Journal of the Mathematical Society of Japan Vol. 11, No. 4, October, 1959

# Some results on arithmetic functions.

By S. A. AMITSUR

(Received April 21, 1959)

**1. Introduction.** Let $g(n)$ be an arithmetic function defined for all positive integers $n$. Following an idea of Yamamoto, we make correspond to $g(n)$ a linear operator $I_g(f)$ acting on the space of all functions $f(x)$ $(x\geqq 1)$, defined by

$$(I_g f)(x)=\sum_{n\leqq x}\frac{g(n)}{n}f\left(\frac{x}{n}\right). \tag{1}$$

The linear operators $I_g$ were dealt with extensively in a previous paper ([**1**]), with the aid of a symbolic calculus introduced and used to approximate $I_g$ for functions $f=f(\log x)$ which are polynomials in $\log x$.

Let $G(D)=\sum_{\nu=-p}^{\infty}g_\nu D^\nu$ be a power series in a symbol $D$ with only a finite number of negative powers of $D$. The symbol $D$ stands for the formal derivative $d/d\log x$. That is we set:

$$D^k\log^n x=\frac{n\,!}{(n-k)\,!}\log^{n-k}x \quad \text{for } n\geqq k$$

and all positive and non-positive values of $k$, $D^k\log^n x=0$ for $n<k$.

The notation $I_g=G(D)+O(\varphi_n)$ serves in [**1**] to denote that $I_g\log^n x-G(D)\log^n x=O(\varphi_n(x))$, where $\varphi_n(x)$, $n\geqq 0$, are non-negative functions. In a more explicit form, the last relation states that

$$I_g\log^n x-G(D)\log^n x=\sum_{\nu\leqq x}\frac{g(\nu)}{\nu}\log^n\frac{x}{\nu}-\sum_{i=-p}^{n}\frac{n\,!}{(n-i)\,!}g_i\log^{n-i}x=O(\varphi_n). \tag{2}$$

It is known that $I_gI_h=I_k$ where $k=g*h$ is the convolution of $g$ and $h$, i.e. $k(n)=\sum_{d|n}g(d)h(n/d)$. Let $I_g=G(D)+O(\varphi_n)$ and $I_h=H(D)+O(\psi_n)$ then it was shown in [**1**, Theorem 4.1] that $I_gI_h=G(D)H(D)+O(\rho_n)$ and a certain bound for $\rho_n$ was given, which was not symmetric in $g$ and $h$; furthermore, an important drawback of that theorem was that $G(D)H(D)f$ had always to be computed as $G(D)[H(D)f]$ and not by the ordinary product of the power series $[G(D)H(D)]f$. This fact caused some complications in the computation in the proof of Theorem 9.1 of [**1**].

The first part of the present paper tries to give a more satisfactory bound for $\rho_n$ and to prove that $I_g I_h = G(D)H(D) + O(\rho_n)$ with $G(D)H(D)$ to be the product in the ring of power series in $D$. The method used to compute the bound for $\rho_n$ is an extension of the Dirichlet's (hyperbola-) method of computing $\sum_{n\leqq x} d(n)$.

The result is applied to obtain some new and old formulas on the asymptotic behaviour of the mean of certain arithmetic functions. Among others we show that

$$\sum_{n\leqq x} n^{-1}d(n) = \frac{1}{2}\log^2 x + 2c_0 \log x + (c_0^2 + c_1) + O(x^{-1/2}),$$

$$\sum_{n\leqq x} n^{-1}d^2(n) = \frac{1}{4}\log^4 x + a_1 \log^3 x + \cdots + a_4 + O(x^{-1/6+\varepsilon}) \quad \text{for every } \varepsilon > 0,$$

$$\sum_{n\leqq x} n^{-1}|\mu(n)| = \frac{6}{\pi^2}\log x + b_0 + O(x^{-1/3}),$$

where $d(n)$ is the number of divisors of $n$, $\mu(n)$ is the Möbius function and $c_0$ is the Euler constant.

The second part of the paper contains improvements and extensions of the main results of [**1**].

The Main Theorem of [**1**] (Theorem 6.1), which yielded the prime number theorem in many of its equivalent forms, is in some sense artificial as it contains the particular function $\Lambda$; furthermore, one of its conditions is superfluous (condition 3) and consequently, corresponding conditions were introduced in the rest of the results of [**1**]. The purpose of the second part of the present paper is to repair and improve the above mentioned result of [**1**] and its consequences. As a by-result we shall obtain among others the fact that

$$\sum_{n\leqq x} n^{-1}\Lambda_2(n)\log(x/n) = (1/3)\log^3 x - c_0 \log^2 x + (c_0^2 - 2c_1)\log x + 2c_2 + o(1)$$

where $\Lambda_2 = \mu * \log^2 x$ is the function used in Selberg's formula.

## 2. The main theorem and some special cases.

Let $G(D) = \sum_{\nu=-p}^{\infty} g_\nu D^\nu$, $H(D) = \sum_{\nu=-q}^{\infty} h_\nu D^\nu$. Put:

$$(3) \quad R_n^g(x) = I_g \log^n x - G(D)\log^n x = \sum_{\nu\leqq x} \nu^{-1}g(\nu)\log^n \frac{x}{\nu} - \sum_{i=-p}^{n}(n-i)!^{-1}n!\, g_i \log^{n-i} x,$$

$$(4) \quad R_n^h(x) = I_h \log^n x - H(D)\log^n x = \sum_{\nu\leqq x} \nu^{-1}h(\nu)\log^n \frac{x}{\nu} - \sum_{i=-q}^{n}(n-i)!^{-1}n!\, h_i \log^{n-i} x,$$

$$(5) \quad R_n^{gh}(x) = I_g I_h \log^n x - [G(D)H(D)]\log^n x.$$

We shall also write $F(D)=G(D)H(D)=\sum_{t=-(p+q)}^{\infty} f_t D^t$ where $f_t=\sum_{i+k=t} g_i h_k$.

