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*Weakly Completely Continuous Banach *-Algebras*

By

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Throughout this paper we shall be only concerned with Banach $*$-algebras with complex scalars. A topological ring is said to be dual [**4**] provided that for every closed right (left) ideal I we have $R(L(I))=I$ and $L(R(I))=I$ respectively, where L and R denote the left and right annihilators. I. Kaplansky [**7**] has shown that the following statements are equivalent for a B^*-algebra A: (1) A is dual. (2) A is a $B^*(\infty)$-sum of C^*-algebras each of which is the algebra of all completely continuous operators on a Hilbert space. (3) A has a faithful $*$-representation by completely continuous operators on a Hilbert space. (4) The socle of A is dense in A. In an earlier paper [**12**] one of the present authors proved that a completely continuous operator on a Hilbert space $\mathfrak{H}$ is characterized as a w. c. c. (= weakly completely continuous) element of the algebra of operators on $\mathfrak{H}$. This leads us to show that (1)—(4) are equivalent to that (5) A is w. c. c. (§**3**).

Kaplansky [**5**] also studied the structure of c. c. B^*-algebras and obtained the result: A c. c. B^*-algebra is a $B^*(\infty)$-sum of full matrix algebras of finite orders over the complex field. This will also follow from our above-mentioned result since the algebra of completely continuous operators on a Hilbert space is finite-dimensional if and only if it is c. c. [**12**]. Various group algebras of a compact group studied by Kaplansky [**4**] are dual A^*-algebras. We show (§**4**) that every semi-simple c. c. Banach $*$-algebra in which $x^*x=0$ implies $x=0$ is an A^*-algebra considered as a dense subalgebra of a c. c. B^*-algebra. The fundamental theorem [**9**] of almost periodic functions in a group is to say that the algebra of a. p. f. is a c. c. dual A^*-algebra. In any c. c. dual A^*-algerbra every closed right ideal is the closure of the union of minimal right ideals contained in it (§**4**). Any dual B^*-algebra is c. c. if and only if it is strongly semi-simple, or the annihilator of the center is zero (§**4**).

In §**2** we treat the uniqueness problem of an auxiliary norm of A^*-algebras. We say that an A^*-algebra A has a unique auxiliary norm in case any two auxiliary norms $|x|$, $|x|_1$ are equivalent, that is, $|x_n|\to 0$ if and only if $|x_n|_1\to 0$. Whether or not every A^*-algebra has a unique auxiliary norm is open for us. We show under certain conditions that an A^*-algebra has a unique auxiliary norm.

We show (§**5**) that any dual A^*-algebra can be embedded as a dense subalgebra of a unique (to within $*$-isomorphism) dual B^*-algebra $\mathfrak{A}$. There then arise two possible cases: A is an ideal of $\mathfrak{A}$ and is called a dual A^*-algebra of the 1st kind, otherwise of the 2nd kind. Any proper H^*-algebra of Ambrose [**1**], if we introduce in it an auxiliary norm $|x|=\text{l. u. b.}_{\|y\|=1}\|xy\|$, is a dual A^*-algebra of the 1st kind. A necessary and sufficient condition for a dual A^*-algebra to be of the 1st kind is given (§**6**). The condition is that $\|x\|_1=\text{l. u. b.}_{\|y\|=1}\|xy\|$ is an auxiliary norm of A, that is, $\|x\|_1^2\leq k\|x^*x\|_1$ for every x and a positive constant k. Any dual A^*-algebra A of the 1st kind is w. c. c. and has the properties: (1) For any maximal family of orthogonal self-adjoint idempotents $\{e_\alpha\}$, $\sum xe_\alpha$ is summable to x in the norm. (2) Every closed $*$-subalgebra is of the 1st kind. (3) Every closed right ideal is the closure of the union of minimal right ideals contained in it (§**6, 7**).

In §**7** we give certain properties of dual A^*-algebras, in intimate connection with the theory of group algebras of compact groups studied by Kaplansky [**4**]. We show that the group algebras C and L of a compact group G are of the 2nd kind unless G is finite.

We conclude the last section § **8** with a short discussion on a commutative dual A^*-algebra.

§ 1. Preliminaries

A Banach algebra A is called a Banach $*$-algebra provided there is defined in A an involution $x\to x^*$ with the following properties: (i) $(x^*)^*=x$. (ii) $(xy)^*=y^*x^*$. (iii) If λ, μ are complex numbers, then $(\lambda x+\mu y)^*=\bar{\lambda}x^*+\bar{\mu}y^*$. If A satisfies also the condition (iv) $\|x\|^2=\|x^*x\|$ for every $x\in A$, then A is called a B^*-algebra. Furthermore if x^*x has a quasi-inverse for every $x\in A$, then A is called a C^*-algebra. We shall say that two Banach $*$-algebras A, B are equivalent provided there exists a $*$-isomorphism ϕ of A onto B such that ϕ and its inverse are continuous.

