Journal of the Mathematical Society of Japan Vol. 1, No. 1, Sept., 1948.

On algebraic Lie algebras.

By Morikuni GOTÔ.

(Recieved Oct. 25, 1947.)

Introduction.

The classical theory of linear algebraic Lie groups and the associated Lie algebras by L. Maurer[1] has been modernized recently by C. Chevalley and H.-F. Tuan[2] by using the concept of "replica" which was invented by the former of them[3].

Let K be a field of characteristic 0. A Lie algebra $\mathfrak{L}$ of matrices over K is called *linear algebraic* (*l-alg.*)[4] if every replica of each matrix $X\in\mathfrak{L}$ belongs to $\mathfrak{L}$. If K is the field of all complex numbers C, any l-alg. Lie algebra is the Lie algebra of a linear algebraic Lie group, and vice versa. This theorem, due to Chevalley and Tuan[5], justifies the definition of l-algebraicity.

The theory of l-alg. Lie algebras has already been established by Chevalley and Tuan. But as their proofs have been reported only in the outline, we shall first give in this note a systematic approach to the theory; our methods will be somewhat different from theirs.

We shall then study the "*algebraic closure*" (see § 4) of any Lie algebra of matrices. Our Theorem 4 is an extension of a theorem of Chevalley and Tuan.

Then we shall give an extension of the theory to, not necessarily matric, *algebraic* (*alg.*) Lie algebras. Namely we shall call a Lie algebra alg. if its regular representation is l-alg. Then as we may easily see that any l-alg. Lie algebra is itself alg., our algebraicity is certainly an extension of l-algebraicity. Our fundamental result is given in our Theorem 5, which indicates the complete connection between alg. and l-alg. Lie algebras. And Theorem 6, which is an immediate consequence of our proof of Theorem 5, gives a characterization of an alg. Lie algebra of matrices.

The present study is closely related to recent works of Y. Matsushima on the similar subject and the writer is indebted to him for various discussions[6]. In particular Theorem 5 was proved by him independently in the case of Lie algebras over C by an analytical method using our Lemma 15; the result was used by him to give the characterization of Lie groups corresponding to alg. Lie algebras.

In conclusion let it be mentioned that our problem is of purely algebraic nature and the writer has tried to treat the problem by algebraic methods.

§ 1. *Preliminaries.* Let K be an algebraically closed[7] field of characteristic 0 and $\mathfrak{M}$ a certain n-dimensional vector space over K. Linear transformations on $\mathfrak{M}$ and the matrices which represent them are conventionally denoted by the same symbols X, Y, ... For simplicity we call a nilpotent matrix an *n-matrix*, a matrix with simple elementary divisors an *s-matrix*, and an s-matrix whose eigenvalues are all rational numbers (elements of the prime field) an *r-matrix*. Then the set $\{X\}$ of all replicas of X is given by the following lemma:

Lemma 1.[8] If

$$X = X^0 + X^s = X^0 + \xi_1 X^1 + \ldots\ldots \xi_k X^k \tag{1}$$

is a decomposition of a matrix X into an n-matrix X^0, an s-matrix X^s and r-matrices X^1, ..., X^k such that

$$X^i X^j = X^j X^i \qquad i, j = 0, 1, 2, \ldots\ldots, k$$

and ξ's are linearly independent with respect to rational numbers, then

$$\{X\} = \{X^0\} + \{X^s\}$$
$$= \text{Abelian linear space spanned by } X^0, X^1, \ldots\ldots, X^k.$$

Remark. We shall call the decomposition such as (1) "*canonical*". A canonical decomposition is given by reducing X to the Jordan normal form, denoting its diagonal part by X^s, and representing its eigen-values as linear combinations of suitably chosen ξ's. Thus X^0 and X^s are determined uniquely by X, while X^1, ..., X^k are not. But if we fix a Hamel basis (linearly independent basis with respect to rational numbers) of K, we get uniquely a canonical decomposition such that ξ's are taken from the basis and all X^1, ..., X^k do not vanish.

Lemma 2.[9] Let $\mathfrak{M}'$ be any subspace of $\mathfrak{M}$ such that $X\mathfrak{M}' \subseteqq \mathfrak{M}'$. Then $\{X\}\mathfrak{M}' \subseteqq \mathfrak{M}'$.

Let $\mathfrak{L}$ be a Lie algebra with a finite basis over K, and for $x, y \in \mathfrak{L}$ let $[x, y]$ denote the commutator product of x and y. Let new $\mathfrak{G}(n, K)$ be the Lie algebra composed of all matrices of degree n over K with the commutator multiplication $[X, Y]=XY-YX$. By a Lie algebra of matrices we mean a Lie subalgebra of a certain $\mathfrak{G}(n, K)$.

Lemma 3.[10] Let $\mathfrak{L}$ be a nilpotent Lie algebra of matrices. Then the space $\mathfrak{M}$ on which $\mathfrak{L}$ operates can be decomposed into the common eigen-spaces: $\mathfrak{M}=\mathfrak{M}_\alpha+\mathfrak{M}_\beta+\ldots\ldots$ (Each $\mathfrak{M}_\alpha$ is contained in an eigen-space of X for every $X\in\mathfrak{L}$.)

Lemma 4. If $[X, Y]=0$. then

$$[\{X\}, \{Y\}]=0 \quad \text{and} \quad \{X+Y\}\subseteqq\{X\}+\{Y\}.$$

Proof. The former part follows easily from Lemma 1 and 3.

To prove the latter, we first fix a Hamel basis $\mathfrak{H}$ of K. Then by the remark above we get canonical decompositions

$$X=X^0+\xi_1X^1+\ldots\ldots+\xi_kX^k,$$
$$Y=Y^0+\xi_1Y^1+\ldots\ldots+\xi_kY^k,$$

where $\xi_1, \ldots\ldots, \xi_k\in\mathfrak{H}$. As $[\{X\}, \{Y\}]=0$,

$$X+Y=(X^0+Y^0)+\xi_1(X^1+Y^1)+\ldots\ldots+\xi_k(X^k+Y^k)$$

is clearly a canonical decomposition of $X+Y$. Hence $\{X+Y\}\subseteqq\{X\}+\{Y\}$, q.e.d.

Lemma 5. If $\mathrm{Sp}X=0$ then $\mathrm{Sp}\{X\}=0$.

Proof. Let $X=X^0+\xi_1X^1+\ldots\ldots+\xi_kX^k$ be a cononical decomposition of X. Then

$$\mathrm{Sp}X=\mathrm{Sp}X^0+\xi_1\mathrm{Sp}X^1+\ldots\ldots+\xi_k\mathrm{Sp}X^k.$$

As $\mathrm{Sp}X=\mathrm{Sp}X^0=0$, we have

$$\xi_1\mathrm{Sp}X^1+\ldots\ldots+\xi_k\mathrm{Sp}X^k=0.$$

From the linear independence of ξ's we get $\mathrm{Sp}X^i=0$, $i=1, 2, \ldots\ldots, k$. Hence $\mathrm{Sp}\{X\}=0$ by Lemma 1, q.e.d.