The main result of the present paper is:

THEOREM 1. A) *For* $1 \leqq \gamma \leqq x$:

$$(6)\qquad R_n^{gh}(x)=\sum_{\nu\leqq\gamma}\frac{g(\nu)}{\nu}R_n^h\left(\frac{x}{\nu}\right)+\sum_{\nu\leqq x/\gamma}\frac{h(\nu)}{\nu}R_n^g\left(\frac{x}{\nu}\right)-\sum_{j=0}^{n}\binom{n}{j}R_{n-j}^g(\gamma)R_j^h\left(\frac{x}{\gamma}\right)$$
$$+\sum_{i=1}^{q}\sum_{j=1}^{i}\frac{n!}{(i-j)!(n+j)!}R_{n+j}^g(\gamma)h_{-i}\log^{i-j}\frac{x}{\gamma}$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{i}\frac{n!}{(i-j)!(n+j)!}R_{n+j}^h\left(\frac{x}{\gamma}\right)g_{-i}\log^{i-j}\gamma\,.$$

B) *And*

$$(6\text{B})\qquad R_n^{gh}(x)=\sum_{\nu\leqq x}\frac{g(\nu)}{\nu}R_n^h\left(\frac{x}{\nu}\right)+\sum_{i=-q}^{n}\frac{n!}{(n-i)!}h_iR_{n-i}^g(x)$$
$$-\sum_{i=1}^{p}\sum_{k=i}^{p}\frac{n!}{(k-i)!}h_{n+i}g_{-k}\log^{k-i}x$$
$$=\sum_{\nu\leqq x}\frac{h(\nu)}{\nu}R_n^g\left(\frac{x}{\nu}\right)+\sum_{i=-p}^{n}\frac{n!}{(n-i)!}g_iR_{n-i}^h(x)$$
$$-\sum_{k=1}^{q}\sum_{i=k}^{q}\frac{n!}{(i-k)!}g_{n+k}h_{-i}\log^{i-k}x\,.$$

The proof of Theorem 1 will be given in section 4. Here we apply this result to some special cases:

THEOREM 2. *Let* $I_g=G(D)+O(x^{-\alpha}\log^{\sigma_n}x)$, $I_h=H(D)+O(x^{-\beta}\log^{\tau_n}x)$ *where* $0<\alpha,\beta$ *and* $\sigma_n,\tau_n$ *are two non-decreasing sequences of non-negative integers. If* $g=O(\log^u x)$, $h=O(\log^v x)$ *then*

$$I_gI_h=G(D)H(D)+O(x^{-\gamma}\log^{\lambda_n}x)$$

*where* $1/\gamma=1/\alpha+1/\beta$ *and* $\lambda_n=\mathrm{Max}(\tau_{n+p}+p-1,\ \tau_n+u,\ \sigma_{n+q}+q-1,\ \sigma_n+v)$.

PROOF. Consider the five terms of $R_n^{gh}(x)$ as given in (6) for $\gamma=x^t$ $0\leqq t\leqq 1$; and note that $R_n^g(x)=O(x^{-\alpha}\log^{\sigma_n}x)$ and $R_n^h(x)=O(x^{-\beta}\log^{\tau_n}x)$.

The first term yields

$$\sum_{\nu\leqq x^t}\nu^{-1}g(\nu)R_n^h(\nu^{-1}x)=O\Big(\sum_{\nu\leqq x^t}\nu^{-1}\log^u\nu[(x\nu^{-1})^{-\beta}\log^{\tau_n}(x\nu^{-1})]\Big)$$
$$=O\Big(x^{-\beta}\log^{\tau_n}x\sum_{\nu\leqq x^t}\nu^{\beta-1}\log^u\nu\Big)=O(x^{\beta(t-1)}\log^{\tau_n+u}x)\,.$$

Similarly the second term is $O(x^{-\alpha t}\log^{\sigma_n+v}x)$. The third term is readily seen to be $O(x^{-\alpha t+\beta(t-1)}\log^{\tau_n+\sigma_n}x)$, which is clearly $O(x^{-\alpha t})$ for $0<t<1$. The other two terms in (6) are readily seen to be $O(x^{-\alpha t}\log^{\sigma_{n+q}+q-1}x)$ and $O(x^{-\beta(1-t)}\log^{\tau_{n+p}+p-1}x)$.

Choose $t$ so that $-\alpha t=\beta(t-1)$ i. e. $t=(\alpha+\beta)^{-1}\beta$ and put $-\gamma=-\alpha t$. Thus $1/\gamma=1/\alpha+1/\beta$, and the rest of the proof follows now easily.

For the purpose of the next theorem we introduce the following notation:

We set $P(x)=O_\varepsilon(x^{-\alpha})$, $\alpha>0$, if $P(x)=O(x^{-\alpha+\varepsilon})$ for every $\varepsilon>0$. Thus $P(x)=O_\varepsilon(1)$ if $P(x)=O(x^\varepsilon)$ for every $\varepsilon>0$.

One would have liked to extend Theorem 2 for the product of $r$ function $g_1*\cdots*g_r$, but it seems to involve too many computation. A somewhat less satisfactory result can be obtained by a relatively simple induction. To this end we first note that:

LEMMA 1. *Let $g_1(n),\cdots,g_r(n)$ be $r$ arithmetic functions for which $g_i(n)=O_\varepsilon(1)$ hold, and let $h=g_i*\cdots*g_r$ be the convolution product of the $g_i$, then $h(n)=O_\varepsilon(1)$.*

The proof follows by induction on $r$, and first we consider the case $r=2$.

For a given $\varepsilon>0$ choose $K>0$ such that $|g_i(n)|<Kn^\varepsilon$. Thus

$$|h(n)|=|\sum_{d|n}g_1(d)g_2(n/d)|\leqq(\sum_{d|n}1)Kn^\varepsilon=Kn^\varepsilon d(n)\,.$$

Now $d(n)=O_\varepsilon(1)$ by [**2**, Theorem 315, p. 260] and the rest of the proof is evident.

We can prove now our next theorem.

THEOREM 3. *Let $I_{g_i}=G_i(D)+O_\varepsilon(x^{-\alpha_i})$, $\alpha_i>0$, and $g_i=O_\varepsilon(1)$, $i=1,2,\cdots,r$, then*

$$I_{g_1}\cdots I_{g_r}=G_1G_2\cdots G_r+O_\varepsilon(x^{-\alpha})$$

*with $1/\alpha=1/\alpha_1+\cdots+1/\alpha_r$.*

In view of Lemma 1, one can use an induction process in the proof of this theorem. We shall consider here only the case $r=2$, where proof is almost identical with the proof of Theorem 2. Indeed, put $g_1=g, g_2=h$, $\alpha_1=\alpha$ and $\alpha_2=\beta$. For a given $\varepsilon>0$, and for $\gamma=x^t$, the term of $R_n^{gh}(x)$ as given in (6) is

$$\sum_{\nu\leqq x^t}\nu^{-1}g(\nu)R_n^h(\nu^{-1}x)=O[\sum_{\nu\leqq x^t}\nu^{-1+\varepsilon}(\nu^{-1}x)^{-\beta+\varepsilon}]=O(x^{\beta(t-1)+\varepsilon})\,.$$

The other terms are readily seen to be either $O(x^{\beta(t-1)+\varepsilon})$ or $O(x^{-\alpha t+\varepsilon})$. Hence, choosing $-\gamma=-\alpha t=\beta(t-1)$, we have $R_n^{gh}(x)=O(x^{-\gamma+\varepsilon})$ which yields $R_n^{gh}(x)=O_\varepsilon(x^{-\gamma})$, q. e. d.