Certain fundamental properties of B^*-algebras have been discussed by Kaplansky [**5**]. Some of them are still valid with modifications for somewhat general Banach $*$-algebras with the condition:

(β_k) $\|x\|^2\leq k\|x^*x\|$, k being a positive constant.

Among them we shall state the following two theorems for our later use. Since they can be proved along the same line as in [**5**], their proofs will be omitted.

THEOREM 1. *Let A be a Banach $*$-algebra in which (β_k) holds for some constant k. Let I be a closed ideal of A, then I is self-adjoint, and A/I is a Banach $*$-algebra satisfying (β_γ) for a positive constant γ, which may be chosen to depend only on k*

continuously and reduces to 1 *for* $k=1$.

THEOREM 2. *Let A, B be Banach *-algebras satisfying* (β_k), $(\beta_{k'})$ *respectively. If there exists an algebraic *-isomorphism ϕ of A into a dense subset of B, then ϕ maps A onto B and A, B are equivalent, more precisely,*

$$\frac{1}{kk'}\|x\|\leq\|\phi(x)\|\leq kk'\|x\|$$

§ 2. Auxiliary norms of A^*-algebras

A Banach *-algebra A is called an A^*-algebra [**13**] provided there exists in A an auxiliary norm $|x|$ (not necessarily complete) which satisfies, in addition to the usual multiplicative property, the condition (β_k) for some constant k. It is noted that a homomorphism ϕ of a Banach algebra into an A^*-algebra A is continuous if its image $\phi(B)$ is self-adjoint [**15**]. An A^*-algebra is said to possess a unique auxiliary norm in case any two auxiliary norms $|x|$ and $|x|_1$, satisfying the above stated conditions, are equivalent; that is, $|x_n|\to 0$ if and only if $|x_n|_1\to 0$. Whether or not every A^*-algebra has a unique auxiliary norm is open for us. We will now prove that under certain conditions an A^*-algebra has a unique auxiliary norm. Before doing this, some lemmas will be considered.

LEMMA 1. *Let A, B be Banach *-algebras satisfying* (β_k), $(\beta_{k'})$ *respectively. Let ϕ denote a *-homomorphism of A into B, which is isomorphic on a dense *-subalgebra A' of A. Then ϕ maps A *-isomorphically into B if any of the following conditions is satisfied;*

(1) *A' is an ideal of A.*

(2) *If I is any closed ideal of A with $I\cap A'=0$, then $IA'=0$.*

PROOF. (1) implies (2) since $IA'\subset I\cap A'=0$. Suppose that (2) holds. Let I denote the kernel of ϕ. Then $I\cap A'=0$ since ϕ is isomorphic on A', and $R(I)$, the right annihilator of I, contains A'. A' being dense in A, we have $R(I)=A$. Owing to (β_k), $x^*x=0$ implies $x=0$. Therefore I must be 0, completing the proof.

LEMMA 2. *Let A, B be Banach *-algebras satisfying* (β_k), $(\beta_{k'})$ *respectively. Let A', B' be dense *-subalgebras of A, B respectively. Assume that A' satisfies any of the conditions* (1), (2) *of Lemma* 1. *If ϕ is a continuous *-isomorphism of A' onto B', then ϕ is uniquely extensible to a *-isomorphism of A onto B, and A, B are equivalent.*

PROOF. Since ϕ is continuous, ϕ can be uniquely extended to a *-homomorphism ϕ' of A into B. It follows from Theorem 2 and Lemma 1 that the statement of our lemma is true.

LEMMA 3. *Let A be an A^*-algebra with an auxiliary norm in which $\|ax\|\leq c\|a\|\,|x|$ for every a, $x\in A$, c being a constant. Let $\mathfrak{A}$ denote the completion of A by $|x|$, then*

A is an ideal dense in $\mathfrak{A}$.

PROOF. Consider the linear mapping $x \to ax$ from a $*$-subalgebra A of $\mathfrak{A}$ to A with its own norm $\|x\|$. It is uniformly continuous, and therefore is uniquely extensible to a linear mapping $z \to az$ from $\mathfrak{A}$ to A with $\|az\| \leq c\|a\| |z|$. This shows that A is a right ideal dense in $\mathfrak{A}$. On the other hand the involution is continuous in both A and $\mathfrak{A}$ [**15**], and therefore $\|xa\| \leq c' |x| \|a\|$ for every $x, a \in A$, c' being a constant. In like manner A is a left ideal of $\mathfrak{A}$. This completes the proof.

LEMMA 4. *Let A' be a dense ideal of an A^*-algebra A with norm $\|x\|$. If A' is a Banach algebra with norm $\|x\|_1$, then $\|ax\|_1 \leq c\|a\|_1\|x\|$, $\|xa\|_1 \leq c\|x\| \|a\|_1$ for every $a \in A'$ and $x \in A$, c being a constant.*

PROOF. By the closed graph theorem [**2**, p.41] it suffices to show that the mapping $x \to ax$ $(a \to ax)$ from A to A' (from A' to itself) is closed. Let $\|x_n - x\| \to 0$ and $\|ax_n - y\|_1 \to 0$. The mapping $a \to a$ from A' to A is continuous [**15**], and therefore $\|ax_n - y\| \to 0$. Hence $ax = y$. In like manner we can show that $a \to ax$ is closed. Thus we have the conclusion.

