J. Sci. Hiroshima Univ. Ser. A-I
30 (1966), 123-130

On Semigroups, Semirings, and Rings of Quotients[*)]

David A. Smith

(Received August 31, 1966)

1. Introduction.

There are many theorems known about the imbedding of algebraic structures in quotient structures, that is, structures with all the properties of the original ones in which suitable candidates (cancellable elements) become invertible, and such that every element of the larger structure is a quotient of elements of the original structure. The best-known classical theorem of this sort asserts that an integral domain may be imbedded in a field of quotients. The construction of such a field, using equivalence classes of ordered pairs, has been adapted to prove a number of generalizations, such as those of Öre [10] and Asano [1] for rings and Vandiver [14] for semigroups and semirings.

In a negative direction, we have the theorem of Malcev [8] that not every ring without zero divisors can be imbedded in a division ring. On the other hand, if one is willing to give up associativity, such an imbedding can always be accomplished (Neumann [9]). We will confine our attention to associative structures.

In [2], Asano generalized his own work with a different kind of construction of quotients using partial endomorphisms (which he called simply "operators"). This in turn was extensively generalized by Findlay and Lambek [3]. In recent years there have been many papers devoted to the subject of rings of quotients: see, for example, [7, 11] and references listed in these.

For the purposes of ring theory, constructions via partial homomorphisms and related ideas are surely more elegant and efficient than the old-fashioned, but more concrete, constructions via equivalence classes of ordered pairs. For example, the verifications of associativity and distributivity are trivial when one uses mappings as elements of the quotient structure.

However, the student is usually introduced first (perhaps solely) to the more concrete construction. Thus it is of interest to see this construction in perhaps its most general form, where only the essential ideas are present at each step. In the process, it becomes clear that for purposes of extending a multiplicative structure to include quotients, the accompanying additive structure (if any) is of little or no consequence. Hence imbedding theorems

*) Supported in part by National Science Foundation grant number GP 4473.

for semigroups and semirings may be obtained in the same way.

The general imbedding theorem for semigroups is presented in section 3 and several special cases are noted. In sections 4 and 5 these results are extended first to semirings and then to rings. In particular, the various imbedding theorems referred to in the first paragraph, as well as others, all appear as more or less immediate corollaries. In the final section, we present an aside on semirings of endomorphisms of additive semigroups, which has as a consequence a different sort of imbedding theorem for finite rings and semirings.

The essential ideas in the general construction were suggested (in the context of rings) by A. W. Goldie in a lecture course. However, this construction was left out of the published version of the lecture notes for the course [5].

The author's interest in semirings was stimulated by a recent collaboration with A. Giovannini [4] on a problem in theoretical physics which gave rise to a semiring in a natural way.

2. Definitions.

A *semigroup* $(S, \cdot)$ is a set S with an associative binary composition $\cdot$. A *semiring* $(S, +, \cdot)$ is a set S with two binary compositions such that $(S, +)$ and $(S, \cdot)$ are semigroups and $\cdot$ distributes over $+$. A *ring* is a semiring $(S, +, \cdot)$ such that $(S, +)$ is an abelian group.

A *cancellable* element of a semigroup $(S, \cdot)$ is an element c such that each of the equalities $ca=cb$ and $ac=bc$ implies that $a=b$. If $(S, +, \cdot)$ is a semiring which has an additive identity 0 (in particular, a ring), an element c is *regular* if each of the equalities $ca=0$ and $ac=0$ implies $a=0$. It is well-known (and easy to see) that an element of a ring is regular if and only if it is cancellable. In a semiring with 0 such that $0a=0=a0$ for all a[1], a cancellable element is clearly regular, but the converse is not true[2]. Since imbedding theorems for rings are usually stated in terms of regular elements, we will use this term in section 5, but otherwise we will use the term cancellable.

Let S be a semigroup which contains cancellable elements. A (*right*) *divisor set* in S is a non-empty multiplicatively closed subset C of cancellable elements satisfying the (*right*) *common multiple property*: Given any $a \in S$ and $\boldsymbol{c} \in C$, there exist $a_1 \in S$ and $\boldsymbol{c}_1 \in C$ such that

$$a\boldsymbol{c}_1 = \boldsymbol{c}a_1. \tag{1}$$

We will consistently use boldface notation for elements of a (fixed, but arbi-

1) This follows from the definition of 0 if addition satisfies the cancellation laws, but not in general.