Definition. A (Lie) subalgebra $\mathfrak{L}$ of $\mathfrak{G}(n, K)$ is called "*l-alg.*" if $\{X\}\subseteqq\mathfrak{L}$ for every $X\in\mathfrak{L}$.

Let now $\mathfrak{D}$ be any distributive algebra (associative or not) with a finite basis over K. If a linear transformation H on $\mathfrak{D}$ satisfies the relation

$$H(x\times y)=Hx\times y+x\times Hy \text{ for every } x, y\in\mathfrak{D},$$

we shall call H a "*derivation*" of $\mathfrak{D}$; $\times$ being the multiplication symbol of $\mathfrak{D}$. Then all derivations of $\mathfrak{D}$ form a Lie algebra $E(\mathfrak{D})$.

Theorem 1.[11] *$E(\mathfrak{D})$ is l-alg. for any disiributive algebra $\mathfrak{D}$.*

Proof. Let $H \epsilon E(\mathfrak{D})$. We shall prove that $\{H\} \subseteqq E(D)$. With respect to the linear transformation H we may decompose $\mathfrak{D}$ into eigen-spaces:

$$\mathfrak{D} = \mathfrak{D}_\alpha + \mathfrak{D}_\beta + \ldots\ldots$$

where α, β, ... are eigen-values of H. Then we get easily

$$\mathfrak{D}_\alpha \times \mathfrak{D}_\beta \subseteqq \mathfrak{D}_{\alpha+\beta} \tag{2}$$

where we put $\mathfrak{D}_{\alpha+\beta} = 0$, in case $\alpha+\beta$ is not an eigen-value. Now let $H = H^0 + H^s = H^0 + \xi_1 H^1 + \ldots\ldots + \xi_k H^k$ be a canonical decomposition of H. By (2) we may show easily that $H^s \epsilon E(\mathfrak{D})$, and therefore $H^0 \epsilon E(\mathfrak{D})$[12]. Hence the problem is reduced to the case when H is an s-matrix. Therefore by a suitably chosen basis x_1, ..., x_r of $\mathfrak{D}$, we may represent H as follows:

$$Hx_i = a_i x_i \qquad i = 1, 2, \ldots\ldots, r. \tag{3}$$

Let the structure of $\mathfrak{D}$ be given by $x_i \times x_j = \sum_h c_{ijh} x_h$. That $H \epsilon E(\mathfrak{D})$ means

$$(a_i + a_j - a_h) c_{ijh} = 0. \tag{4}$$

for all $i, j, h = 1, 2, \ldots\ldots, r$. H^l, $l = 1, 2, \ldots\ldots$, in the canonical decomposition of H may be defined by

$$H^l x_i = r_i^l x_i \qquad a_i = \sum_l \xi_l r_i^l$$

where r's are rational numbers. (4) is a trivial relation for $c_{ijh} = 0$. But, when $c_{ijh} \neq 0$ it gives $a_i + a_j - a_h = 0$, or

$$\sum_l (r_i^l + r_j^l - r_h^l) \xi_l = 0.$$

From the linear independence of ξ's we get then $r_i^l + r_j^l - r_h^l = 0$, or $(r_i^l + r_j^l - r_h^l) c_{ijh} = 0$. Thus in any case the relations (5) are satisfied for $a_i = r_i^l$. This shows nothing but that $H^l \epsilon E(\mathfrak{D})$. Therefore $\{H\} \subseteqq E(\mathfrak{D})$, q.e.d.

Now let $\mathfrak{D}$ be a Lie algebra over K. Then a linear transformation defined by

$$x \to \boldsymbol{y}x = [y, x] \qquad \text{for } x, y \in \mathfrak{D}$$

is a derivation of $\mathfrak{D}$ by virtue of the identities of Jacobi. Such derivations are called "*inner*." All inner derivations of $\mathfrak{D}$ form an ideai $I(\mathfrak{D})$ of $E(\mathfrak{D})$, and the correspondence $y \to \boldsymbol{y}$ is a linear representation of $\mathfrak{D}$, called (left) regular representation.

Now let $X \in \mathfrak{G}(n, K)$. We may easily see that according as X is an n-, s-, or r-matrix, $\boldsymbol{X}$, as a linear transformation of $\mathfrak{G}(n, K)$, is also an n-, s-, or r-matrix respectively[13]. Hence if

$$X = X^0 + \xi_1 X^1 + \dots\dots + \xi_k X^k$$

is a canonical decomposition of X, then

$$\boldsymbol{X} = \boldsymbol{X}^0 + \xi_1 \boldsymbol{X}^1 + \dots\dots + \xi_k \boldsymbol{X}^k$$

is that of $\boldsymbol{X}$. Therefore by Lemma 1 we get

$$\{\boldsymbol{X}\} = \{\boldsymbol{X}\}. \tag{5}$$

From Lemma 2 we may conclude the following

Lemma 6. Let $\mathfrak{L}$ be any subspace of $\mathfrak{G}(n, K)$. If $[X, \mathfrak{L}] \subseteqq \mathfrak{L}$, then $\boldsymbol{X}$ is an n-, s-, or r-matrix according as $\boldsymbol{X}$ is an n-, s-, or r-matrix. We get further $[\{X\}, \mathfrak{L}] \subseteqq \mathfrak{L}$, and

$$\{\boldsymbol{X}\} = \{\boldsymbol{X}\}. \tag{5}$$

Here $\boldsymbol{X}$ etc. are considered as linear transformations on $\mathfrak{L}$.

Lemma 7. Let $\mathfrak{L}$ be an l-alg. Lie algebra and let

$$\mathfrak{L} = \mathfrak{L}_0 + \mathfrak{L}_\alpha + \dots\dots$$

be the decomposition of $\mathfrak{L}$ into eigen-spaces by a certain inner derivation $\boldsymbol{X}$ of $\mathfrak{L}$. Then $\mathfrak{L}_0$ is also l-alg

Proof. Let $X = X^0 + X^s$ be a canonical decomposition. We may easily see that the 0-eigen-space for $\boldsymbol{X}^s$ is equal to the 0-eigen-space for $\boldsymbol{X}$, and as $\boldsymbol{X}^s$ is an s-matrtx with X^s, $(\boldsymbol{X}^s)^m A = 0$, $m = 1, 2, \dots\dots$ implies that $[X^s, A] = 0$. Hence

$$\mathfrak{L}_0 = \{A \,;\, [X^s, A] = 0\},$$

Then the l-algebraicity of $\mathfrak{L}_0$ follows from Lemma 4, q.e.d.

§ 2. *L-alg. Lie algebras, I.* Let $\mathfrak{L}$ be a nilpotent l-alg. Lie algebra.