Making use of these lemmas we will prove the following uniqueness theorem of auxiliary norm of an A^*-algebra.

THEOREM 3. *Let A be an A^*-algebra with an auxiliary norm $|x|$. A has a unique auxiliary norm if any of the following conditions is satisfied:*

(α) $\|ax\| \leq c\|a\| |x|$ *for every $a, x \in A$, c being a constant.*

(β) *The socle ($=$the union of minimal right ideals of A) of A is dense in A.*

PROOF. Let $|x|_1$ be any auxiliary norm of A. Put $|x|_2 = |x| + |x|_1$. It is obvious that A becomes a normed algebra under $|x|_2$. Let $|x|^2 \leq k|x^*x|$ and $|x|_1^2 \leq k_1|x^*x|$, k and k_1 being positive constants. Then

$$|x^*x|_2 = |x^*x| + |x^*x|_1 \geq \frac{1}{k}|x|^2 + \frac{1}{k_1}|x|_1^2 \geq \frac{1}{k_2}(|x| + |x|_1)^2 = \frac{1}{k_2}|x|_2^2\ ,$$

where $k_2 = 2\max(k, k_1)$. This shows that $|x|_2$ is also an auxiliary norm of A. Let $\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2$ denote the completions of A by $|x|$, $|x|_1$, $|x|_2$ respectively.

Case (α). Lemma 3 shows that A is an ideal dense in both $\mathfrak{A}$ and $\mathfrak{A}_2$. It follows from Lemma 2 that $\mathfrak{A}, \mathfrak{A}_2$; $\mathfrak{A}_1, \mathfrak{A}_2$ are equivalent respectively. Hence $\mathfrak{A}, \mathfrak{A}_1$ are equivalent, that is, A has a unique auxiliary norm.

Case (β). Since $x^*x = 0$ implies $x = 0$, every minimal right ideal R of A is generated by a uniquely determined self-adjoint primitive idempotent e of A [**14**]. Let I be any closed ideal of $\mathfrak{A}_2$ such that $I \cap A = 0$. We show that $IA = 0$. For otherwise there exists an R such that $IR \neq 0$, and therefore $Ie \neq 0$. Take $z \in I$ with $ze \neq 0$. Since $eAe = $(the complex field) $\times e$ [**3**], we have $ez^*ze=\lambda e$ $(\lambda\neq 0)$. Hence $e\in I$, which contradicts $I\cap A=0$. Lemma 2 shows that $\mathfrak{A}$, $\mathfrak{A}_2$; $\mathfrak{A}_1$, $\mathfrak{A}_2$ are equivalent respectively, and therefore $\mathfrak{A}$, $\mathfrak{A}_1$ are equivalent, that is, two norms $|x|$, $|x|_1$ are equivalent.

Let A be a proper H^*-algebra, that is, a Banach $*$-algebra whose underlying Banach space is a Hilbert space with inner product (x, y) such that $(xy, z)=(y, x^*z)=(x, zy^*)$ and $xA=0$ implies $x=0$ [**1, 4**]. If we put $|x|=\underset{\|y\|=1}{\mathrm{l.u.b.}}\|xy\|$. Then $|x|^2=|x^*x|$ [**12**], and $\|xy\|\leq|x|\,\|y\|$. Thus A is an A^*-algebra satisfying (α). Therefore A has a unique auxiliary norm, in other words, A can be embedded in a unique (to within $*$-isomorphism) B^*-algebra $\mathfrak{A}$ as its dense ideal.

Theorem 4. *Let A be an A^*-algebra with an auxiliary norm $|x|$ satisfying (α) or (β) of the preceding Theorem 3. Let B be a Banach $*$-algebra with condition (β_k) and ϕ be an algebraic $*$-homomorphism of A into B. Then ϕ can be uniquely extended to a continuous $*$-homomorphism ϕ' of the completion $\mathfrak{A}$ of A by $|x|$ into B and $\phi'(\mathfrak{A})$ is a closed $*$-subalgebra of B. If ϕ is $*$-isomorphism of A into B, then ϕ' is also a $*$-isomorphism of $\mathfrak{A}$ into B and $\mathfrak{A}$, $\phi'(\mathfrak{A})$ are equivalent.*

Proof. Put $|x|_1=|x|+\|\phi(x)\|$. It is easy to see from the proof of Theorem 3 that $|x|_1$ is an auxiliary norm of A. Since A has a unique auxiliary norm, two norms $|x|$, $|x|_1$ are equivalent, and therefore $\|\phi(x)\|\leq c|x|$ for some constant c. Hence ϕ can be uniquely extended to a continuous $*$-homomorphism ϕ' of $\mathfrak{A}$ into B. Theorem 2 shows that $\phi'(\mathfrak{A})$ is a closed $*$-subalgebra of B. Suppose that ϕ is a $*$-isomorphism of A into B. Let I be any closed ideal of $\mathfrak{A}$ with $I\cap A=0$. From the proof of Theorem 3 we see that $IA=0$. Then Lemma 1 shows that ϕ' is a $*$-isomorphism and $\mathfrak{A}$, $\phi'(\mathfrak{A})$ are equivalent.

