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# *Note on Miller's Recurrence Algorithm*

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## 1. Introduction

In this paper, we are concerned with a recurrence algorithm, originated by J. C. P. Miller [1][1], for computing a solution $f_n$ of a second-order difference equation

$$y_{n-1} = a_n y_n + b_n y_{n+1} \qquad (b_n \neq 0;\ n = 1, 2, \cdots), \tag{1.1}$$

in the case where (1.1) has a second solution $g_n$ which ultimately grows much faster than $f_n$ [6]. This algorithm is used for computing Bessel functions [1, 2, 4, 9], Legendre functions [8], repeated integrals of the error function [3], and so on.

Let $P_n(k)$ be defined by the formula

$$P_n(k-1) = a_k P_n(k) + b_k P_n(k+1) \qquad (k = n+1, n, \cdots, 1), \tag{1.2}$$

where

$$P_n(n) = 1, \quad P_n(n+1) = 0, \qquad P_n(n+2) = 1/b_{n+1}. \tag{1.3}$$

Then Miller's algorithm is applied in the following two ways:

1°. when the normalizing condition

$$m_0 f_0 + m_1 f_1 + \cdots = c \qquad (c \neq 0) \tag{1.4}$$

is known, put

$$S_n(k) = \frac{c P_n(k)}{R_n} \qquad (k = 0, 1, \cdots, n), \tag{1.5}$$

where

$$R_n = \sum_{j=0}^{n} m_j P_n(j). \tag{1.6}$$

2°. when $f_0$ is known and $f_0 \neq 0$, put

1) Numbers in square brackets refer to the references listed at the end of this paper.

$$T_n(k)=\frac{f_0 P_n(k)}{P_n(0)} \qquad (k=0, 1, \cdots, n). \tag{1.7}$$

Under suitable conditions, which are reported to have been obtained by Gautschi [3, 6], it is valid that

$$S_n(k),\ T_n(k)\to f_k \quad \text{as} \quad n\to\infty. \tag{1.8}$$

In the sequel, we consider the case where (1.8) holds.

There arises the question how large $n$ should be in order to obtain the approximate values of $f_k$ $(k=0, 1, \cdots, N+1)$ to the desired accuracy. Such a value $n$ will depend on the value $N$, the desired accuracy, the coefficients $a$'s and $b$'s, and so on. Until now theoretical bounds for the starting value $n$ have been obtained for spherical Bessel functions [2], and for the repeated integrals of the error function [3], and empirical bounds have been obtained for $J_k(x)$ [5]. In the case where such a bound is not known, usually Miller's algorithm is applied repeatedly for different values of $n$; the results obtained are compared in accordance with a preassigned tolerence and the process is repeated with $n$ increased by a fixed amount until the criteria for acceptance are satisfied [9].

In the first part of this paper, recurrence formulas are derived for generating $P_n(k)$ and $R_n$ for increasing $n$ with $k$ fixed. By generating $S_n(N)$ and $S_n(N+1)$ for increasing $n$ through these formulas, the approximate values of $f_N$ and $f_{N+1}$ can be obtained to the desired accuracy and then the approximate values of $f_k$ $(k=N-1, \cdots, 1, 0)$ can be generated through (1.2). This process seems to be more efficient than the above iterative process.

In the second part of this paper, we consider the case where $a_r>0$ and $b_r>0$ $(r=1, 2, \cdots)$, and show the methods for generating the approximate values of $f_k$ $(k=0, 1, \cdots, N+1)$ to the desired relative accuracy.

## 2. Recurrence formulas

We shall first show the following

THEOREM 1. *$P_n(k)$ $(n=k-1, k, \cdots)$ satisfy the recurrence formula*

$$P_{n+1}(k)=a_{n+1}P_n(k)+b_nP_{n-1}(k) \qquad (n+1\geqq k\geqq 0), \tag{2.1}$$

*where*

$$P_k(k)=1, \quad P_{k-1}(k)=0, \qquad P_{k-2}(k)=1/b_{k-1}. \tag{2.2}$$

Proof. Since by definition

(2.3) $$P_{n-1}(n+1)=1/b_n,\quad P_n(n+1)=0,\quad P_{n+1}(n+1)=1,$$

and

(2.4) $$P_{n-1}(n)=0,\quad P_n(n)=1,\quad P_{n+1}(n)=a_{n+1},$$

(2.1) ia valid for $k=n+1,\ n$. Hence suppose that (2.1) holds for $k=n+1,\ n,\ \cdots,\ q\ \ (q>0)$. Then we have