The following is very useful in approximating functions $g$:

THEOREM 4. *Let $g(n)$ be an arithmetic function satisfying:*

$$\sum_{\mu\leqq x}\frac{g(\mu)}{\mu}=\frac{\gamma_{-p}}{p!}\log^p x+\frac{\gamma_{-p+1}}{(p-1)!}\log^{p-1}x+\cdots+\gamma_0+\rho(x) \tag{7}$$

*such that* $\int_1^\infty t^{-1}\rho(t)\log^\nu t\,dt$ *converges for all* $\nu \geqq 0$, *then for the power series* $G(D)=\sum_{\nu=-p}^{\infty}\gamma_\nu D^\nu$ *where* $\gamma_\nu$ *for* $\nu\leqq 0$ *are given in* (7), *and*

$$\gamma_\nu=\frac{(-1)^{\nu-1}}{(\nu-1)!}\int_1^\infty \frac{\rho(t)\log^{\nu-1}t}{t}\,dt,\qquad \nu\geqq 1, \tag{8a}$$

*we have* $R_0^g(x)=\rho(x)$ *and*

$$R_n^g(x)=\int_x^\infty \rho(t)\,d\log^n(xt^{-1})=(-1)^n\int_0^\infty \rho(xe^u)du^n\,. \tag{8b}$$

Proof. Clearly we have to deal only with the case $n>0$. First we note that by substituting $u\log x=\log(xt^{-1})$ we have:

$$-\int_1^x \log^\nu t\,d[\log^n(xt^{-1})]=\log^{n+\nu}x\int_0^1(1-u)^\nu du^n=\frac{\nu!\,n!}{(n+\nu)!}\log^{n+\nu}x$$
$$=\nu!\,D^{-\nu}\log^n x\,.$$

It follows now by [2, Theorem 421] that for $n>0$,

$$I_g\log^n x=\sum_{\mu\leqq x}\frac{g(\mu)}{\mu}\log^n\frac{x}{\mu}=-\int_1^x\Big[\sum_{\nu=0}^{p}\frac{\gamma_{-\nu}}{\nu!}\log^\nu t+\rho(t)\Big]d\log^n(x/t)$$
$$=-\sum_{\nu=0}^{p}\frac{\gamma_{-\nu}}{\nu!}\int_1^x\log^\nu t\,d[\log^n(x/t)]$$
$$+\sum_{\nu=1}^{n}(-1)^{\nu-1}\frac{n!}{\nu!(n-\nu)!}\log^{n-\nu}x\Big(\int_1^\infty\rho(t)\,d\log^\nu t-\int_x^\infty\rho(t)\,d\log^\nu t\Big)$$
$$=\sum_{\nu=0}^{p}\gamma_{-\nu}D^{-\nu}\log^n x+\sum_{\nu=1}^{n}\gamma_\nu D^\nu\log^n x+\sum_{\nu=1}^{n}(-1)^\nu\binom{n}{\nu}\log^{n-\nu}x\int_x^\infty\rho(t)\,d\log^\nu t$$
$$=\sum_{\nu=-p}^{\infty}\gamma_\nu D^\nu\log^n x+\int_x^\infty\rho(t)d[\log^n(x/t)]\,,$$

which proves that $R_n^g(x)=\int_x^\infty\rho(t)d[\log^n(x/t)]$. The second form of $R_n^g$ is obtained by substituting $u=-\log(x/t)$. In particular,

Corollary. *If* $\rho(x)=O(x^{-\vartheta})$ *for some* $\vartheta>0$, *then* $R_n^g(x)=O(x^{-\vartheta})$ *for all* $n\geqq 0$.

Indeed, $R_n^g(x)=O\Big(\int_x^\infty(xe^u)^{-\vartheta}du^n\Big)=O(x^{-\vartheta})$ since $\int_0^\infty e^{-\vartheta u}du^n<\infty$.

**3. Applications.**

A) Since $\sum_{n\leq x}\frac{1}{n}=\log x+c_0+O(x^{-1})$, it follows by Theorem 4 that:

$$I_1=D^{-1}+\sum_{\nu=0}^{\infty}c_\nu D^\nu+O(x^{-1}),\ c_0 \text{ is the Euler constant.}$$

In the present paper we shall denote by $\zeta(D)$ this power series which approximates $I_1$.

Now $1*1=d$, where $d(n)$ is the number of divisors of $n$. Thus, $I_1{}^2=I_d$ by [**1**, Theorem 4.1]. In this case conditions of Theorem 2 are satisfied with $\alpha=\beta=1$, $u=v=0$, $p=q=1$. Thus

$$\text{(8c)}\qquad I_d=\zeta(D)^2+O(x^{-1/2}).$$

In particular

$$\sum_{n\leq x}\frac{d(n)}{n}=(D^{-1}+c_0+\cdots)^2 1+O(x^{-1/2})=\frac{1}{2}\log^2 x+2c_0\log x+(c_0{}^2+2c_1)+O(x^{-1/2}).$$

Further results one obtains by computing $I_d\log^n x$, for $n\geqq 1$.

Moreover, let $d_k=1*\cdots*1$, thus $d_k(n)=\sum_{\nu_1\nu_2\cdots\nu_k=n}1$. Here we may apply Theorem 3 and obtain

$$I_{d_k}=I_1{}^k=\zeta(D)^k+O_\varepsilon(x^{-1/k}).$$

In particular,

$$I_{d_k}1=\sum_{\nu\leq x}\nu^{-1}d_k(\nu)=\frac{1}{k!}\log^k x+\frac{k}{(k-1)!}c_0\log^{k-1}x+\cdots+c_k'+O_\varepsilon(x^{-1/k})$$

and one can readily compute the coefficients $c_2',\cdots,c_k'$.

B) As in examples C and D of [**1**, Section 10], we set $\mu_2(n)=(-1)^r$ for $n=(p_1p_2\cdots p_r)^2$ and zero otherwise, $e_2(n)=1$ for square $n$ and zero otherwise. Now,

$$\sum_{n\leq x}\frac{\mu_2(n)}{n}=\sum_{n^2\leq x}\frac{\mu(n)}{n^2}=b_0+O(x^{-1/2}),\ \ b_0=\frac{6}{\pi^2}=\zeta(2)^{-1}.$$

Hence, Theorem 4 implies that

$$I_{\mu_2}=b_0+b_1D+\cdots+O(x^{-1/2}).$$

In this example we use the relation $|\mu|=1*\mu_2$, where $\mu$ is the Möbius function. Hence it follows by Theorem 2 that

$$\text{(8d)}\qquad I_{|\mu|}=(D^{-1}+c_0+\cdots)\Big(\frac{6}{\pi^2}+b_1D+\cdots\Big)+O(x^{-1/3})$$

and in particular

$$I_{|\mu|}1=\sum_{n\leq x}\frac{|\mu(n)|}{n}=\frac{6}{\pi^2}(\log x+c_0)+b_1+O(x^{-1/3})$$

which is a better result than [1, (10D)]. (Note the misprint in 10D of [1]).

