality of the form $|\arg F(e^{i\theta})-\theta| \leqq K\varepsilon$ for a suitable constant K is obtained under weaker hypotheses of D [**3**], [**8**].

4. We can derive following lemmas like Lemma 1, 2 and 3.

LEMMA 5. *Let $\varDelta$ be the intersection of two open circular discs $|w|<1$ and $|w+a|<r$, where $r>1$ and $r-1<a<r$, and $e^{i\alpha}$, $e^{-i\alpha}$ the intersections of two circumferences. Let further $w=f(z)$ be the function mapping $|z|<1$ onto $\varDelta$ such that $f(0)=0$, $f'(0)>0$, and put $f(e^{i\beta})=e^{i\alpha}$, $f(e^{-i\beta})=e^{-i\alpha}$. Then $d\arg f(e^{i\theta})/d\theta$ and $|f'(e^{i\theta})|$ $(-\beta<\theta<\beta)$ attain their minima at $\theta=0$.*

LEMMA 6. *As the function of a, $d\arg f(e^{i\theta})/d\theta|_{\theta=0}$ is strictly decreasing.*

Now we consider, about the domain D, the following condition instead of condition (ii).

(iii) *Through each point $w=F(e^{i\theta})$ on C there exists at least one circle of radius $\sigma(\theta)$, involving D, where $\sigma(\theta)$ is piecewise continuous and $\sigma(\theta)\geqq 1+\varepsilon$.*

From Lemma 4, 5 and 6 we obtain the following theorem by the similar considerations as in Theorem 1.

THEOREM 2. *If a nearly circular domain D satisfies the conditions* (i) *and* (iii), *$[F'(z)]^{-1}$ is continuous in $|z|\leqq 1$ and we have*

$$\frac{d}{d\theta}\arg F(e^{i\theta}) \geqq B(\sigma(\theta)),$$

$$|F'(e^{i\theta})| \geqq (1-\varepsilon)B(\sigma(\theta)),$$

$$-\frac{1}{2\pi}\int_0^{2\pi} t[1-B(\sigma(\theta+t))]dt \leqq \arg F(e^{i\theta})-\theta$$

$$\leqq \frac{1}{2\pi}\int_0^{2\pi}(2\pi - t)\,[1 - B(\sigma(\theta + t))]dt\,,$$

where

(25) $$B(\sigma) = \left(1 - \frac{\delta}{\pi}\right)\frac{\sin\alpha\cot\beta/2}{\cos\alpha + \cos\delta}\,,$$

$$\cos\alpha = \frac{\sigma + 2\varepsilon - \sigma\varepsilon}{\sigma(1+\varepsilon)}\,,$$

$$\cos\delta = \frac{\sigma - 1 - \sigma\varepsilon - \varepsilon^2}{(1+\varepsilon)(\sigma - 1 + \varepsilon)}\,,$$

$$\beta = \alpha \Big/ \left(1 - \frac{\delta}{\pi}\right).$$

Further, putting $\sigma(\theta) = \text{const} = 1 + k'\varepsilon$ $(k' \geqq 1)$, we have the estimates

$$B \geqq 1 - \frac{2}{\pi}(\sqrt{k'} + (k'+1)\cot^{-1}\sqrt{k'})\varepsilon - \frac{2}{\pi}(k'+1)\sqrt{k'}\,\varepsilon^2\,,$$

$$|\arg F(e^{i\theta}) - \theta| \leqq \pi(1 - B)\,,$$

by the similar computations as of A.

If the domain satisfies the conditions (ii) and (iii) at the same time, it necessarily satisfies the condition (i). In fact, then $w'(s)$ exists at each point on C and belongs to H_s^1. Hence we have

THEOREM 3. *When a nearly circular domain satisfies the conditions* (ii) *and* (iii), *it follows that*

$$B(\sigma(\theta)) \leqq \frac{d}{d\theta}\arg F(e^{i\theta}) \leqq A(\rho(\theta))\,,$$

$$-\frac{1}{2\pi}\int_0^{2\pi} Q(\theta, t)dt \leqq \arg F(e^{i\theta}) - \theta \leqq \frac{1}{2\pi}\int_0^{2\pi} P(\theta, t)dt\,,$$

where $A(\rho)$ and $B(\sigma)$ are given by (19) *and* (25), *and $P(\theta, t)$ and $Q(\theta, t)$ are given by* (9) *and* (10) *respectively, putting $p(\theta) = A(\rho(\theta)) - 1$, $q(\theta) = 1 - B(\sigma(\theta))$.*

In this case we can estimate $|F'(z) - 1|$ by the method in [8].