(2.5) $$P_{n+1}(q)=a_{n+1}P_n(q)+b_nP_{n-1}(q),$$

and

(2.6) $$P_{n+1}(q+1)=a_{n+1}P_n(q+1)+b_nP_{n-1}(q+1).$$

On the other hand, from (1.2) it follows that

(2.7) $$P_{n+1}(q-1)=a_qP_{n+1}(q)+b_qP_{n+1}(q+1).$$

Substituting (2.5) and (2.6) into (2.7), we obtain

(2.8) $$\begin{aligned}P_{n+1}(q-1)&=a_{n+1}[a_qP_n(q)+b_qP_n(q+1)]+\\&\quad+b_n[a_qP_{n-1}(q)+b_qP_{n-1}(q+1)]\\&=a_{n+1}P_n(q-1)+b_nP_{n-1}(q-1).\end{aligned}$$

This proves the theorem.

Next put

(2.9) $$U_n(k)=\sum_{j=k}^{n}m_jP_n(j)\qquad (n\geqq k).$$

Then we have the following

THEOREM 2. $U_n(k)\ \ (n=k,\ k+1,\ \cdots)$ *satisfy the recurrence formula*

(2.10) $$U_{n+1}(k)=a_{n+1}U_n(k)+b_nU_{n-1}(k)+m_{n+1}\qquad (n\geqq k),$$

*where*

(2.11) $$U_{k-1}(k)=0,\quad U_k(k)=m_k.$$

Proof. Since

(2.12) $$U_k(k)=m_kP_k(k)=m_k$$

and

$$U_{k+1}(k)=m_kP_{k+1}(k)+m_{k+1}P_{k+1}(k+1)=m_ka_{k+1}+m_{k+1}, \tag{2.13}$$

(2.10) is valid for $n=k$.

For $n>k$, we have

$$\begin{aligned} U_{n+1}(k)&=\sum_{j=k}^{n}m_jP_{n+1}(j)+m_{n+1}P_{n+1}(n+1) \\ &=\sum_{j=k}^{n}m_j\left[a_{n+1}P_n(j)+b_nP_{n-1}(j)\right]+m_{n+1} \\ &=a_{n+1}\sum_{j=k}^{n}m_jP_n(j)+b_n\sum_{j=k}^{n-1}m_jP_{n-1}(j)+m_{n+1} \\ &=a_{n+1}U_n(k)+b_nU_{n-1}(k)+m_{n+1}, \end{aligned} \tag{2.14}$$

because

$$P_{n+1}(n+1)=1, \quad P_{n-1}(n)=0. \tag{2.15}$$

Thus the theorem has been proved.

Since $R_n=U_n(0)$, from this theorem we obtain the following

COROLLARY. $R_n(n=0, 1, 2, \cdots)$ *satisfy the recurrence formula*

$$R_{n+1}=a_{n+1}R_n+b_nR_{n-1}+m_{n+1} \qquad (n\geqq 0), \tag{2.16}$$

*where*

$$R_{-1}=0, \quad R_0=m_0. \tag{2.17}$$

Making use of (2.1) and (2.16), we can generate $P_n(k)$, $S_n(k)$, and $T_n(k)$ for increasing $n$ with $k$ fixed. Hence, for a specified $k$, we can obtain the approximate value of $f_k$ to the desired accuracy by increasing $n$.

When $P_n(N)$ and $P_n(N+1)$ are obtained by means of (2.1), we can use (1.2) to generate $P_n(j)$ $(j=N-1, \cdots, 1, 0)$. In that case, if $U_n(N)$ is computed, $R_n$ can be obtained by the formula

$$R_n=\sum_{j=0}^{N-1}m_jP_n(j)+U_n(N). \tag{2.18}$$

Since by (1.5), (1.7) and (1.8)

$$\frac{P_n(k+1)}{P_n(k)} \to \frac{f_{k+1}}{f_k} \qquad \text{as } n\to\infty \quad (f_k\neq 0), \tag{2.19}$$

we have the following

THEOREM 3. *The ratio* $f_{k+1}/f_k$ $(f_k \neq 0)$ *can be expanded into the continued fraction as follows*:

$$\frac{f_{k+1}}{f_k} = \frac{1}{a_{k+1}+} \frac{b_{k+1}}{a_{k+2}+} \frac{b_{k+2}}{a_{k+3}+} \cdots\cdots \qquad (k \geqq 0). \tag{2.20}$$