C) It is well known that $\zeta^2(s)\zeta^{-1}(2s)=\sum 2^{\nu(n)}n^{-s}$ where $\nu(n)=r$ for $n=p_1^{e_1}\cdots p_r^{e_r}$. Whence one readily verifies that $\rho=\mu_2{*}1{*}1=|\mu|{*}1$, and $\rho(n)=2^{\nu(n)}$.

By the previous example it follows that Theorem 2 is applicable and we have

$$I_\rho=(D^{-1}+c_0+\cdots)^2\left(\frac{6}{\pi^2}+\cdots\right)+O(x^{-1/4}).$$

Thus

$$\sum_{n\leq x}\frac{2^{\nu(n)}}{n}=\frac{3}{\pi^2}\log^2 x+\cdots+O(x^{-1/4}).$$

D) The previous examples show how relations between arithmetic functions and the corresponding Dirichlet series can be utilised to obtain certain asymptotic behaviour. The following is another example.

From the relation $\zeta(2s)^{-1}\zeta^4(s)=\sum d^2(n)n^{-s}$, it follows that $I_{d^2}=I_{\mu_2}I_1^4$. Hence, it follows by Theorem 3 that

$$I_{d^2}=(D^{-1}+c_0+\cdots)^4\left(\frac{6}{\pi^2}+b_1D+\cdots\right)+O_\varepsilon(x^{-1/6}).$$

In particular,

$$I_{d^2}1=\sum_{n\leq x}\frac{d^2(n)}{n}=\frac{1}{4\pi^2}\log^4 x+\cdots+\alpha_4+O_\varepsilon(x^{-1/6}).$$

E) Our last example deals with the function $r(n)$ ([2, p. 256]). It follows by [2, Theorem 306] that

$$\sum r(n)n^{-s}=4\zeta(s)L(s)$$

where $L(s)=\sum \chi(n)n^{-s}$ with $\chi(2n+1)=(-1)^n$ and zero otherwise. Thus $r=4(1{*}\chi)$, and therefore, $I_r=4I_1I_\chi$.

Now one readily verifies that

$$\sum_{n\leq x}\frac{\chi(n)}{n}=\sum_{\nu\leq(x-1)/2}\frac{(-1)^\nu}{2\nu+1}=\gamma_0+O(x^{-1}).$$

Note that

$$\gamma_0=1-\frac{1}{3}+\frac{1}{5}-\cdots=\frac{\pi}{4}.$$

It follows, therefore from Theorem 4 that

$$I_\chi=\gamma_0+\gamma_1D+\cdots+O(x^{-1}).$$

Consequently Theorem 2 yields that:

$$I_r=4(D^{-1}+c_0+\cdots)(\gamma_0+\gamma_1D+\cdots)+O(x^{-1/2})$$

which yields for example:

$$I_r 1 = \sum_{n\leq x} \frac{r(n)}{n} = 4\gamma_0 \log x + (4\gamma_1 + c_0\gamma_0) + O(x^{-1/2}) = \pi \log x + c + O(x^{-1/2}).$$

**4. Proof of Theorem 1.**

(9) $$I_g I_h \log^n x = \sum_{\nu\leq x} \nu^{-1} g(\nu) \sum_{\mu\leq x/\nu} \mu^{-1} h(\mu) \log^n (x\nu^{-1}\mu^{-1})$$

$$= \sum_{\nu\mu\leq x} (\nu\mu)^{-1} g(\nu) h(\mu) \log^n x(\nu\mu)^{-1} = \sum_{\nu\leq\gamma} \nu^{-1} g(\nu) \sum_{\mu\leq x/\nu} \mu^{-1} h(\mu) \log^n [(x\nu^{-1})\mu^{-1}]$$

$$+ \sum_{\mu\leq x/\gamma} \mu^{-1} h(\mu) \sum_{\nu\leq x/\mu} \nu^{-1} g(\nu) \log^n [(x\mu^{-1})\nu^{-1}]) - \sum_{\nu\leq\gamma} \sum_{\mu\leq x/\gamma} (\nu\mu)^{-1} g(\nu) h(\mu) \log^n x(\nu\mu)^{-1}$$

$$= \Sigma_1 + \Sigma_2 - \Sigma_3 .$$

It follows from (4) that:

$$\Sigma_1 = \sum_{\nu\leq\gamma} \nu^{-1} g(\nu) I_h \log^n x\nu^{-1} = \sum_{\nu\leq\gamma} \nu^{-1} g(\nu) [\sum_{i=-q}^{n} (n-i)!^{-1} n!\, h_i \log^{n-i} x\nu^{-1} + R_n^h(x\nu^{-1})]$$

$$= \sum_{i=-q}^{n} (n-i)!^{-1} n!\, h_i \sum_{\nu\leq\gamma} \nu^{-1} g(\nu) \log^{n-i}(\gamma\nu^{-1})(x\gamma^{-1}) + S_{11}$$

$$= \sum_{i=-q}^{n} (n-i)!^{-1} n!\, h_i \sum_{j=0}^{n-i} \binom{n-i}{j} \log^{n-i-j} x\gamma^{-1} \sum_{\nu\leq\gamma} \nu^{-1} g(\nu) \log^j \gamma\nu^{-1} + S_{11}$$

$$= \sum_{i=-q}^{n} \sum_{j=0}^{n-i} \frac{n!}{j!(n-i-j)!} h_i \log^{n-i-j} x\gamma^{-1} \Big[ \sum_{k=-p}^{j} \frac{j!}{(j-k)!} g_k \log^{j-k} \gamma + R_j^g(\gamma) \Big] + S_{11}$$

$$= S_{13} + S_{12} + S_{11}$$

where

$$S_{11} = \sum_{\nu\leq\gamma} \nu^{-1} g(\nu) R_n^h(x\nu^{-1}), \quad S_{13} = \sum_{i=-q}^{n} \sum_{j=0}^{n-i} \frac{n!}{j!(n-i-j)!} R_j^g(\gamma) h_i \log^{n-i-j} x\gamma^{-1}$$

and

$$S_{12} = \sum_{i=-q}^{n} \sum_{j=0}^{n-i} \sum_{k=-p}^{j} \frac{n!}{(j-k)!\,(n-i-j)!} h_i g_k \log^{n-i-j} x\gamma^{-1} \log^{j-k} \gamma .$$