2) Consider 2×2 matrices with non-negative integer entries and the usual addition and multiplication.

trary) divisor set, and ordinary notation for elements not so restricted. Note that C might be the set of all cancellable elements, which is necessarily multiplicatively closed. However, even if the set of all cancellable elements is not a divisor set, there may be smaller subsets which are. One should note that some of the arguments in sections 3 and 4 are a little more complicated than, for instance, those in [1], because some of the cancellable elements that occur need not be in the divisor set.

A *semigroup of* (*right*) *quotients* (of S with respect to C) is a semigroup S_C with identity which contains an isomorphic copy of S (which we identify with S) and which has the properties:

$$\text{Every } \boldsymbol{c}\in C \text{ is invertible in } S_C; \tag{2}$$

$$\text{Every element of } S_C \text{ is of the form } a\boldsymbol{c}^{-1},\ a\in S,\ \boldsymbol{c}\in C. \tag{3}$$

If $(S, +, \cdot)$ is a semiring (in particular, a ring), a *divisor set* for $(S, +, \cdot)$ is a divisor set for $(S, \cdot)$. The definition of *semiring* (*ring*) *of quotients* is exactly the same as for semigroups, but note that the imbedding must preserve both operations.

3. The imbedding theorem for semigroups.

Let $(S, \cdot)$ be a semigroup which contains cancellable elements and C a divisor set in S. The objective of this section is to prove that a semigroup S_C of quotients exists.

Consider the set $S\times C$ of ordered pairs $(a, \boldsymbol{c})$. We will introduce a relation in this set by defining $(a, \boldsymbol{c})\sim(b, \boldsymbol{d})$ if and only if

$$\boldsymbol{c}a_1=\boldsymbol{d}\boldsymbol{c}_1\Rightarrow aa_1=b\boldsymbol{c}_1 \qquad (a_1\in S,\ \boldsymbol{c}_1\in C). \tag{4}$$

Remark. a_1 need not be in C, but it is cancellable. If $a_1x=a_1y$, then $\boldsymbol{c}a_1x=\boldsymbol{c}a_1y\Rightarrow\boldsymbol{d}\boldsymbol{c}_1x=\boldsymbol{d}\boldsymbol{c}_1y\Rightarrow x=y$. By (1), there exist a_2, $\boldsymbol{c}_2$ such that $\boldsymbol{d}\boldsymbol{c}_1a_2=\boldsymbol{c}\boldsymbol{c}_2$. Then $\boldsymbol{c}a_1a_2=\boldsymbol{d}\boldsymbol{c}_1a_2=\boldsymbol{c}\boldsymbol{c}_2\Rightarrow a_1a_2=\boldsymbol{c}_2$. Hence, if $xa_1=ya_1$, then $xa_1a_2=ya_1a_2\Rightarrow x\boldsymbol{c}_2=y\boldsymbol{c}_2\Rightarrow x=y$.

We defer for the moment showing that $\sim$ is a well-defined equivalence relation, and prove first that (4) implies an apparently stronger condition, useful for technical purposes.

Lemma 1. *Suppose* $(a, \boldsymbol{c})\sim(b, \boldsymbol{d})$. *Then*

$$\boldsymbol{c}a_2=\boldsymbol{d}c_2\Rightarrow aa_2=bc_2 \ \textit{for all}\ \ a_2,\ c_2\in S. \tag{5}$$

Proof. Let a_1, $\boldsymbol{c}_1$ be as in (4), and $\boldsymbol{c}a_2=\boldsymbol{d}c_2$, with c_2 not necessarily in C. By (1), there exist $\boldsymbol{e}_1$, e_2 such that $c_2\boldsymbol{e}_1=\boldsymbol{c}_1e_2$. Then $\boldsymbol{c}a_2\boldsymbol{e}_1=\boldsymbol{d}c_2\boldsymbol{e}_1=\boldsymbol{d}\boldsymbol{c}_1e_2=\boldsymbol{c}a_1e_2\Rightarrow a_2\boldsymbol{e}_1=a_1e_2\Rightarrow aa_2\boldsymbol{e}_1=aa_1e_2=b\boldsymbol{c}_1e_2=bc_2\boldsymbol{e}_1\Rightarrow aa_2=bc_2$, as desired.

Corollary. $\sim$ *is a well-defined relation on* $S\times C$.

PROOF. Take $c_2 \in C$ in Lemma 1.

LEMMA 2. $\sim$ *is an equivalence relation on* $S \times C$.