Consider a commutative Banach algebra B. Its Gelfand representation $\hat{B}$ is an algebra of complex-valued continuous functions vanishing at ∞ on a locally compact Hausdorff space Ω (=the set of regular maximal ideals of B). Every homomorphism of B onto the complex numbers is continuous with norm ≤ 1, and the correspondence between such a homomorphism and its kernel determines a one-to-one mapping from the set of such homomorphism onto Ω. B is called *regular* provided, for every closed set $F\subset\Omega$ and $p_0\in\Omega-F$, there exists $x\in B$ such that $\hat{x}(F)=0$ and $\hat{x}(p_0)=1$. Rickart [**16**] has shown that if B is a semi-simple regular Banach algebra which is algebraically embedded in a second Banach algebra $\mathfrak{B}$, then every homomorphism ϕ of B into the complex numbers can be extended to a homomorphism of $\mathfrak{B}$ into the complex numbers. Making use of this result we show the following

Theorem 5. *Let A be an A^*-algebra in which every maximal commutative $*$-subalgebra is regular in the above sense. Then A has a unique auxiliary norm.*

PROOF. Let B denote any maximal commutative $*$-subalgebra of A. Let $|x|$, $|x|_1$ be two auxiliary norms satisfying (β_k), (β_{k_1}) respectively and $\mathfrak{B}$, $\mathfrak{B}_1$ be the completions of B by $|x|$, $|x|_1$ respectively. By a theorem of Rickart's [**16**] we may assume that B, $\mathfrak{B}$, $\mathfrak{B}_1$ has the same representation space Ω. Let $\hat{\mathfrak{B}}$, $\hat{\mathfrak{B}}_1$ denote the algebra of continuous functions on Ω. Theorem 2 shows that $\frac{1}{kk_1}|x|\leq|x|_1\leq kk_1|x|$. Now consider any $x\in A$ and let B be a maximal commutative $*$-subalgebra of A which contains x^*x. Then $\frac{1}{kk_1}|x^*x|\leq|x^*x|_1\leq kk_1|x^*x|$ which in turn implies that $\frac{1}{kk_1}|x|\leq|x|_1\leq kk_1|x|$. This completes the proof.

For example, consider a group algebra L_1 of commutative locally compact group. To each element x of L_1 corresponds the operator on L_2-space of this group which is defined as left multiplication by x, where multiplication means the convolution. Thus L_1 is an A^*-algebra. It is known that L_1 is regular. Therefore L_1 has a unique auxiliary norm.

§ 3. w. c. c. B*-algebras

We say that a Banach algebra A is w. c. c. (=weakly completely continuous) provided the right- and left-multiplications by any element of A are weakly completely continuous operators on A. If A is a B^*-algebra, i. e., a Banach $*$-algebra satisfying (β_k), the left or right multiplication is sufficient to define a w. c. c. algebra. Kaplansky [**5**] has shown that the dual B^*-algebra is a $B^*(\infty)$-sum of C^*-algebras, each of which is the Banach $*$-algebra of all completely continuous operators on a Hilbert space. He has also shown [7] that the followings are equivalent for a B^*-algebra A:

(1) A is dual.

(2) A has a faithful $*$-representation by completely continuous operators on a Hilbert space.

(3) The socle of A is dense in A.

Since every C^*-algebra of completely continuous operators on a Hilbert space is w.c.c. [**12**], we are led to a characterization of a dual B^*-algebra as a w. c. c. B^*-algebra. To show this we need the following

LEMMA 5. *Let Ω be a locally compact Hausdorff space and $C(\Omega)$ be the Banach algebra of complex-valued continuous functions vanishing at* ∞ on Ω. *$C(\Omega)$ is w. c. c. if and only if Ω is discrete.*

PROOF. Let G be any relatively compact open subset of Ω. Let $C(G)$ denote the subalgebra of $C(\Omega)$ consisting of the functions vanishing outside G. $C(G)$ is a closed subalgebra of $C(\Omega)$. Let $k\in C(\Omega)$ be a function equal to 1 on G. Then $kx=x$ for every $x\in C(G)$. It follows that if $C(\Omega)$ is w. c. c., then $C(G)$ is locally weakly compact, therefore G is finite [**11**, p 87], so that Ω is discrete. The converse is evident since $C(\Omega)$ becomes c. c. when Ω is discrete. This completes the proof.

By making use of this lemma we can show

THEOREM 6. *The following statements are equivalent for a B^*-algebra A*:

(1) *A is w. c. c.* .