In the latter we interchange the order of summation and obtain:

(10) $$S_{12} = \sum_{j=0}^{n+q} \sum_{i=-q}^{n-j} \sum_{k=-p}^{j} \frac{n!}{(j-k)!\,(n-i-j)!} h_i g_k \log^{n-i-j} x\gamma^{-1} \log^{j-k} \gamma .$$

Similarly one shows that $\Sigma_2 = S_{23} + S_{22} + S_{21}$, with

$$S_{21} = \sum_{\mu\leq x/\gamma} \mu^{-1} h(\mu) R_n^g(x\mu^{-1}), \quad S_{23} = \sum_{k=-p}^{n} \sum_{j=0}^{n-k} \frac{n!}{j!\,(n-j-k)!} R_j^h(x\gamma^{-1}) g_k \log^{n-j-k} \gamma ,$$

$$S_{22}=\sum_{j=0}^{n+p}\sum_{k=-p}^{n-j}\sum_{i=-q}^{j}\frac{n!}{(n-j-k)!\,(j-i)!}h_i g_k \log^{n-j-k}\gamma \log^{j-i}\gamma^{-1}x.$$

In $S_{22}$ we substitute $j$ by $n-j$ and obtain that

(11) $$S_{22}=\sum_{j=-p}^{n}\sum_{k=-p}^{j}\sum_{i=-q}^{n-j}\frac{n!}{(j-k)!\,(n-j-i)!}h_i g_k \log^{n-j-i}x\gamma^{-1}\log^{j-k}\gamma,$$

which is similar to $S_{12}$ as given in (10), with the exception of the interval of values which $j$ takes.

In computing $\sum_3$ we utilise (3) and (4) as follows:

$$\sum_3=\sum_{\nu\leqq\gamma}\sum_{\mu\leqq x/\gamma}\nu^{-1}g(\nu)\mu^{-1}h(\mu)\log^n[(\gamma\nu^{-1})(x\gamma^{-1}\mu^{-1})]$$

$$=\sum_{j=0}\binom{n}{j}\Big[\sum_{\nu\leqq\gamma}\nu^{-1}g(\nu)\log^j\gamma\nu^{-1}\Big]\Big[\sum_{\mu\leqq x/\gamma}\mu^{-1}h(\mu)\log^{n-j}x\gamma^{-1}\mu^{-1}\Big]$$

$$=\sum_{j=0}^{n}\binom{n}{j}\Big[\sum_{k=-p}^{j}\frac{j!}{(j-k)!}g_k\log^{j-k}\gamma+R_j^g(\gamma)\Big]\Big[\sum_{i=-q}^{n-j}\frac{(n-j)!}{(n-j-i)!}h_i\log^{n-i-j}x\gamma^{-1}$$

$$+R_{n-j}^h(x\gamma^{-1})\Big]=T_4+T_2+T_3+T_4,$$

where

(12a) $$T_4=\sum_{j=0}^{n}\binom{n}{j}R_j^g(\gamma)R_{n-j}^h(x\gamma^{-1}),$$

(12b) $$T_3=\sum_{j=0}^{n}\sum_{k=-p}^{j}\frac{n!}{(n-j)!\,(j-k)!}R_{n-j}^h(x\gamma^{-1})g_k\log^{j-k}\gamma,$$

(12c) $$T_2=\sum_{j=0}^{n}\sum_{i=-q}^{n-j}\frac{n!}{j!\,(n-i-j)!}R_j^g(\gamma)h_i\log^{n-i-j}x\gamma^{-1},$$

(12d) $$T_1=\sum_{j=0}^{n}\sum_{k=-p}^{j}\sum_{i=-q}^{n-j}\frac{n!}{(j-k)!\,(n-i-j)!}h_i g_k\log^{n-i-j}x\gamma^{-1}\log^{j-k}\gamma.$$

Consider now separately the sum $S_{12}+S_{22}-T_1$, $S_{13}-T_2$ and $S_{23}-T_3$. It follows from (12d), (10) and (11) that:

$$S_{12}+S_{22}-T_1=\sum_{j=0}^{n+q}\sum_{i=-q}^{n-j}\sum_{k=-p}^{j}\cdots+\sum_{j=-p}^{n}\sum_{k=-p}^{j}\sum_{i=-q}^{n-j}\cdots-\sum_{j=0}^{n}\sum_{k=-p}^{j}\sum_{i=-q}^{n-j}\cdots=\sum_{j=-p}^{n+q}\sum_{i=-q}^{n-j}\sum_{k=-p}^{j}\cdots.$$

In the last form we set $j-k=s$ and $i+k=t$, and one readily observes that one obtains:

$$S_{12}+S_{22}-T_1=\sum_{t=-(p+q)}^{n}\sum_{s=0}^{n-t}\frac{n!}{s!(n-s-t)!}\Big(\sum_{i+k=t}h_i g_k\Big)\log^{n-s-t}x\gamma^{-1}\log^s\gamma$$

$$= \sum_{t=-(p+q)}^{n} \frac{n!}{(n-t)!} f_t \log^{n-t} x = F(D) \log^n x .$$

We recall that $F(D) = G(D)H(D) = \sum_{t=-(p+q)}^{\infty} f_t D^t$.

For the second sum we obtain:

$$S_{13} - T_2 = \sum_{i=-q}^{n} \sum_{j=0}^{n-i} \cdots - \sum_{j=0}^{n} \sum_{i=-q}^{n-j} \cdots = \sum_{i=-q}^{-1} \sum_{j=n+1}^{n-i} \cdots$$

$$= \sum_{i=1}^{q} \sum_{j=1}^{i} \frac{n!}{(n+j)!\,(i-j)!} R^g_{n+j}(\gamma) h_{-i} \log^{i-j} x \gamma^{-1} ,$$

where in the last step we have substituted $j$ by $j-n$ and $i$ by $-i$.

The rest of the proof of (A) of Theorem 1 is obtained by a similar computation of $S_{23} - T_3$ and from the fact that

$$R^{gh}_n(x) = \Sigma_1 + \Sigma_2 - \Sigma_3 - F(D) \log^n x = S_{11} + S_{21} - T_4 + (S_{13} - T_2) + (S_{23} - T_3)$$

which have been shown to be respectively the five terms given in (6).