PROOF. The relation is clearly reflexive. Suppose $(a, \boldsymbol{c}) \sim (b, \boldsymbol{d})$; to prove symmetry, we have to show that $\boldsymbol{d}a_2 = \boldsymbol{c}\boldsymbol{c}_2$ implies $ba_2 = a\boldsymbol{c}_2$. By ignoring the fact that $\boldsymbol{c}_2$ may be taken in C, this follows from Lemma 1.

Now suppose that $(a, \boldsymbol{b}) \sim (c, \boldsymbol{d})$ and $(c, \boldsymbol{d}) \sim (e, \boldsymbol{f})$. To prove transitivity, we have to show that $\boldsymbol{b}a_1 = \boldsymbol{f}\boldsymbol{c}_1$ implies $aa_1 = e\boldsymbol{c}_1$. Let $\boldsymbol{e}_1$, e_2 be such that $\boldsymbol{d}e_2 = \boldsymbol{f}\boldsymbol{c}_1\boldsymbol{e}_1 = \boldsymbol{b}a_1\boldsymbol{e}_1$. From $(a, \boldsymbol{b}) \sim (c, \boldsymbol{d})$ (and Lemma 1), we have $aa_1\boldsymbol{e}_1 = ce_2$. From $(c, \boldsymbol{d}) \sim (e, \boldsymbol{f})$, we have $ce_2 = e\boldsymbol{c}_1\boldsymbol{e}_1$. Hence $aa_1\boldsymbol{e}_1 = e\boldsymbol{c}_1\boldsymbol{e}_1$ and $aa_1 = e\boldsymbol{c}_1$, as desired.

We now denote the equivalence class of $(a, \boldsymbol{c})$ by $a/\boldsymbol{c}$. Let S_C denote the set of these equivalence classes. A product operation will be defined in S_C by:

$$(a/\boldsymbol{c})(b/\boldsymbol{d}) = aa_1/\boldsymbol{d}\boldsymbol{c}_1, \quad \text{where } \boldsymbol{c}a_1 = b\boldsymbol{c}_1. \tag{6}$$

LEMMA 3. *The product* (6) *is well-defined.*

PROOF. Suppose a, b, $\boldsymbol{c}$, $\boldsymbol{d}$, a_1, $\boldsymbol{c}_1$ are as in (6). If $\boldsymbol{c}a_2 = b\boldsymbol{c}_2$, it is easy to verify that $aa_1/\boldsymbol{d}\boldsymbol{c}_1 = aa_2/\boldsymbol{d}\boldsymbol{c}_2$, and we omit this. Suppose $(a, \boldsymbol{c}) \sim (a', \boldsymbol{c}')$, $(b, \boldsymbol{d}) \sim (b', \boldsymbol{d}')$, and $\boldsymbol{c}'\boldsymbol{a}_2 = b'\boldsymbol{c}_2$. We must show that $\boldsymbol{d}\boldsymbol{c}_1a_3 = \boldsymbol{d}'\boldsymbol{c}_2\boldsymbol{c}_3$ implies $aa_1a_3 = a'a_2\boldsymbol{c}_3$. Using the definitions of a_1, a_2, $\boldsymbol{c}_1$, $\boldsymbol{c}_2$ and $(b, \boldsymbol{d}) \sim (b', \boldsymbol{d}')$, $\boldsymbol{d}\boldsymbol{c}_1a_3 = \boldsymbol{d}'\boldsymbol{c}_2\boldsymbol{c}_3 \Rightarrow b\boldsymbol{c}_1a_3 = b'\boldsymbol{c}_2\boldsymbol{c}_3 \Rightarrow \boldsymbol{c}a_1a_3 = \boldsymbol{c}'a_2\boldsymbol{c}_3$. Then the desired conclusion follows from $(a, \boldsymbol{c}) \sim (a', \boldsymbol{c}')$ and Lemma 1.

THEOREM 1. *If S is any semigroup with at least one cancellable element and C is any divisor set, then S_C is a semigroup of quotients for S with respect to C.*

PROOF. A straightforward verification shows that the product (6) is associative. Similarly, one may verify directly from (6) that any element $\boldsymbol{c}/\boldsymbol{c}$ is both a left and right identity. Since there can be only one identity element, $\boldsymbol{c}/\boldsymbol{c} = \boldsymbol{d}/\boldsymbol{d}$ for all $\boldsymbol{c}, \boldsymbol{d} \in C$.