Proof. By (2.19) it suffices to show that

$$\frac{P_{m+1}(k+1)}{P_{m+1}(k)} = \frac{1}{a_{k+1}+} \frac{b_{k+1}}{a_{k+2}+} \cdots\cdots \frac{b_{m-1}}{a_m+} \frac{b_m}{a_{m+1}} \tag{2.21}$$

$$(m = k+1,\ k+2,\ \cdots\cdots).$$

By (2.1), (2.3) and (2.4) we have

$$\frac{P_{k+2}(k+1)}{P_{k+2}(k)} = \frac{a_{k+2}}{a_{k+2}a_{k+1} + b_{k+1}} = \frac{1}{a_{k+1}+} \frac{b_{k+1}}{a_{k+2}}, \tag{2.22}$$

so that (2.21) is valid for $m = k+1$. Hence suppose that (2.21) holds for $m = k+1, k+2, \cdots, n$. Then it is valid that

$$\frac{a_{n+1}P_n(k+1) + b_n P_{n-1}(k+1)}{a_{n+1}P_n(k) + b_n P_{n-1}(k)} = \frac{1}{a_{k+1}+} \frac{b_{k+1}}{a_{k+2}+} \cdots\cdots \frac{b_{n-1}}{a_n+} \frac{b_n}{a_{n+1}}. \tag{2.23}$$

Replacing $a_{n+1}$ and $b_n$ in (2.23) with $a_{n+2}a_{n+1} + b_{n+1}$ and $a_{n+2}b_n$ respectively, we have

$$\frac{(a_{n+2}a_{n+1} + b_{n+1})P_n(k+1) + a_{n+2}b_n P_{n-1}(k+1)}{(a_{n+2}a_{n+1} + b_{n+1})P_n(k) + a_{n+2}b_n P_{n-1}(k)} \tag{2.24}$$

$$= \frac{1}{a_{k+1}+} \frac{b_{k+1}}{a_{k+2}+} \cdots\cdots \frac{b_{n-1}}{a_n+} \frac{a_{n+2}b_n}{a_{n+2}a_{n+1} + b_{n+1}}.$$

Since by (2.1)

$$(a_{n+2}a_{n+1} + b_{n+1})P_n(r) + a_{n+2}b_n P_{n-1}(r) \qquad (r = k,\ k+1) \tag{2.25}$$

$$= a_{n+2}[a_{n+1}P_n(r) + b_n P_{n-1}(r)] + b_{n+1}P_{n-1}(r)$$

$$= a_{n+2}P_{n+1}(r) + b_{n+1}P_{n+1}(r) = P_{n+2}(r),$$

and

$$\frac{a_{n+2}b_n}{a_{n+2}a_{n+1} + b_{n+1}} = \frac{b_n}{a_{n+1}+} \frac{b_{n+1}}{a_{n+2}}, \tag{2.26}$$

(2.21) is valid also for $m = n+1$.

Now we shall show the examples to which the above results can be applied.

EXAMPLE 1. Bessel functions of the first kind $J_k(x)$ $(k=0, 1, \cdots)$ satisfy the recurrence formula [9]

$$J_{n-1}(x)=\frac{2n}{x}J_n(x)-J_{n+1}(x) \tag{2.27}$$

with the normalizing condition

$$J_0(x)+2\sum_{k=1}^{\infty}J_{2k}(x)=1. \tag{2.28}$$

Hence we can use (2.1) and (2.16) to obtain the approximate values of $J_0(x)$ and $J_1(x)$ to the desired accuracy without knowing previously the starting value $n$. They can be used also to determine the empirical bound for the starting value $n$ for $J_0(x)$ and $J_1(x)$. Once such a bound is obtained, we can use (1.2) to generate the approximate values of $J_0(x)$, $J_1(x)$ and so on efficiently.

EXAMPLE 2. Let

$$i^n \operatorname{erfc} x=\int_x^{\infty} i^{n-1} \operatorname{erfc} t \, dt \qquad (n=0, 1, \cdots), \tag{2.29}$$

where

$$i^{-1} \operatorname{erfc} x=\frac{2}{\sqrt{\pi}}e^{-x^2}, \tag{2.30}$$

and put

$$y_n=i^{n-1} \operatorname{erfc} x. \tag{2.31}$$

Then $y_n$ $(n=0, 1, \cdots)$ satisfy the recurrence formula [3]

$$y_{n-1}=2xy_n+2ny_{n+1}. \tag{2.32}$$

Since $y_1=\operatorname{erfc} x$, it is valid that

$$T_n(1)\to \operatorname{erfc} x \qquad \text{as } n\to\infty, \tag{2.33}$$

where

$$T_n(1)=\frac{2}{\sqrt{\pi}}e^{-x^2}\frac{P_n(1)}{P_n(0)}. \tag{2.34}$$

Hence we can use (2.1) to obtain the approximate value of erfc $x$.