The proof of (6B) follows readily by computing:

$$(I_g I_h) \log^n x = I_g[I_h \log^n x] = I_g[H(D) \log^n x + R^h_n(x)]$$

$$= I_g R^h_n(x) + \sum_{\nu \leqq x} \nu^{-1} g(\nu) \sum_{i=-q}^{n} \frac{n!}{(n-i)!} h_i \log^{n-i} x \nu^{-1}$$

$$= I_g R^h_n(x) + \sum_{i=-q}^{n} \frac{n!}{(n-i)!} h_i I_g \log^{n-i} x$$

$$= I_g R^h_n(x) + \sum_{i=-q}^{n} \frac{n!}{(n-i)!} h_i R^g_{n-i}(x) + \sum_{i=-q}^{n} \sum_{k=-p}^{n-j} \frac{n!}{(n-i-k)!} h_i g_k \log^{n-i-k} x$$

$$= I_g R^h_n(x) + \sum_{i=-q}^{n} \frac{n!}{(n-i)!} h_i R^g_{n-i}(x) + [G(D)H(D)] \log^n x$$

$$- \sum_{i=n+1}^{n+p} \sum_{k=-p}^{n-i} \frac{n!}{(n-i-k)!} h_i g_k \log^{n-i-k} x$$

which proves the first part of (6B). The second part follows similarly.

**5. Asymptotic results.** The methods which have been used in the proof of the Main Theorem of [**1**] may yield far more:

THEOREM 5. *Let $g(n)$ be a non-negative function with the property that:*

(g1) $$\sum_{\nu \leqq x} g(\nu) = a \log^n x + b \log^{n-1} x + o(\log^{n-1} x),\ n \geqq 1,\ a > 0 .$$

*Let* $f(x)$ $(x \geqq 1)$ *be a complex valued function satisfying:*

(1) $$f(x) = O(1)\,,$$

(2) $$\sum_{\nu \leqq x} \frac{f(\nu)}{\nu} = O(1)\,,$$

(3) $$f(tx) - f(x) = o(1) \quad as\ (t, x) \to (1, \infty)\,.$$

*Then the condition*

(4) $$|f(x)| \log^n x \leqq \frac{1}{a} \sum_{\nu \leqq x} g(\nu) \left| f\left(\frac{x}{\nu}\right) \right| + o(\log^n x)$$

*implies that* $f(x) = o(1)$.

Proof. Without loss of generality we may assume that $a = 1$.

From [**1**, Lemmas 6.8 and 6.9] it follows that (2) and (3) imply that following:

Given $\varDelta > 0$, one can find positive numbers $x_0, T$ and $t > 1$ with the property that for every $x > x_0$, the interval $(x, xT)$ contains a subinterval $(y, yt)$ such that for every $z, x \leqq y \leqq z \leqq yt \leqq xT$, we have $|f(z)| < \varDelta$.

From (1) it follows that $|f(x)| < A + \varepsilon$ for all $x > 1$ and $\limsup_{x \to \infty} f(x) = A$. The theorem states that $A = 0$. Suppose $A > 0$. Then choose $0 < \varDelta < A$.

For $\varepsilon > 0$, let $x_1 \geqq x_0$ be such that $|f(x)| < A + \varepsilon$ for all $x \geqq x_1$. Put $i_1 = [\log x_1 / \log T] + 1$ and $j = [\log x / \log T]$. For $i_1 \leqq i \leqq j$, let $(y_i, y_i t)$ be the subinterval of $(T^i, T^{i+1})$ for which $|f(z)| < \varDelta$ for all $T^i \leqq y_i \leqq z \leqq y_i t \leqq T^{i+1}$.

First we observe that since $j = O(\log x)$ and $y_i > T^i$:

$$(*) \quad \sum_{i=i_1}^{j} \sum_{y_i \leqq x\nu^{-1} \leqq y_i t} g(\nu)$$

$$= \sum_{i=i_1}^{j} [\log^n x y_i^{-1} - \log^n x y_i^{-1} t^{-1} + b \log^{n-1} x y_i^{-1} - b \log^{n-1} x y_i^{-1} t^{-1} + o(\log^{n-1} x y_i^{-1} t^{-1})]$$

$$= n \log t \sum_{i=i_1}^{j} \log^{n-1} x y_i^{-1} + O(j \log^{n-2} x) + \sum_{i=i_1}^{j} o(\log^{n-1} x y_i^{-1})$$

$$\geqq n \log t \sum_{i=i_1}^{j} \log^{n-1}(x T^{-(i+1)}) + O(\log^{n-1} x) + o(\log^n x)$$

$$= n \log t \sum_{\rho=0}^{n-1} \binom{n-1}{\rho} (-1)^\rho \log^{n-1-\rho} x \log^\rho T \sum_{i=i_1}^{j} (i+1)^\rho + o(\log^n x)$$

$$= n \log t \sum_{\rho=0}^{n-1} \log^{n-1-\rho} x \log^\rho T (-1)^\rho \binom{n-1}{\rho} \left[ \frac{1}{\rho+1} \left( \frac{\log x}{\log T} \right)^{\rho+1} + O(\log^\rho x) \right] + o(\log^n x)$$

$$= \log^n x \log t / \log T + o(\log^n x) = C \log^n x + o(\log^n x)\,, \quad C > 0\,.$$

Since $\sum_{\rho=0}^{n-1}(-1)^\rho\binom{n-1}{\rho}(\rho+1)^{-1}=n^{-1}$, and $\sum_{i=i_1}^{j}o(\log^{n-1}xy_i^{-1})=o(\log^n x)$. The latter is shown as follows:

Let $R(x)$ be the remainder element in ($g$1), then $|R(x)|\leqq H\log^{n-1}x$ for all $x$, and $|R(x)|<\delta\log^{n-1}x$ for $\delta>0$ and $x>x_\delta$. Thus, the remainder in the last formula is

$$|\sum_{i=i_1}^{j}R(xy_i^{-1})|\leqq H\sum_{1\leqq xy_i^{-1}\leqq x_\delta}\log^{n-1}xy_i^{-1}+\varepsilon\sum_{i=i_1}^{j}\log^{n-1}xy_i^{-1}$$
$$\leqq\varepsilon\log^n x+H\log^{n-1}x\log x_\delta/\log T=\varepsilon\log^n x+o(\log^n x),$$

since the number of the integers $i$ for which $1\leqq xy_i^{-1}\leqq x_\delta$ is the same as those which satisfy $xx_\delta^{-1}\leqq T^i<x$, which is $[\log x_\delta/\log T]$.

