

## A NOTE ON MEAN CURVATURE, MASLOV CLASS AND SYMPLECTIC AREA OF LAGRANGIAN IMMERSIONS

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**In this note we prove a simple relation between the mean curvature form, symplectic area, and the Maslov class of a Lagrangian immersion in a Kähler-Einstein manifold. An immediate consequence is that in Kähler-Einstein manifolds with positive scalar curvature, minimal Lagrangian immersions are monotone.**

### 1. Introduction

Let  $(M, \omega)$  be a Kähler-Einstein manifold whose Ricci curvature is a multiple of the metric by a real number  $\lambda$ . In particular, the Kähler form  $\omega$  and the first Chern class  $c_1(M)$  are related by  $c_1(M) = \frac{\lambda[\omega]}{2\pi}$  (see Section 3). Let  $L$  be an immersed Lagrangian submanifold of  $M$ . Let  $H$  be the trace of the second fundamental form of  $L$  (the mean curvature vector field of  $L$ ). Thus  $H$  is a section of the normal bundle to  $L$  in  $M$  and we have a corresponding 1-form  $\sigma_L := i_H \omega$  on  $L$ , called the *mean curvature form* of  $L$ . Consider a smooth map  $F : \Sigma \rightarrow M$  from a compact oriented surface  $\Sigma$  to  $M$  whose (possibly empty) boundary  $\partial F := F(\partial\Sigma)$  is contained in  $L$ . Let  $\mu(F)$  be the Maslov class of  $F$  (see Section 2) and  $\omega(F) := \int_{\Sigma} F^* \omega$  its symplectic area. The goal of this note is to prove the following simple relation between these quantities:

$$(1) \quad \lambda \omega(F) - \pi \mu(F) = \sigma_L(\partial F).$$

This relation was given in [Mor] for  $\mathbb{C}^n$  and in [Ars] for Calabi-Yau manifolds. Dazord [Daz] showed that the differential of the mean curvature form is the Ricci form restricted to  $L$ , so in the Kähler-Einstein case  $\sigma_L$  is closed (see Section 3). Y.-G. Oh [Oh2] investigated the symplectic area in the case that the mean curvature form is exact.

Lagrangian submanifolds for which  $\mu(F) = a \omega(F)$  on all disks  $F$ , for some  $a > 0$ , are called *monotone* in the symplectic geometry literature, cf. [Oh1].

An immediate consequence of (1) is that in Kähler-Einstein manifolds with positive scalar curvature (i.e.  $\lambda > 0$ ), minimal (i.e.  $\sigma_L \equiv 0$ ) Lagrangian immersions are monotone.

In view of the exact sequence in cohomology (with real coefficients)

$$H^1(M) \longrightarrow H^1(L) \xrightarrow{\delta} H^2(M, L) \longrightarrow H^2(M),$$

formula (1) can be rephrased as

$$\lambda[\omega] - \pi\mu = \delta\sigma_L \in H^2(M, L).$$

Note that the class  $\lambda[\omega] - \pi\mu$  is equivariant under symplectomorphisms of  $M$ . It follows that if the map  $H^1(M) \rightarrow H^1(L)$  is trivial, then the cohomology class of the mean curvature form  $\sigma_L$  is equivariant under symplectomorphisms of  $M$ . This generalizes Oh's observation [Oh2] that the cohomology class is invariant under Hamiltonian deformations.

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## 2. Maslov class

We first recall a definition of the Maslov index that is suitable for our purposes. Let  $V$  be a Hermitian vector space of complex dimension  $n$ . Let  $\Lambda^{(n,0)}V$  be the (one-dimensional) space of holomorphic  $(n,0)$ -forms on  $V$  and set

$$K^2(V) := \Lambda^{(n,0)}V \otimes \Lambda^{(n,0)}V.$$

Let  $L$  be a Lagrangian subspace of  $V$ . We can associate to  $L$  an element  $\kappa(L)$  in  $\Lambda^{(n,0)}V$  of unit length which restricts to a real volume form on  $L$ . This element is unique up to sign and therefore defines a unique element of unit length