5. Next let us consider about the expansion of $F(z)$ by ε. We suppose hereafter that the boundary C is starlike with respect to the origin, and we represent it by the equation

$$r = 1 + \varepsilon h(\theta)$$

in polar coordinates, where $0 < \varepsilon < 1$ and $|h(\theta)| \leqq 1$. Following lemma is proved by the inequality of Carathéodory, as shown in [7].

LEMMA 7. *If $G(z)$ is holomorphic in $|z| < 1$ and satisfies the conditions*

$$|\operatorname{Re} G(z)| \leqq \eta(< 1), \quad G(0) = real,$$

$$|G(z_1) - G(z_2)| \leqq k|z_1 - z_2|^{\alpha} \qquad (0 < \alpha \leqq 1)$$

for each z_1, z_2, *then*

$$|\operatorname{Im} G(z)| \leqq \left(k + \frac{2 \log 2}{\pi}\right)\eta + \frac{2}{\pi\alpha}\eta \log \frac{1}{\eta}.$$

Using this lemma we can prove

THEOREM 4. *When* $h(\theta)$ *is* n *times differentiable and* $h^{(n)}(\theta) \in H_\theta^\alpha$ *for* $0 < \alpha < 1$, *we have the expansion*

$$\log \frac{F(z)}{z} = \sum_{\nu=1}^{n} F_\nu(z)\varepsilon^\nu + O\left(\varepsilon^{n+1}\left(\log \frac{1}{\varepsilon}\right)^n\right), \tag{26}$$

where the first derivative of the residual term is continuous in $|z| \leqq 1$, $F_\nu(z)$ $(\nu = 1, 2, \cdots, n)$ *are holomorphic in* $|z| < 1$ *and independent of* ε, *and* $F_\nu(0) = real$. *Further* $F_\nu(z), F_\nu'(z), \cdots, F_\nu^{(n-\nu+1)}(z)$ *are continuous in* $|z| \leqq 1$ *and* $F_\nu^{(n-\nu+1)}(e^{i\theta}) \in H_\theta^\alpha$.

PROOF. It is clear that such expansion is uniquely determined, if it is possible.

We first consider the case $n = 1$, and so

$$h'(\theta) \in H_\theta^\alpha. \tag{27}$$

Now the relation

$$|\varphi(\theta) - \theta| \leqq k_1\varepsilon$$

holds for some constant k_1, where $\varphi(\theta) = \arg F(e^{i\theta})$ [**3**], [**8**]. Hence

$$\begin{aligned} \log \left|\frac{F(e^{i\theta})}{e^{i\theta}}\right| &= \log[1 + \varepsilon h(\varphi(\theta))] \\ &= \log(1 + \varepsilon h(\theta)) + O(\varepsilon^2) \\ &= \varepsilon h(\theta) + O(\varepsilon^2). \end{aligned}$$

Therefore, putting

$$F_1(z) = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta,$$

$$G(z) = \log \frac{F(z)}{z} - \varepsilon F_1(z),$$

we have

$$|\operatorname{Re} G(z)| \leqq k_2\varepsilon^2, \qquad G(0) = \text{real}.$$

Further we see in virtue of (27) that $F'(z)$ and $F_1'(z)$ are continuous in $|z| \leqq 1$ [**6**], [**2**], and so $G'(z)$ also, and $G'(z)$ is bounded with respect to ε. Hence we have by Lemma 7

$$|G(z)| \leqq k_3\varepsilon^2 \log \frac{1}{\varepsilon}.$$

Next let us suppose that

(28) $$h^{(n+1)}(\theta) \in H_\theta^\alpha$$

and that the results of the theorem hold. Then putting

$$F_\nu(e^{i\theta}) = u_\nu(\theta) + iv_\nu(\theta) \qquad (\nu = 1, 2, \cdots, n),$$

we have

(29) $$u_\nu^{(n-\nu+1)}(\theta) \in H_\theta^\alpha,\ v_\nu^{(n-\nu+1)}(\theta) \in H_\theta^\alpha \quad (\nu = 1, 2, \cdots, n).$$