Then the totality of s-matrices in $\mathfrak{L}$ forms a central ideal $\mathfrak{A}$, and that of n-matrices in $\mathfrak{L}$ forms also an ideal $\mathfrak{B}$ by virtue of Lemma 3. As $\mathfrak{L}$ is l-alg. $\mathfrak{L}$ is clearly a direct sum of ideals $\mathfrak{A}$ and $\mathfrak{B}$.

Let now $\mathfrak{R}$ be a solvable l-alg. Lie algebra and let

$$\mathfrak{R}=\mathfrak{R}_0+\mathfrak{R}_\alpha+\ldots\ldots \tag{1}$$

be its decomposition by a certain regular[14] inner derivation. Then as $\mathfrak{R}_0$ is nilpotent and by Lemma 7 l-alg. we get

$$\mathfrak{R}_0=\mathfrak{A}+\mathfrak{B}.$$

Hence

$$\mathfrak{N}=\mathfrak{B}+\mathfrak{R}_\alpha+\mathfrak{R}_\beta+\ldots\ldots$$

is the largest ideal composed of n-matrices. Therefore we get

$$\mathfrak{R}=\mathfrak{A}+\mathfrak{N} \qquad \mathfrak{A}\cap\mathfrak{N}=0\,;$$

here $\mathfrak{A}$ is an abelian l-alg. subalgebra of $\mathfrak{R}$ composed of s-matrices. We note also that $\mathfrak{A}$ may be considered as $\{A\}$ for some $A\in\mathfrak{A}$.

Lemma 8. Let $\mathfrak{L}$ be l-alg. Then the radical $\mathfrak{R}$ of $\mathfrak{L}$ is also l-alg.

Proof. Let $X=X^0+X^s$ be any element of $\mathfrak{R}$. We decompose $\mathfrak{L}$ by $\boldsymbol{X}$:

$$\mathfrak{L}=\mathfrak{L}_0+\mathfrak{L}_\alpha+\mathfrak{L}_\beta+\ldots\ldots \tag{2}$$

By Lemma 7 from $X\in\mathfrak{L}_0$ follows $\{X\}\subseteqq\mathfrak{L}_0$. As $X\mathfrak{R}\subseteqq\mathfrak{R}$ we get the decomposition of $\mathfrak{R}$ by $\boldsymbol{X}$ as folloms:

$$\mathfrak{R}=\mathfrak{R}_0+\mathfrak{R}_\alpha+\mathfrak{R}_\beta+\ldots\ldots \tag{1}$$

As $\boldsymbol{X}\mathfrak{L}\subseteqq\mathfrak{R}$ and $\boldsymbol{X}\mathfrak{L}_\alpha=\mathfrak{L}_\alpha$, $\boldsymbol{X}\mathfrak{L}_\beta=\mathfrak{L}_\beta$ etc., me have $\mathfrak{L}_\alpha=\mathfrak{R}_\alpha$, $\mathfrak{L}_\beta=\mathfrak{R}_\beta$, ..., whence $\mathfrak{L}_0+\mathfrak{R}=\mathfrak{L}$. As $\mathfrak{R}_0=\mathfrak{L}_0\cap\mathfrak{R}$ we get by the isomorphism theorem

$$\mathfrak{L}_0/\mathfrak{R}_0\cong\mathfrak{L}/\mathfrak{R}$$

The right side being semi-simple, solvable $\mathfrak{R}_0$ must be the radical of $\mathfrak{L}_0$.

On the other hand, we have as in the proof of Lemma 7 $[\{X^s\}, \mathfrak{L}_0]=0$, i.e. $\{X^s\}$ is contained in the centre of $\mathfrak{L}_0$. So $\{X_s\}\subseteqq\mathfrak{R}_0$, since the centre is contained in the radical. Consequently $\{X\}\subseteqq\mathfrak{R}_0\subseteqq\mathfrak{R}$, q.e.d.

Lemma 9. Let the radical $\mathfrak{R}$ of $\mathfrak{L}$ be l-alg. Denote by $\mathfrak{N}$ the largest ideal composed of n-matrices. (The existence of such $\mathfrak{N}$ is clear, and $\mathfrak{N}$, being nilpotent, belongs to $\mathfrak{R}$). Then there exists a semi-simple subalgebra $\mathfrak{S}$ such that

$$\mathfrak{L}=\mathfrak{S}+\mathfrak{R}, \quad \mathfrak{R}=\mathfrak{A}+\mathfrak{N}, \quad [\mathfrak{S}, \mathfrak{A}]=0,$$
$$\mathfrak{S} \cap \mathfrak{R}=0, \quad \mathfrak{A} \cap \mathfrak{N}=0,$$

where $\mathfrak{A}$ is an abelian subalgebra composed of s-matrices.

Proof. As $\mathfrak{R}$ is solvable and l-alg.,

$$\mathfrak{R}=\mathfrak{A}+\mathfrak{N} \quad \mathfrak{A}=\{X\}.$$

Suppose (1) and (2) represent the decom ositions by X of $\mathfrak{R}$ and $\mathfrak{L}$ respectively. The proof of Lemma 8 shows that $\mathfrak{R}_0$ is the radical of $\mathfrak{L}_0$. Hence we have by a well-known theorem of Levi

$$\mathfrak{L}_0=\mathfrak{S}+\mathfrak{R}_0,$$

where $\mathfrak{S}$ is a maximal semi-simple subalgebra of $\mathfrak{L}_0$. Then it is clear that

$$\mathfrak{L}=\mathfrak{S}+\mathfrak{R},$$

or $\mathfrak{S}+\mathfrak{R}$ is a Levi decomposition of $\mathfrak{L}$.

On the other hand Lemma 4 and $[X, \mathfrak{L}_0]=0$ imply $[\{X\}, \mathfrak{L}_0]=0$. Hence we get $[\mathfrak{S}, \mathfrak{A}]=0$, q.e.d.

We shall call a Lie algebra $\mathfrak{L}(\subseteq \mathfrak{G}(n, K))$ as in Lemma 9 "*normal.*" Combining above lemmas we get

Theorem 2[15]. *Any l-alg. Lie algebra is normal.*

Now, let $\mathfrak{L}$ be a Lie algebra over K and $H \in E(\mathfrak{L})$. If H is an n-matrix[16],

$$\exp H=I+\frac{1}{1!}H+\frac{1}{2!}H^2+\ldots\ldots,$$

a polynomial of H, is an automorphism of $\mathfrak{L}$ as we may easily verify. If moreover $H=A \in I(\mathfrak{L})$, the automorphism $\exp H=\exp A$ is called "*inner*".