$$\kappa^2(L) := \kappa(L) \otimes \kappa(L) \in K^2(V).$$

Thus we get a map  $\kappa^2$  from the Grassmanian  $Gr_{\text{Lag}}(V)$  of Lagrangian planes to the unit circle in  $K^2(V)$ . This map induces a homomorphism  $\kappa_*^2$  of fundamental groups

$$\kappa_*^2 : \pi_1(Gr_{\text{Lag}}(V)) \rightarrow \mathbb{Z}.$$

To understand the map  $\kappa_*^2$ , let  $L$  be a Lagrangian subspace and let  $v_1, \dots, v_n$  be an orthonormal basis for  $L$ . For  $0 \leq t \leq 1$  consider the subspace

$$L_t = \text{span}\{v_1, \dots, v_{n-1}, e^{\pi it}v_n\}.$$

This loop  $\{L_t\}$  is the standard generator of  $\pi_1(Gr_{\text{Lag}}(V))$ . The induced elements in  $\Lambda^{(n,0)}V$  are related by  $\kappa(L_t) = \pm e^{-\pi it}\kappa(L)$ , so  $\kappa^2(L_t) = e^{-2\pi it}\kappa^2(L)$  and  $\kappa_*^2(\{L_t\}) = -1$ . Thus we see that the homomorphism  $\kappa_*^2$  is related to the Maslov index  $\mu$  (as defined, e.g., in [AuLa]) by

$$\kappa_*^2 = -\mu : \pi_1(Gr_{\text{Lag}}(V)) \rightarrow \mathbb{Z}.$$

Now let  $(M, \omega)$  be a symplectic manifold of dimension  $2n$ . Pick an almost complex structure  $J$  on  $M$  such that  $\omega(\cdot, J\cdot)$  defines a Riemannian metric and let  $K(M) := \Lambda^{(n,0)}T^*M$  be the canonical bundle of  $M$ , i.e., the bundle of  $(n, 0)$ -forms on  $M$ . Note that  $c_1(K(M)) = -c_1(M)$ . Let  $K^2(M) := K(M) \otimes K(M)$  be the square of the canonical bundle.

Let  $L$  be an immersed Lagrangian submanifold of  $M$ . For any point  $l \in L$  there is an element of unit length  $\kappa(l)$  of  $K(M)$  over  $l$ , unique up to sign, which restricts to a real volume form on the tangent space  $T_lL$ . The squares of these elements give rise to a section of unit length

$$\kappa_L^2 : L \rightarrow K^2(M).$$

Note that if  $L$  is oriented, then  $\kappa_L^2$  is the square of the unit length section  $\kappa_L : L \rightarrow K(M)$  defined by picking the volume forms  $\kappa(l)|_L$  positive with respect to the orientation.

Now let  $F : \Sigma \rightarrow M$  be a smooth map from a compact oriented surface to  $M$  with boundary  $\partial F = F(\partial\Sigma)$  on  $L$ . To define the Maslov class  $\mu(F)$ , assume first that  $\Sigma$  is connected and  $\partial\Sigma$  is nonempty. Then  $H^2(\Sigma; \mathbb{Z}) = 0$ , hence the pullback  $F^*K(M)$  to  $\Sigma$  is a trivial bundle and we can pick a unit length section  $\kappa_F$  of  $K(M)$  over  $\Sigma$ . Now on the boundary  $\partial F$  we also have the section  $\kappa_L^2$  defined above. We can uniquely write

$$\kappa_L^2 = e^{i\theta} \kappa_F^2$$

for a function  $e^{i\theta} : \partial\Sigma \rightarrow S^1$  to the unit circle. We define the Maslov class  $\mu(F)$  as minus its winding number,

$$\mu(F) := \frac{-1}{2\pi} \int_{\partial F} d\theta.$$

If  $\Sigma$  is closed replace some point of  $\Sigma$  by a new boundary circle  $\partial\Sigma$  which gets mapped under  $F$  to a point  $x \in M$ . Pick a unit length element  $\kappa_x$  of  $K(M)$  at  $x$  and a unit length section  $\kappa_F$  of  $K(M)$  over  $\Sigma$  (which is possible since  $\Sigma$  now has nonempty boundary). Now write  $\kappa_x^2 = e^{i\theta} \kappa_F^2$  over  $\partial\Sigma$  and define  $\mu(F) := \frac{-1}{2\pi} \int_{\partial F} d\theta$  as above. For disconnected  $\Sigma$  define  $\mu(F)$  as the sum over all connected components.