On the other hand, from (6.20) it follows that

$$\text{(2.35)} \qquad \operatorname{erfc} x = \frac{2}{\sqrt{\pi}} e^{-x^2} \left[ \frac{1}{2x+} \frac{2}{2x+} \frac{4}{2x+} \frac{6}{2x+} \cdots\cdots \right].$$

J. Patry and J. Keller [7] obtained the expansion

$$\text{(2.36)} \qquad \operatorname{erfc} x = e^{-x^2} \left[ \frac{1}{c_0 x+} \frac{1}{c_1 x+} \frac{1}{c_2 x+} \cdots\cdots \right]$$

where

$$\text{(2.37)} \qquad c_0 = \sqrt{\pi}, \quad c_1 = \frac{2}{\sqrt{\pi}}, \quad c_{n+1} = \frac{2}{c_n +} \frac{2}{c_{n-1}}.$$

As is easily seen, this is equivalent to (2.35), but (2.35) is simpler than (2.36).

## 3. Case of positive coefficients

In this paragraph, we are concerned with the case where

$$\text{(3.1)} \qquad a_n > 0, \quad b_n > 0 \qquad (n = 1, 2, \cdots\cdots).$$

This condition is satisfied, for instance, by the recurrence formulas for $I_n(x)$, $i_n(x)$ and $i^n$ *erfc* $x$. Our problem is how to generate $f_k$ $(k = 0, 1, \cdots, N+1)$ to the desired relative accuracy. To that end we need the following

LEMMA. *Put*

$$\text{(3.2)} \qquad \frac{f_{N+1}}{f_N} = r$$

*and*

$$\text{(3.3)} \qquad \frac{P_n(N+1)}{P_n(N)} = r_n = r(1 + e_n) \qquad (n \geqq N+1).$$

*Then it is valid that*

$$\text{(3.4)} \qquad \frac{P_n(k)}{f_k} = (1 + d_k e_n) \frac{P_n(N)}{f_N} \qquad (k = 0, 1, \cdots, N+1),$$

$$\text{(3.5)} \qquad 0 = d_N < d_{N-2} < \cdots < d_0 < \cdots < d_{N-1} < d_{N+1} = 1,$$

*and*

(3.6) $$-1=e_N<e_{N+2}<\cdots<0<\cdots<e_{N+3}<e_{N+1},$$

*where*

(3.7) $$d_k=\frac{rb_NP_{N-1}(k)}{P_N(k)+rb_NP_{N-1}(k)}.$$

Proof. It is easy to show by induction that

(3.8) $$P_n(k)=P_n(N)P_N(k)+b_NP_n(N+1)P_{N-1}(k)$$

and

(3.9) $$f_k=f_NP_N(k)+b_Nf_{N+1}P_{N-1}(k) \qquad (k=0,1,\cdots,N+1).$$

From these, (3.2) and (3.3) we have

(3.10) $$\frac{P_n(k)}{f_k}=\frac{P_n(N)}{f_N}\cdot\frac{P_N(k)+r(1+e_n)b_NP_{N-1}(k)}{P_N(k)+rb_NP_{N-1}(k)}$$
$$=\frac{P_n(N)}{f_N}\left[1+\frac{rb_NP_{N-1}(k)}{P_N(k)+rb_NP_{N-1}(k)}\cdot e_n\right].$$

This proves (3.4).

Next, substituting (3.3) into

(3.11) $$P_{n+1}(N+1)=a_{n+1}P_n(N+1)+b_nP_{n-1}(N+1),$$

we have

(3.12) $$r_{n+1}P_{n+1}(N)=r_na_{n+1}P_n(N)+r_{n-1}b_nP_{n-1}(N).$$

From this and

(3.13) $$P_{n+1}(N)=a_{n+1}P_n(N)+b_nP_{n-1}(N),$$

it follows that

(3.14) $$r_{n+1}-r_n=(r_{n-1}-r_n)\frac{b_nP_{n-1}(N)}{P_{n+1}(N)}$$

and

(3.15) $$r_{n+1}-r_{n-1}=(r_n-r_{n-1})\frac{a_{n+1}P_n(N)}{P_{n+1}(N)}.$$

Since

(3.16) $$r_N=0,\quad r_{N+1}=\frac{1}{a_{N+1}},\quad P_m(N)>0 \qquad (m\geqq N),$$

from (3.14) and (3.15) we have

$$0=r_N<r_{N+2}<\cdots<r<\cdots<r_{N+3}<r_{N+1}. \tag{3.17}$$

This proves (3.6).