Let now $\mathfrak{L}$ be a subalgebra of $\mathfrak{G}(n, K)$ and A be an n-matrix of $\mathfrak{L}$. For any $X \in \mathfrak{L}$ we have easily

$$(\exp A)X=(\exp A)X(\exp(-A)). \tag{3}$$

Modifying a theorem due to A. Malcev, we can prove in a purely algebraic way the following

Lemma 10[17]. Let

$$\mathfrak{L}=\mathfrak{S}+\mathfrak{R}=\mathfrak{S}^*+\mathfrak{R}$$

be two Levi decompositions of $\mathfrak{L}$. Then there exists an element $A\in\mathfrak{R}$ such that $\boldsymbol{A}$ is an n-matrix and

$$\mathfrak{S}^*=\exp \boldsymbol{A}\mathfrak{S} \tag{4}$$

If in particular $\mathfrak{L}$ is a subalgebra of $\mathfrak{G}(n, K)$ we get

$$\mathfrak{S}^*=(\exp A)\mathfrak{S}(\exp (-A)) \quad (=\exp \boldsymbol{A}\mathfrak{S}) \tag{5}$$

Proof. Let us call the ideal $D(\mathfrak{L})=[\mathfrak{L},\mathfrak{L}]$ of a Lie algebra $\mathfrak{L}$ the "derived (Lie) algebra" of $\mathfrak{L}$. As any semi-simple Lie algebra is equal to its derived algebra, $\mathfrak{S}$ and $\mathfrak{S}^*$ are both contained in $D(\mathfrak{L})$. Hence we may prove the Lemma in $D(\mathfrak{L})$. We note here the fact that the radical of $D(\mathfrak{L})$ is composed of n-matrices in any linear representation of $\mathfrak{L}$, hence of course it ts nilpotent[18]. Next we note an obvious fact that if $\mathfrak{N}$ is a Lie algebra composed of n-matrices and $X, Y\in\mathfrak{N}$ then $\exp X \exp Y$ can be written as $\exp Z$ for some $Z\in\mathfrak{N}$.

Then a method, similar to Malcev's, of induction with respect to the degree of the radical of $D(\mathfrak{L})$, applied to $D(\mathfrak{L})$, would easily establish the lemma, q.e.d.

Remark[19]. Lemma 10 (5) shows that for any Levi decomposition $\mathfrak{L}=\mathfrak{S}+\mathfrak{R}$ of a normal Lie algebra there exists $\mathfrak{A}$ such as in Lemma 9.

§ 3. *L-alg. Lie algebras, II.* In this paragraph we shall prove the following

Lemma 11. Any normal Lie algebra is l-alg.

When this lemma is proved, from the results of § 2 we get immediately

Theorem 3. *$\mathfrak{L}$ is l-alg. if and only if $\mathfrak{L}$ is normal. And the radical $\mathfrak{R}$ of $\mathfrak{L}$ is l-alg. if and only if $\mathfrak{L}$ is l-alg..*

Proof of Lemma 11. Let $\mathfrak{L}$ be an irreducible Lie subalgebra of $\mathfrak{G}(n, K)$, and $\mathfrak{L}=\mathfrak{S}+\mathfrak{R}$ be its Levi decomposition. Then by a theorem of E. Cartan[20], either $\mathfrak{R}=0$ or $\mathfrak{R}$ is the one-dimensional centre composed of scalar matrices. Since $\mathfrak{S}$ is semi-simp e $I(\mathfrak{S})=E(\mathfrak{S})$ as is well known. As $E(\mathfrak{S})$ is l-alg. by Theorem 1, $I(\mathfrak{S})$ is of course l-alg. Then $I(\mathfrak{L})$ is also l-alg. as we may see easily. Let $X\in\mathfrak{L}$, $Y\in\{X\}$. Then by Lemma 6 there exists $Y'\in\mathfrak{L}$ such that $\boldsymbol{Y}-\boldsymbol{Y}'=0$, or $[(Y-Y'), \mathfrak{L}]=0$. Since $\mathfrak{L}$ is irreducible $Y-Y'=\lambda 1$, $\lambda\in K$, by so-called Schur's lemma. Now if $\mathfrak{L}$ is not semi-simple, $\lambda 1\in\mathfrak{L}$, hence $Y\in\mathfrak{L}$. For semi-simple $\mathfrak{L}$, we get $\mathrm{Sp}\, \boldsymbol{X}=0$,

$\mathrm{Sp}Y'=0$ and by Lemma 5 $\mathrm{Sp}Y=0$ also. Therefore $\lambda=0$, or $Y(=Y')\in\mathfrak{L}$. Hence in any case $X\in\mathfrak{L}$ implies $\{X\}\subseteqq\mathfrak{L}$, in other words any irreducible Lie algebta is l-alg.

Next we shall show that any semi-simple Lie algebra of matrices is l-alg. Let $\mathfrak{L}$ be such a Lie algebra. By the well-known complete reducibility of semi-simple Lie algebras the space $\mathfrak{M}$ on which $\mathfrak{L}$ operates can be decomposed into the direct sum of irreducible $\mathfrak{L}$-moduli

$$\mathfrak{M}=\sum \mathfrak{M}_i .$$

As in the irreducible case, from $X\in\mathfrak{L}$, $Y\in\{X\}$ follows $[(Y-Y'), \mathfrak{L}]=0$ for some $Y'\in\mathfrak{L}$. But as $X\mathfrak{M}_i\subseteqq\mathfrak{M}_i$ we get $Y\mathfrak{M}_i\subseteqq\mathfrak{M}_i$ by Lemma 2. Hence $Y-Y'$ is a scalar matrix on every $\mathfrak{M}_i$ by Schur's lemma. On the other hand, from $\mathrm{Sp}_{\mathfrak{M}_i}X=0$ we get $\mathrm{Sp}_{\mathfrak{M}_i}Y=0$ by Lemma 5. Therefore

$$\mathrm{Sp}_{\mathfrak{M}_i}(Y-Y')=0, \quad i=1,2, \ldots\ldots,$$

which shows that $Y=Y'\in\mathfrak{L}$. Therefore $\mathfrak{L}$ is l-alg.

Let now $\mathfrak{L}$ be a normal Lie algebra such that the largest ideal composed of n-matrices vanishes. Then

$$\mathfrak{L}=\mathfrak{S}+\mathfrak{A}, \quad [\mathfrak{S}, \mathfrak{A}]=0,$$

and $\mathfrak{S}$ and $\mathfrak{A}$ are both l-alg. Let $X\in\mathfrak{L}$. Then since

$$X=S+A, \quad [S, A]=0, \quad S\in\mathfrak{S}, \quad A\in\mathfrak{A},$$
$$\{X\}\subseteqq\{S\}+\{A\}\subseteqq\mathfrak{S}+\mathfrak{A}\subseteqq\mathfrak{L}$$

by Lemma 4, which shows that $\mathfrak{L}$ is l-alg.

Next, let $\mathfrak{L}$ be a general normal Lie algebra. In order to establish the lemma in this case we shall use the following

Lemma 12[22]. Any Lie algebra composed of n-matrices can be defined by its tensor invariants.

Let X be an n-, s-, or r-matrix and let $\mathfrak{T}$ be any tensor space allowable by X. Then the induced matrix $\overline{X}$ on $\mathfrak{T}$ is n-, s-, or r-matrix respectively. Hence we get $\{\overline{X}\}=\{\overline{X}\}$.