(2) *A is a $B^*(\infty)$-sum of C^*-algebras, each of which consists of the set of all completely continuous operators on a Hilbert space.*

PROOF. (1)→(2). Let B be a maximal commutative *-subalgebra of A. B is necessarily a closed subalgebra of A and is isomorphic with $C(\Omega)$ considered in the preceding lemma. Since B is closed in the w. c. c. algebra A, B is w. c. c. as well. Lemma 5 shows that Ω is discrete. Let e_α be the elements of B corresponding to the characteristic functions of the points $\alpha \in \Omega$. Then $\{e_\alpha\}$ is an orthogonal family of self-adjoint primitive idempotents of B such that, for every $x \in B$, we can write $x = \sum \lambda_\alpha e_\alpha$ where the right hand series is summable in the norm i. e., for any given positive number ε the number of α's such that $|\lambda_\alpha| \geq \varepsilon$ is finite. Conversely if $\sum \lambda_\alpha e_\alpha$ is summable, it clearly represents an element of B. We show that $e_\alpha A$ is a minimal right ideal of A. Let a be any self-adjoint element of A. $e_\alpha a e_\alpha$ is self-adjoint and commutative with every e_β and therefore $e_\alpha a e_\alpha \in B$ so that $e_\alpha a e_\alpha = \lambda e_\alpha$, λ being real. It follows from this $e_\alpha A e_\alpha =$ (the complex field) $\times e_\alpha$. Since A contains no nilpotent ideal, $e_\alpha A$ is a minimal right ideal.

If, for any given $z \in A$, $e_\alpha z = 0$ for every e_α, then $z = 0$. For $e_\alpha z = 0$ implies $e_\alpha z z^* = 0$ and therefore $z z^* e_\alpha = 0$. Since B coincides with its commuter, $z z^* \in B$. Hence $z z^* e_\alpha = 0$ for every e_α implies $z z^* = 0$ so that $z = 0$. Let us consider the directed set of finite sums $e_{\alpha_1} + \cdots + e_{\alpha_n}$ of mutually orthogonal e_α. Let z be any element of A. Since A is w. c. c., the unit sphere of A is transformed by right multiplication by z into a relatively weakly compact subset of A. Let z' be any limitting point (in the weak topology) of a directed set $\{(e_{\alpha_1} + \cdots + e_{\alpha_n})z\}$. It is clear that $e_\alpha z = e_\alpha z'$ so that $z = z'$. This implies that $\{(e_{\alpha_1} + \cdots + e_{\alpha_n})z\}$ converges weakly to z. In like manner we ean show that if, in this discussion, e_α is confined to any subfamily, $\{(e_{\alpha_1} + \cdots + e_{\alpha_n})z\}$ converges weakly to an element of A. By a theorem of Orlicz's [**2**, p. 240] $\sum e_\alpha z$ is summable to z. That is, $e_\alpha z \neq 0$ for only a countable number of e_α's, and $\{(e_{\alpha_1} + \cdots + e_{\alpha_n})z\}$ converges to z in the norm.

For any given closed ideal I of A, $e_\alpha A \cap I = 0$ or $e_\alpha A \subset I$. Indeed, $e_\alpha A \cap I \neq 0$ implies $0 \neq e_\alpha a \in I$ for some $a \in A$ and therefore $e_\alpha a a^* e_\alpha = \lambda e_\alpha$ $(\lambda \neq 0)$ implies $e_\alpha \in I$ so that $e_\alpha A \subset I$. Let $\{e_{\alpha'}\}$ be the set of e_α's with $e_{\alpha'} A \subset I$ and $\{e_{\alpha''}\}$ the rest of e_α's. Denote by H', H'' the closed subspaces of A spanned by $\{e_{\alpha'} A\}$, $\{e_{\alpha''} A\}$ respectively. Then A is a direct sum of H' and H''. Evidently $H' \subset I$. We show that $H' = I$ and $H'' = L(I)$.

Let z be any element of I. $z=\sum e_\alpha z$ where $e_\alpha z \neq 0$ implies $e_\alpha z \in I$. Hence $H'=I$ $e_{\alpha''}AI \subset e_{\alpha''}A \cap I=0$ implies $H' \subset L(I)$. Conversely if z is any element of $L(I)$, then $z^* \in L(I)$ since $L(I)$ is a closed ideal of A and therefore it is self-adjoint. This implies that $z^* e_{\alpha'}=0$ so that $e_{\alpha'}z=0$. Hence $z \in H''$, so that $H''=L(I)$. Thus any closed ideal is a direct summand with the supplementary ideal $L(I)$. Let I be any primitive ideal of A. $L(I)$ is then a primitive algebra [**3**] with minimal right ideals since $L(I)$ is isomorphic with A/I. Take an e_α such that $e_\alpha \in L(I)$. We have $Ae_\alpha A \cdot Ae_\beta A \neq 0$ for any $e_\beta \in L(I)$ since $Ae_\alpha A$ and $Ae_\beta A$ are non zero ideals of $L(I)$. We can take $a \in A$ such that $e_\alpha a e_\beta \neq 0$. Hence $e_\beta a^* e_\alpha a e_\beta = \lambda e_\beta$ $(\lambda \neq 0)$. This shows that $e_\beta \in Ae_\alpha A$ so that $Ae_\alpha A$ is the socle of $L(I)$ and $L(I)$ is the closure of $Ae_\alpha A$. Conversely for any e_α it is easy to see that the closure J_α of $Ae_\alpha A$ is $L(I)$ for a primitive ideal I. Consider the maximal set $\{J_\alpha\}$ chosen in such a way that $J_\alpha \cap J_\beta = 0$ for $\alpha \neq \beta$. The direct sum of J_α's is dense in A. Since J_α is $*$-isomorphic with a C^*-algebra of all completely continuous operators on a Hilbert speace [**5, 14**], it is easy to see that A is $*$-isomorphic with a $B^*(\infty)$-sum of C^*-algebras stated in (2).