Let ϕ: $S \to S_C$ be defined by $\phi(a) = a\boldsymbol{c}/\boldsymbol{c}$. This is clearly well-defined. If $\phi(a) = \phi(b)$, then $a\boldsymbol{c}^2 = b\boldsymbol{c}^2$, hence $a = b$. Also $(a\boldsymbol{c}/\boldsymbol{c})(b\boldsymbol{c}/\boldsymbol{c}) = a\boldsymbol{c}a_1/\boldsymbol{c}\boldsymbol{c}_1$, where $\boldsymbol{c}a_1 = b\boldsymbol{c}\boldsymbol{c}_1$, so the product is $ab\boldsymbol{c}\boldsymbol{c}_1/\boldsymbol{c}\boldsymbol{c}_1$, hence ϕ is a monomorphism. Since $(\boldsymbol{c}^2/\boldsymbol{c})(\boldsymbol{c}/\boldsymbol{c}^2) = \boldsymbol{c}^3/\boldsymbol{c}^3 = \boldsymbol{c}/\boldsymbol{c}$, $\phi(\boldsymbol{c})^{-1} = \boldsymbol{c}/\boldsymbol{c}^2$. Finally, $a/\boldsymbol{c} = (a\boldsymbol{c}/\boldsymbol{c})(\boldsymbol{c}/\boldsymbol{c}^2) = \phi(a)\phi(\boldsymbol{c})^{-1}$. Hence conditions (2) and (3) are satisfied.

COROLLARY 1. *If S is a semigroup in which the set of cancellable elements has the common multiple property* (1), *then S can be imbedded in a semigroup in which all its cancellable elements become invertible.*

COROLLARY 2. (Vandiver [14]) *If S is a semigroup with cancellable elements, all of which lie in the center of S, then S can be imbedded in a semi-*

group with identity in which the cancellable elements form an abelian group.

COROLLARY 3. [15, Theorem 20.3]. *If S is a commutative semigroup with cancellable elements the conclusion of Corollary 2 holds.*

COROLLARY 4. [15, Theorem 20.2]. *A commutative semigroup, all of whose elements are cancellable, may be imbedded in a commutative group.*

4. Applications to semirings.

Let $(S, +, \cdot)$ be a semiring, and let C be a divisor set in S. Let S_C be defined as in the previous section. We define addition in S_C by:

$$a/\boldsymbol{c}+b/\boldsymbol{d}=(ad_1+b\boldsymbol{c}_1)\boldsymbol{n}, \tag{7}$$

where $\boldsymbol{d}\boldsymbol{c}_1=\boldsymbol{c}d_1=\boldsymbol{n}$.

LEMMA 4. *The sum* (7) *is well-defined.*

PROOF. The proof that the sum is independent of the choice of $\boldsymbol{c}_1$ and d_1 is straightforward, and is omitted. We give the rest of the proof to indicate the use of Lemma 1.

Suppose $(a, \boldsymbol{c})\sim(a', \boldsymbol{c}')$ and $(b, \boldsymbol{d})\sim(b', \boldsymbol{d}')$. Let $\boldsymbol{n}'=\boldsymbol{d}'\boldsymbol{c}_2=\boldsymbol{c}'d_2$. We have to show that $\boldsymbol{n}e_1=\boldsymbol{n}_2\boldsymbol{e}_2$ implies $(ad_1+b\boldsymbol{c}_1)e_1=(a'd_2+b'\boldsymbol{c}_2)\boldsymbol{e}_2$. The hypothesis of this statement can be written $\boldsymbol{d}\boldsymbol{c}_1e_1=\boldsymbol{c}d_1e_1=\boldsymbol{d}'\boldsymbol{c}_2\boldsymbol{e}_2=\boldsymbol{c}'d_2\boldsymbol{e}_2$. From the first and third terms, since $(b, \boldsymbol{d})\sim(b', \boldsymbol{d}')$, we have $b\boldsymbol{c}_1e_1=b'\boldsymbol{c}_2\boldsymbol{e}_2$. From the other two terms, using $(a, \boldsymbol{c})\sim(a', \boldsymbol{c}')$ and Lemma 1, we have $ad_1e_1=a'd_2\boldsymbol{e}_2$. The desired conclusion follows.