Lastly, from (3.7) we can easily deduce the relations

$$d_k-d_{k-1}=(d_k-d_{k+1})b_k\frac{P_N(k+1)+rb_NP_{N-1}(k+1)}{P_n(k-1)+rb_NP_{N-1}(k-1)} \tag{3.18}$$

and

$$d_{k+1}-d_{k-1}=(d_{k-1}-d_k)a_k\frac{P_N(k)+rb_NP_{N-1}(k)}{P_N(k+1)+rb_NP_{N-1}(k+1)}. \tag{3.19}$$

Since $d_N=0$ and $d_{N+1}=1$, from (3.18) and (3.19) follows (3.5). This completes the proof of the lemma.

Now we shall show the following

THEOREM 4. *Let*

$$f_0^*=f_0(1+c) \qquad (|c|\leqq c_0<1) \tag{3.20}$$

*and*

$$T_n^*(k)=\frac{f_0^*P_n(k)}{P_n(0)}=(1+e_{n,k})f_k \qquad (k=0, 1,\dots, N+1), \tag{3.21}$$

*where $f_0^*$ is an approximate value of $f_0$. Then, for $n$ such that*

$$n=N+1+2q \qquad (q\geqq 1), \tag{3.22}$$

*it is valid that*

$$(1+c)(1-e_n)<1+e_{n,k}<(1+c)(1+e_n) \qquad (k=0, 1,\dots, N+1) \tag{3.23}$$

*and*

$$\frac{r_n-r_{n+2}}{r_{n+2}}<e_n<\frac{r_n-r_{n-1}}{r_{n-1}}. \tag{3.24}$$

Proof. From (3.20), (3.21) and (3.4), it follows that

$$1+e_{n,k}=(1+c)\frac{1+d_ke_n}{1+d_0e_n}=(1+c)\left[1+\frac{(d_k-d_0)}{1+d_0e_n}\cdot e_n\right]. \tag{3.25}$$

We consider the case where $n$ satisfies (3.22). Then it is valid that

(3.26) $$1+d_0e_n>1,$$

because $d_0>0$ and $e_n>0$ by (3.5) and (3.6). Further, from (3.5), it follows that

(3.27) $$|d_k-d_0|<1.$$

Hence we have the inequality

(3.28) $$\left|\frac{d_k-d_0}{1+d_0e_n}\right|<1,$$

and (3.23) is proved.

On the other hand, since by (3.6)

(3.29) $$-1<e_{n-1}<0, \qquad 0<e_{n+2}<e_n,$$

we have

(3.30) $$\frac{r_n-r_{n-1}}{r_{n-1}}=\frac{e_n-e_{n-1}}{1+e_{n-1}}>e_n,$$

and

(3.31) $$\frac{r_n-r_{n+2}}{r_{n+2}}=\frac{e_n-e_{n+2}}{e_{n+2}}<e_n.$$

Thus the theorem has been proved.

Now, by (3.24), it holds that

(3.32) $$e_{n,k}<e_n(1+c)+c\leqq e_n(1+c_0)+c_0$$

and

(3.33) $$e_{n,k}>-e_n(1+c)+c\geqq -e_n(1+c_0)-c_0.$$

Hence we have the following

COROLLARY. *Under the condition* (3.22), *if for a positive number* $\mu$ ($\mu>c_0$)

(3.34) $$e_n\leqq\frac{\mu-c_0}{1+c_0},$$

*then the inequality*

(3.35) $$|e_{n,k}|<\mu$$

*is valid for* $k=0, 1,\cdots, N+1$.

Next we shall show the following

THEOREM 5. *Let*

$$S_n(k)=f_k(1+s_{n,k}) \tag{3.36}$$

*and suppose that, for a positive number* $\mu$,

$$|s_{n,N}|\leqq\mu, \qquad |s_{n,N+1}|\leqq\mu. \tag{3.37}$$

*Then the inequality*

$$|s_{n,k}|\leqq\mu \tag{3.38}$$

*is valid for* $k=0, 1,\cdots, N+1$.