(2)$\rightarrow$(1). This is evident from the fact that a C^*-algebra of completely continuous operators on a Hilbert space is w. c. c. [**12**].

Corollary 1. *Let A be a w. c. c. B^*-algebra. Let $\{e_\alpha\}$ be an orthogonal family of self-adjoint idempotents of A. Then $\sum e_\alpha z$ is summable in the norm for every $z \in A$.*

Proof. Consider a maximal commutative $*$-subalgebra B containing $\{e_\alpha\}$. It is then clear from the proof of Theorem 6 that each e_α is a sum of finite number of primitive idempotents of B, and therefore $\sum e_\alpha z$ is summable in the norm.

Corollary 2. *Let A be a w.c.c. B^*-algebra. Let e denote a self-adjoint idempotent of A. Then eAe is $*$-isomorphic with a direct sum of full matrix algebras of finite orders over the complex field.*

Proof. eAe is a w. c. c. B^*-algebra with a unit e. Hence eAe is weakly complete. This implies [**12**] that the conclusion of this lemma is valid.

In this corollary if e is primitive in the center of A, eAe is $*$-isomorphic with a full matrix algebra of finite order over the complex field.

w. c. c. B^*-algebras contains c. c. B^*-algebras as a special case. The C^*-algebra of completely continuous operators on a Hilbert space $\mathfrak{H}$ contains a non trivial c. c. element if and only if $\mathfrak{H}$ is finite-dimensional [**12**]. From the proof of Theorem 6 we see that every closed ideal of a w. c. c. B^*-algebra A is the closure of the union of minimal closed ideals J contained in it. On account of Corollary 2 it is clear that the following conditions for J are equivalent:

(1) J has a unit.

(2) J is a full matrix algebra of finite order over the complex field.

(3) J is finite-dimensional.

(4) J contains a non-zero central element.

(5) J is c. c. .

A Banach algebra is said *strongly semi-simple* [**18**] provided the intersection of regular maximal ideals is the zero element. Among w. c. c. B^*-algebras c. c. B^*-algebras are characterized by various algebraic properties as stated in the following

THEOREM 7. *Let A be a w. c. c. B^*-algebra. A is c. c. if and only if any of the following conditions is satisfied:*

(1) *A is a $B^*(\infty)$-sum of finite-dimensional B^*-algebras.*

(2) *A is strongly semi-simple.*

(3) *The annihilator of the center of A is* 0.

PROOF. Proof is omitted since it is clear from the above discussion.

We remark that a w. c. c. B^*-algebra is finite-dimensional if and only if it is regular in the sense of J. v. Neumann [**10**]. We recall the definition: A ring is *regular* if every element x has a relative inverse x' such that $xx'x=x$. This is a special case of a result obtained by Kaplansky [**6**] to the effect that a Banach algebra is finite-dimensional provided it is regular.

It follows from a theorem due to Kaplansky [**7**] concerning the structure of dual B^*-algebras that a B^*-algebra A is dual if and only if it is w. c. c.. We shall give here a direct proof that a w. c. c. B^*-algebra is dual. To this end we need a

LEMMA 6. *Let I be a closed right ideal of a w. c. c. B^*-algebra A and $\{e_{\alpha'}\}$ be a maximal family of orthogonal self-adjoint primitive idempotents contained in I. Then for every $z\in I$ we have $z=\sum e_{\alpha'}z$.*

PROOF. Put $z'=\sum e_{\alpha'}z$, where the right hand series is summable by Corollary 1 to Theorem 6. $e_{\alpha'}(z-z')=0$ and therefore $e_{\alpha'}(z-z')(z-z')^*=0$. The closed subalgebra B generated by $(z-z')(z-z')^*$ and $\{e_{\alpha'}\}$ is a closed commutative $*$-subalgebra contained in I and therefore it is generated by self-adjoint idempotents by Lemma 5 since it is w. c. c. . $\{e_{\alpha'}\}$ being a maximal family of orthogonal self-adjoint primitive idempotents contained in B, we have $(z-z')(z-z')^*=0$ so that $z=z'$, completing the proof.