This definition is independent of the choice of  $\kappa_F$  and defines a map

$$\mu : H_2(M, L; \mathbb{Z}) \rightarrow \mathbb{Z}.$$

To see this, first note that any other unit length section  $\kappa'_F$  of  $K(M)$  over  $F$  is related to  $\kappa_F$  by a multiple  $e^{i\phi} : \Sigma \rightarrow S^1$ . So on  $F(\partial\Sigma)$  we have  $\kappa_L^2 = e^{i\theta'} (\kappa'_F)^2$  with  $e^{i\theta'} = e^{-2i\phi} e^{i\theta} : \partial\Sigma \rightarrow S^1$ . By Stokes' theorem, this implies  $\int_{\partial F} d\theta' = \int_{\partial F} d\theta$ . Next suppose that  $F$  and  $F'$  have the same boundary  $\partial F = \partial F' =: \gamma$  and  $[F \cup_\gamma -F'] = 0 \in H_2(M; \mathbb{Z})$ . Then the pullback of  $K(M)$  to  $[F \cup_\gamma -F']$  is a trivial bundle and there is a unit length section  $\kappa$  of  $K(M)$  over  $[F \cup_\gamma -F']$ . If we take the restriction of  $\kappa$  to  $F$  as  $\kappa_F$  and the restriction of  $\kappa$  to  $F'$  as  $\kappa'_F$  we get  $e^{i\theta} = e^{i\theta'}$ , and hence  $\mu(F) = \mu(F')$ .

In particular, if  $[F] = 0 \in H_2(M, L; \mathbb{Z})$  we find an  $F' : \Sigma' \rightarrow L$  with  $\partial F = \partial F' = \gamma$  and  $[F \cup_\gamma - F'] = 0 \in H_2(M; \mathbb{Z})$ , and thus  $\mu(F) = \mu(F') = 0$ . This shows that  $\mu(F)$  depends only on  $[F] \in H_2(M, L; \mathbb{Z})$ .

In view of the discussion above, our definition of  $\mu$  agrees with the usual definition of the Maslov class, cf. [AuLa].

### 3. Proof

Now assume that  $(M, \omega)$  is Kähler with complex structure  $J$  and Kähler metric  $\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot)$ . We denote by  $\nabla$  the Levi-Civita connection, as well as the induced connections on  $K(M)$  and  $K^2(M)$ . Let us briefly review the geometry of  $K(M)$ , following the notations in [Bes], pp. 81-82. Any local non-vanishing section  $\kappa$  of  $K(M)$  over an open subset  $U$  of  $M$  defines a (complex valued) connection one form  $\eta$  on  $U$  by  $\nabla \kappa = \eta \otimes \kappa$ . The curvature of  $K(M)$  is defined to be  $R_K := -d\eta$ ; it is a global closed imaginary valued  $(1, 1)$ -form on  $M$ . By [Bes], Prop. 2.45, the Ricci tensor  $Ric$  of  $M$  is a symmetric bilinear form of type  $(1, 1)$ ; the associated 2-form  $\rho(\cdot, \cdot) := Ric(J\cdot, \cdot)$  is called the *Ricci form* of  $M$ . By [Bes], Prop. 2.96, the Ricci form satisfies

$$\rho = iR_K.$$

It follows (cf. [Bes], Prop. 2.75) that the first Chern class  $c_1(M)$  is represented by  $\frac{\rho}{2\pi}$ . Note that in the Kähler-Einstein case,  $\rho = \lambda\omega$ .