Now, let $\mathfrak{T}$ be a sufficiently large space of tensor invariants of the largest n-matric ideal $\mathfrak{N}$ such that any matrix which induces 0 matrix on $\mathfrak{T}$ is contained in $\mathfrak{N}$. Then $\mathfrak{L}$ induces a representation on $\mathfrak{T}$. Since $\mathfrak{S}+\mathfrak{A}$ is l-alg,, $\mathfrak{S}+\mathfrak{A}$ induces an l-alg. representation on $\mathfrak{T}$, which is identical to the representation of $\mathfrak{L}$. Let $X\epsilon\mathfrak{L}$, $Y\epsilon\{X\}$. Then the induced matrix of Y on $\mathfrak{T}$ is a replica of that of X, and it is contained in the representation because of the l-algebraicity of our representation of $\mathfrak{L}$. Hence there exists $Y'\epsilon\mathfrak{L}$ such that $Y-Y'$ induces 0 matrix on $\mathfrak{T}$. Since $\mathfrak{T}$ is chosen sufficiently large we get $Y-Y'\epsilon\mathfrak{R}$, or $Y\epsilon\mathfrak{L}$. This completes the proof of Lemma 11.

§ 4. *Algebraic closure of a matric Lie algebra.* Let $\mathfrak{L}$ be a Lie subalgebra of $\mathfrak{G}(n, K)$. Then as $\mathfrak{G}(n, K)$ is l-alg. and the intersection of any number of l-alg. Lie algebras is also l-alg., there exists the smallest l-alg. Lie algebra which contains $\mathfrak{L}$. We shall call it the "*algebraic closure*[23]" of $\mathfrak{L}$ and denote it by $\{\mathfrak{L}\}$.

Lemma 13. Let $\mathfrak{L}$ be a nilpotent subalgebra of $\mathfrak{G}(n, K)$. Then

$$\{\mathfrak{L}\}=\mathfrak{L}+\mathfrak{A}, \qquad \mathfrak{L}\cap\mathfrak{A}=0,$$

where $\mathfrak{A}$ is a central ideal of $\{\mathfrak{L}\}$ composed of s-matrices and $\mathfrak{L}$ is an ideal of $\{\mathfrak{L}\}$.

Proof. Let $X=X^0+X^s$ be a canonical decomposition of a matrix $X\epsilon\mathfrak{L}$. Then by Lemma 3 the totality of such X^s's forms an abelian Lie algebra $\mathfrak{A}_1$. Then $\{\mathfrak{A}_1\}$ is clearly an abelian Lie algebra composed of s-matrices. Since for $X=X^0+X^s\epsilon\mathfrak{L}$ and $Y\epsilon\mathfrak{L}$, $[X^s, Y]=0$, we get $[\mathfrak{A}_1, \mathfrak{L}]=0$. Then Lemma 4 implies $[\{\mathfrak{A}_1\}, \mathfrak{L}]=0$. Hence $\mathfrak{L}_1=\mathfrak{L}+\{\mathfrak{A}_1\}$ forms a Lie algebra.

Let us show that $\mathfrak{L}_1$ is l-alg. If $X_1\epsilon\mathfrak{L}$, then

$$X_1=X+A, \qquad X\epsilon\mathfrak{L}, \qquad A\epsilon\{\mathfrak{A}_1\},$$
$$=X^0+X^s+A, \qquad [X^0, X^s+A]=0,$$

where $X=X^0+X^s$ is a canonical decomposition of X. Since $X^s+A\epsilon\{\mathfrak{A}_1\}$, we get $\{X^s+A\}\subseteqq\{\mathfrak{A}_1\}$. Hence $X^0\epsilon\mathfrak{L}_1$. As $X=X^0+(X^s+A)$ is a canonical decomposition of X_1, we get $\{X_1\}\subseteqq\mathfrak{L}_1$, i.e. $\mathfrak{L}_1$ is l-alg. Therefore clearly $\mathfrak{L}_1=\{\mathfrak{L}\}$.

Putting

$$\mathfrak{A}\cap\{\mathfrak{A}_1\}=\mathfrak{A}_2, \qquad \{\mathfrak{A}_1\}=\mathfrak{A}_2+\mathfrak{A}, \qquad \mathfrak{A}_2\cap\mathfrak{A}=0,$$

we get

$$\mathfrak{L}_1=\mathfrak{L}+\mathfrak{A}, \qquad \mathfrak{L}\cap\mathfrak{A}=0, \qquad \text{q.e.d.}$$

Lemma 14. If there exists an element $X\epsilon\mathfrak{L}$ such that $\mathfrak{L}_0$, the 0-eigenspace of X on $\mathfrak{L}$, is l-alg. and nilpotent[24], then $\mathfrak{L}$ is itseif l-alg..

Proof. Let

$$\mathfrak{L}=\mathfrak{L}_0+\mathfrak{L}_\alpha+\ldots\ldots,$$
$$\mathfrak{R}=\mathfrak{R}_0+\mathfrak{R}_\alpha+\ldots\ldots,$$

be the decompositions of $\mathfrak{L}$ and the radical $\mathfrak{R}$ by an inner derivation $\boldsymbol{X}$. Suppose that $\mathfrak{L}$ is l-alg. and nilpotent. We shall first prove that $\mathfrak{R}_0$ is l-alg. Let Y be an arbitrary element of $\mathfrak{R}_0$. Then, by the nilpotency of $\mathfrak{L}_0$ the 0-eigen-space $\mathfrak{L}_0'$ of Y will contain $\mathfrak{L}_0$. Hence $\{Y\}\subseteqq\mathfrak{L}_0\subseteqq\mathfrak{L}_0'$. Then by a method analogous to the proof of Lemma 8 we may easily conclude that $\{Y\}\subseteqq\mathfrak{R}$. Hence $\{Y\}\subseteqq\mathfrak{R}_0$, i.e. $\mathfrak{R}_0$ is l-alg.

Then since $\mathfrak{R}_\alpha+\ldots\ldots$ is contained in the radical of $D(\mathfrak{L})$, it is composed of n-matrices. Therefore, $\mathfrak{R}=\mathfrak{R}_0+\mathfrak{R}_\alpha+\ldots\ldots$ is normal, and by Theorem 3 $\mathfrak{L}$ is l-alg., q.e.d.

Theorem 4[25]. *For any* $\mathfrak{L}\subseteqq\mathfrak{G}(n,K)$,

$$\{\mathfrak{L}\}=\mathfrak{L}+\mathfrak{A},\qquad \mathfrak{L}\cap\mathfrak{A}=0,$$

where $\mathfrak{A}$ *is an abelian Lie algebra composed of s-matrices. Any ideal of* $\mathfrak{L}$ *is also an ideal of* $\{\mathfrak{L}\}$, *and*

$$P^h(\{\mathfrak{L}\})=P^h(\mathfrak{L}),\qquad h=2,3,\ldots\ldots \tag{2}$$

where $P^h(\mathfrak{L})$ *denotes the ideal spanned by all bracket polynomials of degree at least* h.