The proof of Theorem 5 follows now readily as in [**1**, Theorem 6.1]: Indeed, it follows by (4) and in view of the fact that $0<\Delta<A+\varepsilon$:

$$|f(x)|\log^n x\leqq K\sum_{1\leqq x\nu^{-1}\leqq x_1}g(\nu)+(A+\varepsilon)\sum_{x_1<x\nu^{-1}\leqq x}g(\nu)$$
$$+\sum_{i=i_1}^{j}\sum_{y_i<x\nu^{-1}\leqq y_it}g(\nu)\left(\left|f\left(\frac{x}{\nu}\right)\right|-A-\varepsilon\right)+o(\log^n x)\leqq K(\log^n x-\log^n xx_1^{-1})$$
$$+O(\log^{n-1}x)+(A+\varepsilon)\log^n x+O(\log^{n-1}x)+(\Delta-A-\varepsilon)C\log^n x+o(\log^n x)$$
$$=[(A+\varepsilon)+(\Delta-A-\varepsilon)C]\log^n x+o(\log^n x).$$

Thus

$$|f(x)|\leqq A+\varepsilon+(\Delta-A-\varepsilon)C+o(1)$$

and as $x\to\infty$ we have

$$\limsup|f(x)|=A<A+\varepsilon+(\Delta-A-\varepsilon)C$$

which being true for all $\varepsilon>0$ yields $A\leqq A+(\Delta-A)C$. But this is impossible since $C>0$ and $\Delta-A<0$, and the proof of the theorem is concluded.

REMARK 1. The preceding theorem does not yield immediately the Main Theorem of [**1**, 6.1] since the latter requires that $|f(x)|\log x\leqq I_\Lambda|f(x)|+o(\log x)$ and if one chooses $g(\nu)=\Lambda(\nu)\nu^{-1}$, then $g$ does not satisfy ($g$1). Nevertheless our theorem is applicable for $g(\nu)=\Lambda_2(\nu)\nu^{-1}$ and condition (2) of [**1**, Theorem 6.4] yields by [**1**, Lemma 6.6] that $|f(x)|\log^2 x\leqq I_{\Lambda_2}|f(x)|+o(\log^2 x)$ and our theorem yields the Main Theorem of [**1**].

REMARK 2. Theorem 5 can be extended to a wider class of functions $g(n)$, for which one has to assume instead of ($g$1):

($g$2) $$\sum_{\nu\leqq x}g(\nu)=G(x)+o(G(x)/\log x)$$

with $G(x)$ a non-decreasing function, and

$$(4') \qquad |f(x)|G(x) \leqq \sum_{\nu\leqq x} g(\nu)\left|f\left(\frac{x}{\nu}\right)\right| + o(G(x)).$$

The proof can be carried over to this case if $g(\nu)$ will satisfy a condition similar to (*). Namely that:

$$(**) \qquad \sum_{i=i_1}^{j} \sum_{y_i < x\nu^{-1} \leqq y_i t} g(\nu) \geqq CG(x) + o(G(x)), \quad \text{for some } C > 0.$$

In particular, note that once we have shown that $\sum_{\nu\leqq x} \Lambda(\nu)/\nu = \log x + c + o(1)$ ([**1**, (7.3), p. 289]), then clearly ($g2$) holds for $g(n) = \Lambda(n)/n$, and $G(x) = \log x + c$. Furthermore, (**) is valid since $\sum_{y_i < \nu \leqq y_i t} \Lambda(\nu)/\nu = \log xy_i^{-1} - \log xy_i^{-1}t^{-1} + o(1) = \log t + o(1)$ and the rest is easily verified.

The generality of Theorem 5 and the fact that we have got rid of condition (3) of [**1**, Theorem 6.1], enables us to modify the conditions in [**1**, Theorem 9.1]. Namely we have:

THEOREM 6. *Let* $I_h = H(D) + O(1)$, $h(n) = O(n^{\vartheta})$ *for* $\vartheta < 1$, *then* $f_n(x) = I_h \log^n x - H(D)\log^n x$ *satisfies* (1) *and* (2) *of Theorem* 5.

[**1**, Theorem 9.1] was proved with the assumption that $I_h 1 = O(\log^s x)$. This assumption was first used to prove condition (3) of [**1**, Theorem 9.1, p. 306] which is not necessary in view of Theorem 5. Next it was used on [**1**, p. 307] but there one readily observes that it sufficies to assume that $h(n) = O(n^{\vartheta})$, $\vartheta < 1$ since one has only to verify that

$$\sum_{t\leqq x} \frac{|h(t)|}{t} \cdot \frac{\log^k t}{t} = O(1)$$

which is true in our case.

We conclude the paper with an extension of the last general result of [**1**] (Theorem 9.1). Namely, we show that

THEOREM 7. *Let g be an arithmetic function satisfying the following conditions:*

$$I_g = G(D) + O(\varphi_n); \quad I_1\varphi_n = O(1) \text{ and } I_1\varphi_n \log x = o(\log x)$$

*and let* $h = \mu * g$ *satisfy* $h(n) = O(n^{\vartheta})$ *for some* $\vartheta < 1$, *then*

$$I_h \log^n x = [\zeta^{-1}(D)G(D)] \log^n x + o(1) \quad \text{for } n > 0,$$

*and it holds also for* $n = 0$, *if*

$$\sum_{x<\nu\leqq tx} \frac{h(\nu)}{\nu} - \sum_{j=1}^{p} \sum_{i=j}^{p} \frac{1}{j!\,(i-j)!} h_{-i} \log^{i-j} x \log^j t = R_0^h(tx) - R_0^h(x) = o(1)$$

*as* $(t, x) \to (1, \infty)$, *where* $\zeta(D)^{-1}G(D) = \sum_{\nu=-p}^{\infty} h_\nu D^\nu$.

PROOF. First we prove that $R_n^h(x)=I_h\log^n x-[\zeta^{-1}(D)G(D)]\log^n x=O(1)$ for all $n\geqq 0$, and to this end we use (6B). Indeed, by Theorem 1 it follows that:

$$R_n^h(x)=I_\mu R_n^g(x)+O(1)=O(I_1\varphi_n)+O(1)=O(1)\,.$$

Thus

$$I_1R_n^h=I_1(I_\mu I_g-\zeta^{-1}(D)G(D))\log^n x=[I_g-\zeta(D)(\zeta^{-1}(D)G(D))]\log^n x+O(x^{-1})$$

since $I_1=\zeta(D)+O(x^{-1})$. Now,

$$\zeta(D)[\zeta^{-1}(D)G(D)]\log^n x=[\zeta\zeta^{-1}G]\log^n x-n!^{-1}h_{n+1}\,.$$

Hence,

$$I_1R_n^h=O(\varphi_n+x^{-1})-n!^{-1}h_{n+1}\,.$$

Consider the function $f_n(x)=R_n^h(x)+(n!x)^{-1}h_{n+1}$, which will satisfy $I_1f_n=O(\varphi_n+x^{-1})$. As in the proof of Theorem 9.2 of [**1**, p. 309] one verifies that $f_n(x)$ satisfies (1) and (2) of Theorem 5 and (2) of [**1**, Theorem 6.1]. This theorem is applicable in view of Remark 2 if we show that: $f_n(tx)-f_n(x)=o(1)$ as $(t,x)\to(1,\infty)$, or equivalently $R_n^h(tx)-R_n^h(x)=o(1)$.