Now let  $L$  be an immersed Lagrangian submanifold of  $M$  and let  $\kappa_L^2$  be the canonical section of  $K^2(M)$  over  $L$  as above. The section  $\kappa_L^2$  defines a connection 1-form  $\eta_L$  for  $K^2(M)$  over  $L$  by the condition  $\nabla \kappa_L^2 = \eta_L \otimes \kappa_L^2$ . Since  $\kappa_L^2$  has constant length 1,  $\eta_L$  is an imaginary valued 1-form on  $L$ . Let  $\sigma_L = i_H\omega$  be the mean curvature form of  $L$  as in Section 1. The following fact goes back to [Oh2], Prop. 2.2:

$$(2) \quad \eta_L = 2i\sigma_L.$$

Here the factor 2 is due to the fact that  $\eta_L$  is a connection 1-form for  $K^2(M)$  rather than  $K(M)$ . In particular, since  $d\eta_L = -2R_K = 2i\rho$ , this formula implies  $d\sigma_L = \rho|_L$ , so in the Kähler-Einstein case  $\sigma_L$  is closed.

For the convenience of the reader, we recall the proof of formula (2) from [Gol] (where, however, the formula is stated with the wrong sign). Pick a point  $l \in L$  and let  $e_1, \dots, e_n$  be a local orthonormal frame tangent to  $L$ . Orient  $L$  locally by this frame. Then  $\kappa_L(e_1, \dots, e_n) \equiv 1$ , and hence for every local vector field  $v$  tangent to  $L$ ,

$$0 = v(\kappa_L(e_1, \dots, e_n)) = (\nabla_v \kappa_L)(e_1, \dots, e_n) + \sum_{j=1}^n \kappa_L(e_1, \dots, \nabla_v e_j, \dots, e_n).$$

Since the complex structure  $J$  is parallel (see [KoNo], Ch. IX Thm. 4.3), the  $j$ -th term in the last sum equals

$$i\langle \nabla_v e_j, J e_j \rangle = i\langle \nabla_{e_j} v, J e_j \rangle = -i\langle v, J \nabla_{e_j} e_j \rangle = i\omega(v, \nabla_{e_j} e_j).$$

Summing over  $j$  and inserting  $H = \sum_{j=1}^n \nabla_{e_j} e_j$ , we find

$$\nabla_v \kappa_L(e_1, \dots, e_n) = i \sum_{j=1}^n \omega(\nabla_{e_j} e_j, v) = i\omega(H, v) = i\sigma_L(v).$$

Now formula (2) follows from  $\nabla \kappa_L^2 = \eta_L \otimes \kappa_L^2$  via

$$\eta_L = 2\nabla \kappa_L(e_1, \dots, e_n) = 2i\sigma_L.$$

Now let  $F : \Sigma \rightarrow M$  be a smooth map from a compact oriented surface with boundary on  $L$ . We will prove the following identity in any Kähler manifold:

$$(3) \quad \rho(F) - \pi\mu(F) = \sigma_L(\partial F).$$

Note that in general the form  $\sigma_L$  need not be closed on  $L$ . It is closed in the Kähler-Einstein case, in which  $\rho = \lambda\omega$  and (3) implies formula (1) in the introduction.

To prove identity (3), assume that every connected component of  $\Sigma$  has nonempty boundary (closed components are treated similiary, see Section 2). Define the section  $\kappa_F$  of  $K(M)$  over  $F$  as in Section 2. Let  $\eta_F$  be the connection 1-form along  $F$  defined by  $\nabla \kappa_F^2 = \eta_F \otimes \kappa_F^2$ . By the discussion in the beginning of this section,  $d\eta_F = 2iF^*\rho$ . Stokes' theorem implies

$$2\rho(F) = \int_{\partial F} -i\eta_F.$$

Recall from Section 2 that along  $\partial F$  we have  $\kappa_L^2 = e^{i\theta} \kappa_F^2$  for a function  $e^{i\theta} : \partial\Sigma \rightarrow S^1$ , and the Maslov class is given by

$$\mu(F) = \frac{-1}{2\pi} \int_{\partial F} d\theta.$$

The connection 1-forms  $\eta_F$  and  $\eta_L$  are related by

$$\eta_L = \eta_F + i d\theta$$

on  $\partial F$ . Combining the equations above and formula (2), we find

$$\sigma_L(\partial F) = \int_{\partial F} \frac{-i\eta_L}{2} = \int_{\partial F} \frac{-i\eta_F}{2} + \int_{\partial F} \frac{d\theta}{2} = \rho(F) - \pi\mu(F).$$

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