Now, since

$$\log \frac{F(e^{i\theta})}{e^{i\theta}} = \log[1 + \varepsilon h(\varphi(\theta))] + i(\varphi(\theta) - \theta),$$

we have the relations

(30) $$\log[1 + \varepsilon h(\varphi(\theta))] = \sum_{\nu=1}^{n} u_\nu(\theta)\varepsilon^\nu + O\left(\varepsilon^{n+1}\left(\log \frac{1}{\varepsilon}\right)^n\right),$$

(31) $$\varphi(\theta) - \theta = \sum_{\nu=1}^{n} v_\nu(\theta)\varepsilon^\nu + O\left(\varepsilon^{n+1}\left(\log \frac{1}{\varepsilon}\right)^n\right).$$

It follows from (31), noticing (28) and (29), that

$$\begin{aligned}\log[1 + \varepsilon h(\varphi(\theta))] &= \log[1 + \varepsilon h(\theta + \sum_{\nu=1}^{n} v_\nu(\theta)\varepsilon^\nu)] + O\left(\varepsilon^{n+2}\left(\log \frac{1}{\varepsilon}\right)^n\right) \\ &= \sum_{\nu=1}^{n+1} u_\nu^*(\theta)\varepsilon^\nu + O\left(\varepsilon^{n+2}\left(\log \frac{1}{\varepsilon}\right)^n\right),\end{aligned}$$

where $u_\nu^*(\theta)$ $(\nu = 1, 2, \cdots, n+1)$ are such functions that

(32) $$u_\nu^{*(n-\nu+2)}(\theta) \in H_\theta^\alpha \quad (\nu = 1, 2, \cdots, n+1).$$

But, comparing with (30), we find the relations

(33) $$u_\nu(\theta) = u_\nu^*(\theta) \quad (\nu = 1, 2, \cdots, n).$$

We put further

(34) $$F_{n+1}(z) = \frac{1}{2\pi}\int_0^{2\pi} u_{n+1}^*(\theta) \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\theta.$$

Then for the function

$$G(z) = \log \frac{F(z)}{z} - \sum_{\nu=1}^{n+1} F_\nu(z)\varepsilon^\nu$$

we see that

$$|\operatorname{Re} G(z)| \leqq k_4 \varepsilon^{n+2}\left(\log \frac{1}{\varepsilon}\right)^n, \quad G(0) = \text{real},$$

and that $G'(z)$ is continuous in $|z| \leqq 1$, as before. Hence we have

$$|G(z)| \leqq k_5 \varepsilon^{n+2}\left(\log \frac{1}{\varepsilon}\right)^{n+1},$$

and therefore

$$\log \frac{F(z)}{z} = \sum_{\nu=1}^{n+1} F_\nu(z)\varepsilon^\nu + O\Big(\varepsilon^{n+2}\Big(\log \frac{1}{\varepsilon}\Big)^{n+1}\Big),$$

where $F_\nu(0) = \text{real}$ ($\nu = 1, 2, \cdots, n+1$), $F_\nu(z)$, $F_\nu'(z), \cdots, F_\nu^{(n-\nu+2)}(z)$ are continuous in $|z| \leqq 1$ and $F_\nu^{(n-\nu+2)}(e^{i\theta}) \in H_\theta^\alpha$ because of (32), (33) and (34). The conclusion with respect to the residual term is now clear. Thus Theorem 4 is proved.

Theorem 4 gives immediately

THEOREM 5. *If $h(\theta)$ is indefinitely differentiable, the asymptotic expansion*

$$\log \frac{F(z)}{z} \sim \sum_{\nu=1}^{\infty} F_\nu(z)\varepsilon^\nu$$

holds, where the derivatives of $F_\nu(z)$ of each order are continuous in $|z| \leqq 1$ and $F_\nu(0) = real$.

6. Given the function $h(\theta)$, we can compute the functions $F_\nu(z)$ ($\nu = 1, 2, \cdots$) practically as follows. We expand the right-hand side of

$$\sum_{\nu=1}^{\infty} u_\nu(\theta)\varepsilon^\nu = \log[1 + \varepsilon h(\theta + \sum_{\nu=1}^{\infty} v_\nu(\theta)\varepsilon^\nu)] \tag{35}$$

formally by ε and compare the coefficients of both sides. It follows then that

$$u_1(\theta) = h(\theta),$$

$$u_2(\theta) = h'(\theta)v_1(\theta) - \frac{1}{2} h(\theta)^2,$$

$$u_3(\theta) = h'(\theta)v_2(\theta) + \frac{1}{2} h''(\theta)v_1(\theta)^2 - h(\theta)h'(\theta)v_1(\theta) + \frac{1}{3} h(\theta)^3,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.$$