If $\mathfrak{R}$ *denotes the radical of* $\mathfrak{L}$, *then* $\{\mathfrak{R}\}$ *is the radical of* $\mathfrak{L}$.

Proof. Let

$$\mathfrak{L}=\mathfrak{L}_0+\mathfrak{L}_\alpha+\ldots\ldots$$

be the decomposition of $\mathfrak{L}$ by a regular inner derivation $\boldsymbol{X}$. Since

$$[\{\mathfrak{L}_0\},\ \mathfrak{L}_\alpha]=[\mathfrak{L}_0,\ \mathfrak{L}_\alpha]=\mathfrak{L}_\alpha, \tag{3}$$

we see that

$$\mathfrak{L}_1=\{\mathfrak{L}_0\}+\mathfrak{L}_\alpha+\ldots\ldots$$

constitutes a Lie algebra.

We shall first show that $\{\mathfrak{L}_0\}\cap\mathfrak{L}=\mathfrak{L}_0$. Since by Lemma 14 $\mathfrak{L}_0$ is an ideal of $\{\mathfrak{L}_0\}$, $\mathfrak{L}_0$ is also an ideal of $\{\mathfrak{L}_0\}\cap\mathfrak{L}$. But as we may see easily that $\mathfrak{L}_0$ itself is the only subalgebra of $\mathfrak{L}$ which contains $\mathfrak{L}_0$ as an ideal, we get $\{\mathfrak{L}_0\}\cap\mathfrak{L}=\mathfrak{L}_0$. Since $\mathfrak{L}_0$ is nilpotent we have from Lemma 13 $\{\mathfrak{L}_0\}=\mathfrak{L}_0+\mathfrak{A}$, $\mathfrak{A}\cap\mathfrak{L}_0=0$. Therefore $\mathfrak{A}\cap\mathfrak{L}=0$, and

$$\mathfrak{L}_1=\mathfrak{L}+\mathfrak{A}.$$

Next as $[X, \mathfrak{A}]=0$, $\{\mathfrak{L}_0\}$ is equal to the 0-eigen-space of $\boldsymbol{X}$ in $\mathfrak{L}_1$, which is of course l-alg. and nilpotent. Hence $\mathfrak{L}_1$ is also l-alg. by Lemma 14. Clearly $\{\mathfrak{L}\} \geqq \mathfrak{L}_1$ and so we have that $\{\mathfrak{L}\}=\mathfrak{L}_1$, or

$$\{\mathfrak{L}\}=\mathfrak{L}+\mathfrak{A}, \quad \mathfrak{L} \cap \mathfrak{A}=0. \tag{1}$$

Now, Lemma 13 implies

$$P^h(\{\mathfrak{L}_0\})=P^h(\mathfrak{L}_0), \qquad h=2, 3, \ldots\ldots \tag{4}$$

From (3) and (4) we get easily by mathematical induction with respect to h

$$P^h(\{\mathfrak{L}\})=P^h(\mathfrak{L}), \qquad h=2, 3, \ldots\ldots \tag{2}$$

Next, let $\mathfrak{L}=\mathfrak{S}+\mathfrak{R}$ be a Levi decomposition of $\mathfrak{L}$. Then $\mathfrak{L}'=\mathfrak{S}+\{\mathfrak{R}\} \geqq D(\{\mathfrak{L}\})$ $(=D(\mathfrak{L}))$ constitutes a Lie algebra. Since the radical $\mathfrak{R}_1$ of $\{\mathfrak{L}\}$ is l-alg. and contains $\mathfrak{R}$, it must contain $\{\mathfrak{R}\}$. Hence we have $\mathfrak{L}' \cap \mathfrak{R}_1=\{\mathfrak{R}\}$. This implies that $\{\mathfrak{R}\}$ is an ideal of $\mathfrak{L}'$ As $(\mathfrak{R})$ is solvable with $\mathfrak{R}$ it must be the radical of $\mathfrak{L}'$. Then the l-algebraicity of the radical implies that of $\mathfrak{L}'$. Thus we get $\mathfrak{L}'=\{\mathfrak{L}\}$; i.e. $\{\mathfrak{R}\}$ is the radical of $\{\mathfrak{L}\}$, q.e.d.

§ 5. *Alg. Lie algebras.*

Definition. Let $\mathfrak{L}$ be a Lie algebra over K. We shall call $\mathfrak{L}$ "*algebraic*" (alg.) if $I(\mathfrak{L})$ is l-alg.

For example any Lie algebra $\mathfrak{L}$ such that $I(\mathfrak{L})=E(\mathfrak{L})$ is alg. by Theorem 1. If a subalgebra $\mathfrak{L}$ of $\mathfrak{G}(n, K)$ is l-alg., it is also alg. by virtue of Lemma 6. Thus the algebraicity is an extension of l-algebraicity, and any non-alg. Lie algebra has no l-alg. faithful representation. The analogue to Theorem 2 is given by

Lemma 15. Let $\mathfrak{L}$ be alg. By Levi's theorem $\mathfrak{L}$ is a direct sum of a semi-simple subalgebra $\mathfrak{S}$ and its radical $\mathfrak{R}$: $\mathfrak{L}=\mathfrak{S}+\mathfrak{R}$. Let $\mathfrak{N}$ be the largest nilpotent ideal. There exists an abelian subalgebra $\mathfrak{A}$ such that

$$\mathfrak{R}=\mathfrak{A}+\mathfrak{N}, \qquad \mathfrak{A} \cap \mathfrak{N}=0, \qquad [\mathfrak{S}, \mathfrak{A}]=0,$$

and in the regular representation of $\mathfrak{L}$, $\mathfrak{A}$ is represented faithfully by an l-alg. Lie algebra composed of s-matrices.

Proof. Let $\mathfrak{Z}$ be the centre of $\mathfrak{L}$. Put $\mathfrak{S}+\mathfrak{Z} \equiv \mathfrak{S}_1 \pmod{\mathfrak{Z}}$. Then as $I(\mathfrak{L})$ is l-alg., we obtain from Theorem 2 and Remark to Lemma 10

$$I(\mathfrak{L}) = \mathfrak{S}_1 + \mathfrak{A}_1 + \mathfrak{N}_1,$$

where $\mathfrak{A}_1$ is an abelian subalgebra composed of s-matrices and $\mathfrak{N}_1$ is the largest ideal composed of n-matrices. Let $\mathfrak{A}_2$, $\mathfrak{N}$ be the complete inverse images of $\mathfrak{A}_1$, $\mathfrak{N}_1$, respectively. Then $\mathfrak{N}$ is clearly the largest nilpotent ideal. Since $[\mathfrak{S}_1, \mathfrak{A}_1]=0$ we have $[\mathfrak{S}, \mathfrak{A}_2] \subseteqq \mathfrak{Z}$, so $[\mathfrak{S}, [\mathfrak{S}, \mathfrak{A}_2]]=0$. Hence considering $\mathfrak{A}_2$ as an $\mathfrak{S}$-module we get a nilpotent representation of $\mathfrak{S}$, which must be 0-representation because $\mathfrak{S}$ is semi-simple. Therefore $[\mathfrak{S}, \mathfrak{A}_2]=0$. Now since $\mathfrak{L}$ is completely reducible into one-dimensional irreducible $\mathfrak{A}$-submoduli we may conclude that $\mathfrak{A}_2$ is abelian. As $\mathfrak{A}_2 \cap \mathfrak{N} = \mathfrak{Z}$, putting $\mathfrak{A}_2 = \mathfrak{Z} + \mathfrak{A}$, $\mathfrak{Z} \cap \mathfrak{A} = 0$, we get the result, q.e.d.