Proof. From (3.4) and (1.5) it follows that

$$s_{n,k}=s_{n,N}+d_k e_n(1+s_{n,N}) \qquad (k=0, 1,\cdots, N+1) \tag{3.39}$$

and

$$e_n(1+s_{n,N})=s_{n,N+1}-s_{n,N}, \tag{3.40}$$

because $d_{N+1}=1$. Substituting (3.40) into (3.39), we obtain

$$s_{n,k}=d_k s_{n,N+1}+(1-d_k)s_{n,N}. \tag{3.41}$$

Since by (3.5)

$$0\leqq d_k\leqq 1 \qquad (k=0, 1,\cdots, N+1), \tag{3.42}$$

from (3.41) and (3.37) follows (3.38).

Now we are in a position to apply theorems 4 and 5 for generating the approximate values of $f_k$ $(k=0, 1,\cdots, N+1)$ such that

$$|s_{n,k}|\leqq\mu \qquad (k=0, 1,\cdots, N+1) \tag{3.43}$$

for a preassigned positive number $\mu$. For this purpose, the following three methods can be considered.

*Method* 1. Generate $P_n(0)$, $R_n$, $P_n(N)$ and $P_n(N+1)$ for increasing $n$ until the inequalities

$$|s_{n,0}| \leqq \frac{\mu}{2} \tag{3.44}$$

and

$$|e_n| \leqq \frac{\mu}{2+\mu} \tag{3.45}$$

are valid for $n$ satisfying (3.22), and then compute $S_n(k)$ $(k=0, 1,\cdots, N+1)$ by (1.2).

*Method* 2. Generate $R_n$, $P_n(N)$ and $P_n(N+1)$ until the condition (3.37) is satisfied, and then compute $S_n(k)$ $(k=0, 1,\cdots, N+1)$ by (1.2).

When there is known a bound $M(\mu)$ such that the inequality

$$n \geqq M(\mu) \tag{3.46}$$

implies (3.44), the following method becomes possible.

*Method* 3. Generate $P_n(N)$, $P_n(N+1)$ and $U_n(N)$ until (3.45) and (3.46) are valid for $n$ satisfying (3.22), and then compute $S_n(k)$ $(k=0, 1,\cdots, N+1)$ by (1.2) and (2.18).

Among the three methods, the last one seems to be the most efficient, and the methods 1 and 2 can be applied for determining the empirical bound $M(\mu)$ with $N=0$.

EXAMPLE 3. Let $I_n(x)$ $(n=0, 1,\cdots)$ be the modified Bessel functions of the first kind and put

$$y_n = e^{-x} I_n(x) \tag{3.47}$$

for a fixed value of $x$. Then they satisfy the recurrence formula [9]

$$y_{n-1} = \frac{2n}{x} y_n + y_{n+1} \tag{3.48}$$

with the normalizing condition

$$y_0 + 2\sum_{j=1}^{\infty} y_j = 1. \tag{3.49}$$

Generating $S_n(0)$ and $S_n(1)$ for increasing $n$ until they were in the state of numerical convergence [10] for $x=0.01, 0.05(0.05)1.0, 1.5(0.5)10, 15(5)100$, and $110(10)500$, we obtained the following empirical bounds for a digital computer with 39 bits mantissa:

$$M(x, 10^{-6}) = \begin{cases} x+9-\dfrac{62}{39x+10} & (0<x\leqq 10) \\ 0.1x+74-\dfrac{6270}{x+105} & (10<x\leqq 500), \end{cases} \tag{3.50}$$

$$M(x, 10^{-8}) = \begin{cases} x+12-\dfrac{83}{27x+10} & (0<x\leqq 10) \\ 0.1x+99-\dfrac{10800}{x+130} & (10<x\leqq 500), \end{cases} \tag{3.51}$$

$$M(x, 10^{-10}) = \begin{cases} x+16-\dfrac{44}{5x+4} & (0<x\leqq 10) \\ 0.15x+83-\dfrac{4514}{x+65} & (10<x\leqq 500). \end{cases} \tag{3.52}$$

These bounds mean that the inequalities

$$|s^*_{n,0}| \leqq \frac{\mu}{2}, \qquad |s^*_{n,1}| \leqq \frac{\mu}{2} \tag{3.53}$$

are valid approximately provided $n \geqq M(x, \mu)$, where $s^*_{n,0}$ and $s^*_{n,1}$ are the relative errors of $S_n(0)$ and $S_n(1)$ to the computed values of $e^{-x}I_0(x)$ and $e^{-x}I_1(x)$ respectively.

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