Repeating the computation of [**1**, p. 309] we obtain

$$\begin{aligned}
R_n^h(tx)-R_n^h(x)&=\Big[\sum_{\nu\leqq tx}\frac{h(\nu)}{\nu}\log^n\frac{tx}{\nu}-\sum_{i=-p}^{n}\frac{n!}{(n-i)!}h_i\log^{n-i}tx\Big]\\
&\quad-\Big[\sum_{\nu\leqq x}\frac{h(\nu)}{\nu}\log^n\frac{x}{\nu}-\sum_{i=-p}^{n}\frac{n!}{(n-i)!}h_i\log^{n-i}x\Big]\\
&=\sum_{x<\nu\leqq tx}\frac{h(\nu)}{\nu}\log^n\frac{tx}{\nu}+\sum_{j=1}^{n}\binom{n}{j}\log^j t\sum_{\nu\leqq x}\frac{h(\nu)}{\nu}\log^{n-j}\frac{x}{\nu}\\
&\quad-\sum_{i=-p}^{n}\sum_{j=1}^{n-i}\frac{n!}{(n-i)!}\binom{n-i}{j}h_i\log^j t\log^{n-i-j}x\\
&=\sum_{x<\nu\leqq tx}\frac{h(\nu)}{\nu}\log^n\frac{tx}{\nu}+\sum_{j=1}^{n}\binom{n}{j}\log^j t\Big[\sum_{\nu\leqq x}\frac{h(\nu)}{\nu}\log^{n-j}\frac{x}{\nu}\\
&\quad-\sum_{i=-p}^{n-j}\frac{(n-j)!}{(n-i-j)!}h_i\log^{n-i-j}x\Big]-\sum_{j=n+1}^{n+p}\sum_{i=-p}^{n-j}\frac{n!}{j!(n-i-j)}h_i\log^{n-i-j}x\log^j t\\
&=\Delta_n+\Delta_n'\,,
\end{aligned}$$

and $\Delta_n'=\sum_{j=1}^{n}\binom{n}{j}\log^j tR_{n-j}^h(x)\,,$

$$\Delta_n=\sum_{x<\nu\leqq tx}\frac{h(\nu)}{\nu}\log^n\frac{tx}{\nu}-\sum_{j=1}^{p}\sum_{i=j}^{p}\frac{n!}{(n+j)!(i-j)!}h_{-i}\log^{i-j}x\log^{n+j}t\,.$$

The latter is obtained by replacing $j$ and $i$ by $j-n$ and $-i$ respectively in the corresponding part of the preceding formula.

Clearly, if $\Delta_n = o(1)$ as $(t, x) \to (1, \infty)$ then since $\Delta_n' = O(\log t) = o(1)$, it follows that $R_n^h(tx) - R_n^h(x) = o(1)$ and our theorem is proved. For $n=0, \Delta_n'=0$, and $\Delta_0 = o(1)$ is the condition given in the statement of the theorem. For $n > 0$ we have by [2, Theorem 421, p. 346]:

$$\sum_{x<\nu\leqq tx} \frac{h(\nu)}{\nu} \log^n \frac{tx}{\nu} = -\int_x^{tx} [H(u)-H(x)] \Big(\log^n \frac{tx}{u}\Big)' du$$

where $H(u) = I_1 h = \sum_{\nu\leqq u} \nu^{-1} h(\nu) = \sum_{i=-p}^{0} (-i)!^{-1} h_i \log^{-i} u + R_0^h(u)$. Hence, by setting $-i$ instead of $i$, we obtain that the last integral is equal to:

$$-\sum_{i=1}^{p} i!^{-1} h_{-i} \int_x^{tx} [\log^i u - \log^i x](\log^n txu^{-1})' du - \int_x^{tx} [R_0^h(u) - R_0^h(x)](\log^n txu^{-1})' du$$

$$= -\sum_{i=1}^{p} i!^{-1} h_{-i} \Big[(\log^i u - \log^i x) \log^n txu^{-1}\Big]_x^{tx} + \sum_{i=1}^{p} (i-1)!^{-1} h_{-i} \int_x^{tx} \log^n txu^{-1} \frac{\log^{i-1} u}{u} du$$
$$+ O(\log^n t).$$

The latter follows by integration by part and from the fact that $R_0^h = O(1)$. Now, the first term in the last formula is zero, and in the second term we obtain by setting $v = (\log txu^{-1})/\log t$,

$$\int_x^{tx} \log^n txu^{-1} \frac{\log^{i-1} u}{u} du = \int_0^1 (\log^{n+1} t) v^n (\log tx - v \log t)^{i-1} dv$$

$$= \sum_{j=1}^{i-1} \binom{i-1}{j} \log^{n+j+1} t \log^{i-1-j} x \int_0^1 v^n (1-v)^j dv$$

$$= \sum_{j=0}^{i-1} \binom{i-1}{j} \frac{n!\,j!}{(n+j+1)!} \log^{n+j+1} t \log^{i-(j+1)} x = \sum_{j=1}^{i} \frac{(i-1)!\,n!}{(n+j)!\,(i-j)!} \log^{n+j} t \log^{i-j} x.$$

From which one observes that:

$$\Delta_n = \sum_{i=1}^{p} \sum_{j=1}^{i} \frac{n!}{(n+j)!\,(i-j)!} h_{-i} \log^{n+j} t \log^{i-j} x$$

$$-\sum_{j=1}^{p} \sum_{i=j}^{p} \frac{n!}{(n+j)!\,(i-j)!} h_{-i} \log^{n+j} t \log^{i-j} x + O(\log^n t) = O(\log^n t) = o(1)$$

as $t \to 1$ (independent of $x$), when $n > 0$.

We conclude with an application:

The function $\Lambda_2 = \mu * \log^2 x$ satisfies the requirement of our theorem. Hence, we obtain from (5.12b) of [1] that

$$I_{\Lambda_2} \log x = \sum_{\nu \leqq x} \frac{\Lambda_2(\nu)}{\nu} \log \frac{x}{\nu} = \frac{1}{3} \log^3 x - c_0 \log^2 x + (c_0{}^2 - 2c_1) \log x + 2c_2 + o(1)\,,$$

instead of $O(1)$.

Hebrew University,
Jerusalem, Israel.

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