Remark. The converse of Lemma 15 is trivial; i.e. any Lie algebra which allows a decomposition as in Lemma 15 is alg.

Theorem 5[26]. *Every alg. Lie algebra has a faithful l-alg. representation (and conversely).*

Proof. Let $\mathfrak{L}$ be an alg. Lie algebra. If the centre $\mathfrak{Z}$ of $\mathfrak{L}$ does not belong to $D(\mathfrak{L})$, we put

$$D(\mathfrak{L}) \cap \mathfrak{Z} = \mathfrak{Z}_1, \quad \mathfrak{Z} = \mathfrak{Z}_1 + \mathfrak{Z}', \quad \mathfrak{Z}_1 \cap \mathfrak{Z}' = 0,$$

and obtain

$$D(\mathfrak{L}) \cap \mathfrak{Z}' = 0.$$

Since any linear space in $\mathfrak{L}$ containing $D(\mathfrak{L})$ is an ideal of $\mathfrak{L}$, we get a direct decomposition of $\mathfrak{L}$ into ideals:

$$\mathfrak{L} = \mathfrak{L}' + \mathfrak{Z}'$$

where $\mathfrak{Z}'$ is a central ideal and $\mathfrak{L}'$ is an alg. ideal such that $D(\mathfrak{L}')$ contains its centre. As an abelian Lie algebra clearly has a faithful l-alg. representation, we have only to prove the theorem under the condition that the centre of $\mathfrak{L}$ belongs to $D(\mathfrak{L})$.

As is well-known any Lie algebra over K has a faithful representation by finite matrices[27]. Hence for simplicity we shall identify the representation with the given Lie algebra. In other words we shall suppose $\mathfrak{L}$ to be a Lie subalgebra of $\mathfrak{G}(n, K)$, which is alg. and such that $D(\mathfrak{L})$ contains its centre. We note here the fact that the centre is composed of n-matrices, because the radical of $D(\mathfrak{L})$ is composed of n-matrices.

Let $\mathfrak{L}=\mathfrak{S}+\mathfrak{A}+\mathfrak{N}$ be a direct decomposition of $\mathfrak{L}$ indicated by Lemma 15. We choose a basis $A_1, \ldots\ldots, A_n$ of $\mathfrak{A}$ such that $\boldsymbol{A}_h$'s are r-matrices, linearly independent with respect to K. The 1-algebraicity of $\mathfrak{A}$ on $\mathfrak{L}$ makes the choice possible. For a certain Hamel basis of K

$$\xi_1=1, \ \ldots\ldots, \ \xi_\alpha, \ \ldots\ldots, \ \xi_\beta, \ \ldots\ldots,$$

we get uniquely canonical decompositions

$$A_h=A_h^0+\xi_1A_h^1+\xi_2A_h^2+\ldots\ldots, \qquad h=1, 2, \ldots\ldots, n,$$

Hence we have

$$\boldsymbol{A}_h=\boldsymbol{A}_h^0+\xi_1\boldsymbol{A}_h^1+\xi_2\boldsymbol{A}_h^2+\ldots\ldots, \qquad h=1, 2, \ldots\ldots, n,$$

where $\boldsymbol{A}_h$, $\boldsymbol{A}_h^1$, $\boldsymbol{A}_h^2$, ... are r-matrices and $\boldsymbol{A}_h^0$ is an n-matrix and they are commutative with each other. On the other hand, if a linear combination $\xi_1 r_1+\xi_2 r_2+\ldots\ldots$ with rational coefficients is a rational number, then $r_2=\ldots\ldots=0$. Therefore we get

$$\boldsymbol{A}_h^0=0, \qquad \boldsymbol{A}_h^1=\boldsymbol{A}_h, \qquad \boldsymbol{A}_h^2=\ldots\ldots=0. \qquad (1)$$

Now since $\mathfrak{A}$ is abelian Lemma 4 implies that $[A_i^1, A_j^1]=0$, and from the linear independence of $\boldsymbol{A}_h$'s follows that of $\boldsymbol{A}_h^1$'s. Therefore A_h^1's are also linearly independent. Thus

$$A_h \rightarrow A_h^1, \qquad h=1, 2, \ldots\ldots, n,$$

gives a faithful representation of $\mathfrak{A}$ by 1-alg. $\mathfrak{A}^*$, which is spanned by A_1^1, ..., A_n^1.

Next $N=N^0+N^s\in\mathfrak{N}$. Since $\mathfrak{N}$ is nilpotent the totality of such N^0's forms a Lie algebra $\mathfrak{N}^*$ and

$$\mathfrak{N}\ni N \rightarrow N^0\in\mathfrak{N}^*$$

gives a representation of $\mathfrak{N}$. Since $\boldsymbol{N}=\boldsymbol{N}^0+\boldsymbol{N}^s$ and $\boldsymbol{N}$ is an n-matrix we get easily from Lemma 6

$$\boldsymbol{N}^s=0, \qquad \boldsymbol{N}^0=\boldsymbol{N}. \qquad (2)$$

If $N^0=0$ for some $N\in\mathfrak{N}$ we get from (2) $\boldsymbol{N}=0$ i.e. N belongs to the centre, whence N is an n-matrix, or $N=N^0=0$. This implies that the representation $\mathfrak{N}\rightarrow\mathfrak{N}^*$ is faithful.

Let us now consider the correspondence

$$\mathfrak{S}\ni S\rightarrow S^{*}=S$$
$$\mathfrak{A}\ni A_{h}\rightarrow A_{h}^{*}=A_{h}^{1}\in\mathfrak{A}^{*}$$
$$\mathfrak{N}\ni N\rightarrow N^{*}=N^{0}\in\mathfrak{N}^{*}$$

By this correspondence any element of $D(\mathfrak{L})$ is represented by itself. Hence from (1) and (2) we may easily conclude that the correspondence gives a faithful representation of $\mathfrak{L}$ by a certain normal Lie algebra, which is l-alg. by Theorem 3, q.e.d.

Remark. Let $\mathfrak{L}$ be any Lie algebra over K, and $\mathfrak{N}$ be the largest nilpotent ideal. Then there exists a faithful representation of $\mathfrak{L}$ such that $\mathfrak{N}$ be represented by a Lie algebra composed of n-matrices.