THEOREM 2. *If S is any semiring with at least one* (*multiplicatively*) *cancellable element, and C is any divisor set, then S_C* (*with the operations* (6) *and* (7)) *is a semiring of quotients for S with respect to C.*

PROOF. Again we omit the proof of associativity of the extended addition and distributivity of the product (6) over the sum (7). We have a one-to-one mapping ϕ: $S\rightarrow S_C$, as in the proof of Theorem 1, which preserves products, and S_C has properties (2) and (3). It remains to show that ϕ preserves sums. From (7) we have immediately $\phi(a)+\phi(b)=(a\boldsymbol{c}/\boldsymbol{c})+(b\boldsymbol{c}/\boldsymbol{c})=(a\boldsymbol{c}^2+b\boldsymbol{c}^2)/\boldsymbol{c}^2=\phi(a+b)$.

COROLLARY 5. *If S is a semiring in which the set of cancellable elements has the common multiple property* (1), *then S can be imbedded in a semiring in which all its cancellable elements become invertible.*

COROLLARY 6. (Vandiver [14]). *If S is a semiring with cancellable elements, all of which lie in the center of* $(S, \cdot)$, *then S can be imbedded in a semi-*

ring with identity in which the cancellable elements form an abelian group under multiplication.

COROLLARY 7. *If S is a semiring with cancellable elements and if multiplication in S is commutative, then the conclusion of Corollary* 6 *holds.*

The results of section 3 can also be applied to semirings by considering the additive semigroup. For example, Corollary 4 can be used to prove the following classical result.

THEOREM 3 [15, Theorem 20.8]. *If S is a semiring such that addition is commutative and satisfies the cancellation law, then S can be imbedded in a ring.*

PROOF. Corollary 4 states $(S, +)$ may be imbedded in a commutative group $(S_C, +)$ in which every element has the form $a-b$, $a, b \in S$. We extend the product operation to S_C in the obvious way:

$$(a-b)(c-d)=ac+bd-(ad+bc). \tag{8}$$

Associativity and distributivity of this product over addition are easy to check, but the fact that it is well-defined is not quite trivial.

One verifies easily that condition (4) for the equivalence relation can be translated, under the present hypotheses, into $a-b=a'-b'$ if and only if $a+b'=a'+b$. Suppose we have $a+b'=a'+b$ and $c+d'=c'+d$. These equations give

$$ac+b'c=a'c+bc, \tag{9a}$$

$$ad+b'd=a'd+bd, \tag{9b}$$

$$a'c+a'd'=a'c'+a'd, \tag{9c}$$

$$b'c+b'd'=b'c'+b'd. \tag{9d}$$

Then $ac+bd+a'd'+b'c'$ may be shown to equal $a'c'+b'd'+ad+bc$ by carrying out the following steps: (i) replace ac and bd by equivalent expressions from (9a) and (9b); (ii) replace $a'c$ and $b'd$ using (9c) and (9d); (iii) make appropriate cancellations.

In connection with this theorem, it is easy to see that not every semiring can be imbedded in a ring [13].

5. Applications to rings.

Theorem 2 has several immediate corollaries in the case where the semiring S is actually a ring.

THEOREM 4 (Asano [2]). *If S is a ring with at least one regular element, and C is any divisor set, then S_C is a ring of quotients for S with respect to C.*

PROOF. The proof that the sum (7) is commutative when $(S, +)$ is commutative is, like many of the other necessary verifications, straightforward and omitted. To complete the proof, it suffices to observe that if S has an additive identy 0, then $0/\boldsymbol{c}$ is an additive identity in S_C, and if $a+(-a)=0$ in S, then $(a/\boldsymbol{c})+(-a/\boldsymbol{c})=0/\boldsymbol{c}$ in S_C, both of which are trivial.

COROLLARY 8 (Asano [1]). *If S is a ring in which the set of regular elements has the common multiple property* (1), *then S can be imbedded in a ring in which all its regular elements become invertible.*

COROLLARY 9 (Öre [10]). *If S is a ring in which the non-zero elements are all regular and have the common multiple, property, then S can be imbedded in a division ring.*

COROLLARY 10 [15, Theorem 23.9]. *Any integral domain can be imbedded in a field.*

6. Endomorphisms of additive semigroups.

Let $(S, +)$ be a semigroup, $\mathscr{M}(S)$ the set of mappings of S into itself, and $\mathscr{E}(S)$ the subset of endomorphisms of S. If $\mathscr{M}(S)$ is equipped with the operations of pointwise addition and composition as sum and product, respectively, it fails to be a semiring only in that one of the distributive laws is not always satisfied. $\mathscr{E}(S)$ does not have this difficulty, but is not necessarily closed under addition. However, if S is commutative, $\mathscr{E}(S)$ is closed under addition, and so is a semiring. Furthermore, if S is an abelian group, $\mathscr{E}(S)$ is a ring. (See [6, pp. 1–2] for the details of some of these remarks.) Whether or not $\mathscr{E}(S)$ is a semiring, we may speak of *a semiring* (*ring*) *of endomorphisms* of S, meaning a subset of $\mathscr{E}(S)$ which forms a semiring (ring) with respect to the indicated operations.