Let $\{e_{\alpha''}\}$ be a maximal family of orthogonal self-adjoint primitive idempotents orthogonal to $\{e_{\alpha'}\}$ in the above lemma. It is clear that $\{e_{\alpha'}\}$ together with $\{e_{\alpha''}\}$ becomes a maximal family of orthogonal self-adjoint primitive idempotents in A. It follows from Lemma 6 and the proof of Theorem 6 that $L(I)$ is spanned by $\{Ae_{\alpha''}\}$ and $R(L(I))$ is spanned by $\{e_{\alpha'}A\}$, that is, $R(L(I))=I$. This shows that A is dual. The converse will be treated in the next §.

We remark also that it follows from Lemma 6 by the translation by duality that every closed right ideal of a dual B^*-algebra is the intersection of regular maximal right ideals containing it [**7**].

Our discussion hitherto given in this § will be applied with slight modifications to Banach $*$-algebras satisfying (β_k) as well. Hence we have

THEOREM 8. *The following statements are equivalent for a Banach $*$-algebra A satisfying (β_k):*

(α) A is w. c. c..

(β) A is equivalent to a $B^*(\infty)$-sum of C^*-algebra each of which is the set of all completely continuous operators on a Hilbert space.

(γ) A is dual.

As a closed subalgebra of a w. c. c. algebra is w. c. c., we have the following generalization of a result due to Kaplansky [**5**].

COROLLARY. *A closed $*$-subalgebra of a dual Banach $*$-algebra satisfying (β_k) is dual.*

§ 4. c. c. Banach $*$-algebras

First we shall be concerned with the conditions under which a c. c. Banach $*$-algebra becomes an A^*-algebra.

THEOREM 9. *The following statements are equivalent for a c.c. Banach $*$-algebra A.*

(1) *A is semi-simple and $x^*x=0$ implies $x=0$.*

(2) *A is a dense subalgebra of a c.c. B^*-algebra $\mathfrak{A}$.*

(3) *A is an A^*-algebra.*

PROOF. (1)$\rightarrow$(2). First we show that A is symmetric [**5**], that is, xx^* has a quasi-inverse for every $x \in A$. To this end suppose that the contrary holds for some $x \in A$. -1 is then a proper value of the c. c. operator defined by the left multiplication by xx^*, and its proper space, being finite-dimensional [**2**, p. 160], contains a minimal right ideal R. Non existence of nilpotent ideals in A allows us to write $R=eA$ [**14**], where e is a self-adjoint primitive idempotent. As $xx^*e=-e$, xx^* commutes with e. Since $eAe=$(the complex field)$\times e$ [**14**], and exe, ex^*e are mutually conjugate complex multiples of e, the identities $-e=zz^*+exex^*e$, $z=ex-exe$ show that $zz^*=-\kappa^2e$ for some $\kappa>0$. This turns out to $(z+\kappa e)(z+\kappa e)^*=0$, and therefore $z=-\kappa e$ and $zz^*=\kappa^2 e$, a contradiction. Thus the symmetry of A is assured. Let M be a primitive ideal. It is self-adjoint, since A is symmetric [**5**]. Moreover A is a direct sum of M and a full matrix algebra B of finite order over the complex field, which is a self-adjoint simple ideal of A. Denote the B-component of x by x_B and its factor space norm by $\|x_B\|_{A/M}$. Since B is $*$-isomorphic with a C^*-algebra with C^*-norm $|x_B|$, the $*$-homomorphism $x\to x_B$ is continuous from A onto B with norm $|x_B|$ [15]. We are now to show that the involution $x\to x^*$ is continuous. Assume that $\|x_n-x\|\to 0$ and $\|x_n^*-y\|\to 0$. Put $x_n=b_n+m_n$, $x=b+m$ and $y=b'+m'$, where b_n, b, $b'\in B$ and m_n, m, $m'\in M$. $b_n\to b$, $b_n^*\to b'$ by continuity of $x\to x_B$, and therefore $b^*=b'$ and $y-x^*\in M$. This being true for every primitive M, we have $y=x^*$ since A is semi-simple. It follows from the closed graph theorem [2, p. 41] that $x\to x^*$ is continuous. Therefore we may assume that $\|x\|=\|x^*\|$, since, if necessary, we may take an equivalent norm $\|x\|+\|x^*\|$ instead of $\|x\|$. Put $\|x_B\|_l=\underset{\|y_B\|=1}{\text{l. u. b.}}\|x_By_B\|$, $\|x_B\|_r=\underset{\|y_B\|=1}{\text{l. u. b.}}\|y_Bx_B\|$ and $|||x_B|||=\|x_B\|_l+\|x_B\|_r$; From the minimal character of the usual norm in an algebra $C(\Omega)$ [5], we have $|||x_B|||^2\geq|||x_B^*x_B|||\geq|x_B^*x_B|=|x_B|^2$. Suppose that there exists an infinite number of B's, such that $|x_B|>\varepsilon$ and, a fortiori, $|||x_B|||>\varepsilon$, ε being a given positive number. Then we may assume that $\|x_{B_n}\|_l>\frac{\varepsilon}{2}$, $n=1, 2, 3, \cdots$. Choose $y_n\in B_n$ such that $\|xy_n\|=\|x_{B_n}y_n\|>\frac{\varepsilon}{2}$, $\|y_n\|=1$. Since A is c. c., there exists a subsequence $\{xy_{n'}\}$ converging to an element $x'\in A$ and $\|x'\|\geq\frac{\varepsilon}{2}$. But $xy_ny_B=0$ for $B_n\neq B$ since B's are simple ideals. This implies $x'y_B=0$. Hence $x'_Bx'^*_B=0$, and therefore $x'_B=0$ for every B, that is, $x'\in M$ for every M. This implies $x'=0$, a contradiction. It follows that $\{x_B\}$ is an element of the $B^*(\infty)$-sum $\mathfrak{A}$ of B^*-algebras B with norm $|x_B|$. A is mapped by $x\to\{x_B\}$ into a dense subalgebra of the B^*-algebra $\mathfrak{A}$ which is c. c.. This completes the proof of (1)→(2).