Proof. For an abelian Lie algebra it is obvious. For $\mathfrak{L}$ such that the centre belongs to $D(\mathfrak{L})$ it is also valid as we may easily see from our above proof. Combining these we get the result, q.e.d.

Now, an alg. Lie algebra of matrices is not necessarily l-alg., and there arises a question what sort of matric Lie algebra alg. is. An answer to this question is given by the following

Theorem 6. *Let* $\mathfrak{L}$ *be a subalgebra of* $\mathfrak{G}(n, K)$. $\mathfrak{L}$ *is alg. if and only if there exists an ideal* $\mathfrak{A}$ *of* $\mathfrak{L}$ *such that*

$$\{\mathfrak{L}\}=\mathfrak{A}+\mathfrak{L},\qquad \mathfrak{A}\quad\mathfrak{L}=0\ \text{(direct sum of ideals)}.$$

Proof. Let $\mathfrak{L}$ be an alg. subalgebra of $\mathfrak{G}(n, K)$. From the proof of Theorem 5 we have a direct decomposition

$$\mathfrak{L}=\mathfrak{L}'+\mathfrak{Z}',\qquad \mathfrak{L}'\cap\mathfrak{Z}'=0,$$

where $\mathfrak{Z}'$ is a central ideal of $\mathfrak{L}$ and $D(\mathfrak{L}')$ contains the centre of $\mathfrak{L}'$. By Lemma 4 we get easily that

$$\{\mathfrak{L}\}=\{\mathfrak{L}'\}+\{\mathfrak{Z}'\},\qquad [\{\mathfrak{L}'\},\{\mathfrak{Z}'\}]=0.$$

Hence if the theorem is proved for $\mathfrak{L}'$, It is also true for $\mathfrak{L}$. Thus the problem is reduced again to the case when $D(\mathfrak{L})\supseteqq$ centre of $\mathfrak{L}$.

Now as in the proof of Theorem 5 $\{N^{s}\}$, $N\in\mathfrak{N}$ and A_{h}^{0}, A_{h}^{2}, A_{h}^{3}, ... ($h=1, 2, \ldots\ldots, n$) in (1) and (2) span an abelian Lie algebra $\mathfrak{A}'$ such that

$$[\mathfrak{L}, \mathfrak{A}']=0.$$

Let $\mathfrak{R}$ be the radical of $\mathfrak{L}$. Then as $\mathfrak{R}+\mathfrak{A}'$ is clearly a normal Lie algebra, it coincides with $\{\mathfrak{R}\}$ by Theorem 3. From Theorem 5 this implies $\mathfrak{L}+\mathfrak{A}'=\{\mathfrak{L}\}$ and we get a desired decomposition easily.

Conversely, if

$$\{\mathfrak{L}\}=\mathfrak{L}+\mathfrak{A}, \qquad \mathfrak{L}\cap\mathfrak{A}=0, \qquad [\mathfrak{L},\mathfrak{A}]=0,$$

then $I(\{\mathfrak{L}\})$ is essentially identical with $I(\mathfrak{L})$. On the other hand, as any l-alg. Lie algebra is also alg., $I(\mathfrak{L})$ is l-alg. and so is $I(\mathfrak{L})$, or what is the same $\mathfrak{L}$ is alg., q.e.d.

Bibliography.

Ado, I. [1]: "On the representation of finite continuous groups by means of linear transformations" Izvestia Kazan, **7** (1934–35) (in Russian).

Cartan, E. [1]: Thèse, Paris (1894).

Cartan, E. [2]: "Les groupes de transformations continues, infinis, simples," Ann. Ec. Norm. Sup., 3e série, t. XXVI, (1909).

Cartan, E. [3]: "Les representations linéaires des groupes de Lie," Jour. math. pures et app., **17** (1938).

Chevalley, C. [3]: "A new kind of relationship between matrices," Amer. J. of Math., **65** (1943).

Chevalley, C. and Tuan, H.-F. [1]: "On algebraic Lie algebras," Proc. Nat. Acad. Sci. U.S.A.

Gantmacher, F. [1]: "Canonical representation of automorphisms of a complex semi-simple Lie group," Recuil Math. Moscou, **5** (1939).

Gotô, M. [1]: "On the replicas of nilpotent matrices," forthcoming in Proc. Imp. Acad. Tokyo.

Malcev, A. [1]: "On the representation of an algebra as a direct sum of the radical and a semi-simple subalgebra," Comptes Rendus (Doklady) (1942).

Matsushima, Y. [1]: "Note on the replicas of matrices," forthcoming in Proc. Imp. Acad. Tokyo.

Matsushima, Y [2]: "On algebraic Lie groups and algebras," forthcoming in this journal.

Maurer, L. [1]: "Zur Theorie der continuierlichen, homogenen und linearen Gruppen," Sitz. d. Bayer. Acad., **24** (1894).

Zassenhaus, H. [1]: "Ueber Lie'sche Ringe mit Primzahlcharakteristik." Hamb. Abh. **13** (1939).

Mathematical Institute,
Nagoya University.

Revised December 11, 1947.

References.

1) Maurer [1].

2) Chevalley and Tuan [1].

3) Chevalley [1].

4) Our l-algebraicity is the same as Chevalley and Tuan's algebraicity.

5) Chevalley and Tuan [1]. Cf. also Matsushima [2].

6) Matsushima [2].

7) The condition of algebraic closedness is conventional, and all our theorems are valid for any field of characteristic 0.

8) Matsushima (in [1]) formulated this lemma in a somewhat different manner. The equivalence follows from the remark below.

9) Matsushima [1].

10) E. g. Zassenhaus [1].

11) Chevalley and Tuan [1].

12) Gantmacher [1].

13) Gantmacher (in [1]) gave an analogous theorem with respect to an element of a matric Lie group and the inner automorphism defined by the element.

14) An inner derivation is called regular if it has as many different eigenvalues as possible.

15) Chevalley and Tuan l. c.

16) For ordinary real (or complex) Lie algebras nilpotency is of course unnecessary.

17) Malcev. [1]. It seems to the writer that Malcev's definition of inner automorphism is not very clear when K is a general field of characteristic 0.

18) Cartan [1].

19) Chevalley and Tuan l. c.

20) Cartan [2].

21) Cartan [1].

22) Chevalley and Tuan [1]; cf also Gotô [1].

23) This notion is due to Chevalley and Tuan.

24) Nilpotency is in fact unnecessary for our Lemma, and combining this to Lemma 7 we get a characterization of l-alg. Lie algebras. And analogous characterization of alg. Lie algebras (see § 5) may also be given easily. But the proof in the general case being rather cumbersome and superfluous for our present purpose, we limit ourselves to the nilpotent case.

25) Chevalley and Tuan [1].

26) See "Introduction".

27) Ado [1]; Cartan [3]. Recently K. Iwasawa gave a new proof on the basis of his theory of splitting Lie algebras. See his forthcoming paper in Jap. J. of Math.