THEOREM 5. (See also [6, p. 54, Theorem 1]). *If $(S, +, \cdot)$ is a semiring* (*ring*) *with at least one cancellable element, then S is isomorphic to a semiring* (*ring*) *of endomorphisms of $(S, +)$.*

PROOF. For each $a \in S$, let R_a denote the operation of right multiplication by a in S:

$$bR_a = ba, \quad \text{for all} \quad b \in S. \tag{10}$$

The right distributive law says that $R_a \in \mathscr{E}(S)$ (meaning, of course, endomorphisms of $(S, +)$, not of $(S, +, \cdot)$). The left distributive law can be written

(11) $$R_{a+b}=R_a+R_b,$$

and associativity of multiplication implies

(12) $$R_{ab}=R_aR_b.$$

Thus R: $S\to\mathscr{E}(S)$, where $R(a)=R_a$, is a homomorphism. If c is a cancellable element in S, then $R_a=R_b\Rightarrow ca=cb\Rightarrow a=b$, so R is one-to-one, and the image $R(S)$ is a semiring of endomorphisms isomorphic to S.

We next observe that c is a cancellable element of S if and only if R_c is one-to-one. If S is a finite semiring (for example, one of those studied by Vandiver in [12]), then a mapping of S into itself is one-to-one if and only if it is onto, so that c is cancellable if and only if R_c is an automorphism. Thus Theorem 5 yields the following analog of Corollaries 5 and 8 via a quite different approach:

COROLLARY 11. *If* $(S, +, \cdot)$ *is a finite semiring in which addition is commutative (in particular, a finite ring) and which has at least one cancellable element, then* S *can be imbedded in a finite semiring (ring) in which its cancellable elements become invertible.*

We remark that a finite semiring (in fact, a finite semigroup) has a cancellable element if and only if it has an identity [12, Theorem 1].

References

[1] K. Asano, *Arithmetische Idealtheorie in nichtkommutativen Ringen*, Japanese J. of Math., **16** (1939-40), 1-36.

[2] ———, *Uber die Quotientenbildung von Schiefringen*, J. Math. Soc. Japan, **1** (1949), 73-78.

[3] G. D. Findlay and J. Lambek, *A generalized ring of quotients* I, II, Canadian Math. Bull., **1** (1958), 77-85, 155-167.

[4] A. Giovannini and D. A. Smith, *On the projective geometry of the* 3-*j symbols*, to appear.

[5] A. W. Goldie, *Rings with maximum condition*, multilithed lecture notes, Yale University, 1961.

[6] N. Jacobson, *The Theory of Rings*, New York: American Mathematical Society, 1943.

[7] R. E. Johnson, *Quotient rings of rings with zero singular ideal*, Pacific J. Math., **11** (1961), 1385-1392.

[8] A. Malcev, *On the immersion of an algebraic ring into a field*, Math. Ann., **113** (1936), 686-691.

[9] B. H. Neumann, *Embedding non-associative rings in division rings*, Proc. London Math. Soc. (3), **1** (1951), 241-256.

[10] O. Öre, *Linear equations in non-commutative fields*, Annals of Math., **32** (1931), 463-477.

[11] L. W. Small, *Orders in Artinian rings*, J. Algebra, **4** (1966), 13-41.

[12] H. S. Vandiver, *Note on a simple type of algebra in which the cancellation law of addition does not hold*, Bull. Amer. Math. Soc., **40** (1934), 914-920.

[13] ———, *On some simple types of semi-rings*, Amer. Math. Monthly, **46** (1939), 22-26.

[14] ———, *On the imbedding of one semi-group in another, with application to semi-rings*, Amer. J. Math., **62** (1940), 72-78.

[15] Seth Warner, *Modern Algebra*, vol. I, Englewood Cliffs: Prentice-Hall, 1965.

Duke University
Durham, North Carolina
U. S. A.