From these and the relations

$$v_\nu(\theta) = -\frac{1}{2\pi}\int_0^{2\pi}(u_\nu(\theta) - u_\nu(\tau)) \cot \frac{\theta-\tau}{2}\, d\tau \quad (\nu = 1, 2, \cdots),$$

we obtain the desired functions

$$F_\nu(z) = \frac{1}{2\pi}\int_0^{2\pi} u_\nu(\theta) \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\theta \quad (\nu = 1, 2, \cdots).$$

Therefore we find that the functions $F_\nu(z)$ become polynomials, when $h(\theta)$ is a trigonometric polynomial. Putting for example $h(\theta) = \cos\theta$, we have

$$F_1(z) = z, \quad F_2(z) = \frac{1}{4}(-3 + z^2),$$

$$F_3(z) = \frac{1}{12}(3z + z^3),$$

$$F_4(z)=\frac{1}{32}(-13+4z^2+z^4),$$

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Faculty of Engineering,
Kyushu University.

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Addendum

When the boundary C of the domain is represented by the equation such as

$$r=h(\theta,\varepsilon),$$

we obtain, with respect to the mapping function $F(z)$, the following theorems corresponding to Theorem 4 and 5.

THEOREM 4′. *Let $h(\theta,\varepsilon)$ be expanded so that*

$$h(\theta,\varepsilon)=1+\sum_{\nu=1}^{n}h_\nu(\theta)\varepsilon^\nu+O\left(\varepsilon^{n+1}\left(\log\frac{1}{\varepsilon}\right)^{n-1}\right)$$

for sufficiently small ε, where

$$h_\nu^{(n-\nu+1)}(\theta)\in H_\theta^\alpha \qquad (\nu=1,2,\cdots,n)$$

for $0<\alpha<1$. Let further $\partial h/\partial\theta=O(\varepsilon)$ and $\partial h/\partial\theta\in H_\theta^\alpha$. Then we have the expansion

$$\log\frac{F(z)}{z}=\sum_{\nu=1}^{n}F_\nu(z)\varepsilon^\nu+O\left(\varepsilon^{n+1}\left(\log\frac{1}{\varepsilon}\right)^{n}\right),$$

where the functions $F_\nu(z)$ $(\nu=1,2,\cdots,n)$ *and the residual term satisfy the same properties as in Theorem* 4.

THEOREM 5'. *Let* $h(\theta,\varepsilon)$ *be expanded asymptotically so that*

$$h(\theta,\varepsilon)\sim 1+\sum_{\nu=1}^{\infty}h_\nu(\theta)\varepsilon^\nu$$

for sufficiently small ε, *where* $h_\nu(\theta)$ $(\nu=1,2,\cdots)$ *are indefinitely differentiable functions. Let further* $\partial h/\partial\theta=O(\varepsilon)$ *and* $\partial h/\partial\theta\in H_\theta^\alpha$ *for* $0<\alpha<1$. *Then we have the asymptotic expansion*

$$\log\frac{F(z)}{z}\sim\sum_{\nu=1}^{\infty}F_\nu(z)\varepsilon^\nu\,,$$

where $F_\nu(z)$ $(\nu=1,2,\cdots)$ *satisfy the same properties as in Theorem* 5.

The proofs of these theorems proceed similarly as of Theorem 4 and 5. The functions $F_\nu(z)$ $(\nu=1,2,\cdots)$ are obtained practically from the formula

$$\sum_{\nu=1}^{\infty}u_\nu(\theta)\varepsilon^\nu=\log h(\theta+\sum_{\nu=1}^{\infty}v_\nu(\theta)\varepsilon^\nu,\,\varepsilon)\,,$$

instead of (35). Taking, for example, by C an ellipse

$$r=(1-e^2\cos^2\theta)^{-1/2}$$

with small eccentricity e, we can expand $\log(F(z)/z)$ asymptotically by e^2. In this case also all the coefficients become polynomials of z. That is,

$$F_1(z)=\frac{1}{4}(1+z^2),\quad F_2(z)=\frac{1}{32}(1+4z^2+3z^4)\,,$$

$$F_3(z)=\frac{1}{96}(-1+3z^2+9z^4+5z^6)\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,.$$