(2)→(3) is obvious since the B^*-norm of $\mathfrak{A}$ serves as an auxiliary norm of A.

(3)→(1) follows from a theorem of Rickart's [15].

Thus we have proved the theorem.

If A is a c. c. B^*-algebra (Banach $*$-algebra satisfying (β_k)), then the map $x\to\{x_B\}$ becomes automatically isometric (bicontinuous) from A onto $\mathfrak{A}$. Hence we have incidentally a theorem due to Kaplansky [5] concerning the structure of c. c. B^*-algebras.

Corollary. *A c. c. A^*-algebra has a unique auxiliary norm.*

Proof. Let A be a *c. c.* A^*-algebra. Let B denote any maximal commutative $*$-subalgebra of A. The structure space Ω after the manner of Gelfand is discrete [5] and it is clear from our proof of Theorem 6 that for any point $p_0\in\Omega$ the characteristic function of a single point set $\{p_0\}$ is a Gelfand representation of a self-adjoint idempotent of B. Hence B is regular in the sense described in § 2. Theorem 5 shows that A has a unique auxiliary norm.

Next we consider the conditions under which a c. c. A^*-algebra becomes dual.

THEOREM 10. *Let A be a c. c. A^*-algebra. A is dual if and only if the socle is dense in A and, for every $x \in A$, the closure of xA contains x.*

PROOF. If A is dual, the socle of A is dense in A and the closure of xA contains x [4]. Suppose that the converse holds. It follows from the proof of the preceding Theorem 9 that the socle of A is the direct sum of dual simple algebras B, and therefore dual in relative topology induced by A [4]. Moreover it is clearly a dense ideal of A. Hence by a theorem due to Kaplansky [4] we see that A is dual. Thus the proof is completed.

We remarked in the preceding § that every closed right ideal of a dual B^*-algebra is the intersection of regular maximal right ideals containing it. We shall show that this property holds also for a c. c. dual A^*-algebra.

THEOREM 11. *Let A be a c. c. dual A^*-algebra. Then every closed right ideal of A is the intersection of regular maximal right ideals of A containing it.*

PROOF. A is a dense subalgebra of a c. c. B^*-algebra $\mathfrak{A}$ as stated in the proof of Theorem 9. Denote by $\bar{E}$ the closure of a subset $E \subset \mathfrak{A}$. Let M be any regular maximal right ideal of A. We show that $\bar{M}$ is a regular maximal right ideal of $\mathfrak{A}$. $L_A(M)$, the left annihilator of M in A, is a closed minimal left ideal generated by a self-adjoint primitive idempotent $e \in A$, and $M = \{x;\ ex = 0,\ x \in A\}$. It is clear that $\bar{M} = \{z;\ ez = 0, z \in \mathfrak{A}\}$ and $\mathfrak{A}e$ is a minimal left ideal. Therefore $\bar{M}$ is a regular maximal right ideal of $\mathfrak{A}$. Conversely if $\mathfrak{M}$ is a regular maximal right ideal of $\mathfrak{A}$ and e is a self-adjoint primitive idempotent generating $L(\mathfrak{M})$, then clearly $e \in A$ and $\mathfrak{M} \cap A = \{x\,;\ ex = 0, x \in A\}$. Since e is a self-adjoint primitive idempotent of A, $\mathfrak{M} \cap A$ is a regular maximal right ideal of A. Let N be any closed right ideal of A. $N = R_A(L_A(N)) = R_A\{L(N) \cap A\} = R\{L(N) \cap A\} \cap A \supset R(L(N)) \cap A = \bar{N} \cap A$. Hence $N = \bar{N} \cap A$. As remarked above, $\bar{N}$ is the intersection of regular maximal right ideals $\mathfrak{M}$ of $\mathfrak{A}$ containing $\bar{N}$. Therefore $N = (\cap \mathfrak{M}) \cap A = \cap (\mathfrak{M} \cap A)$, that is, N is the intersection of regular maximal right ideals of A containing N. This completes the proof.

We remark here that if A is a c. c. dual A^*-algebra and A is dense in a B^*-algebra $\mathfrak{A}$, then the correspondence $N \rightarrow \bar{N}$ establishes a one-to-one mapping between the families of closed right ideals of A and $\mathfrak{A}$.

THEOREM 12. *Let A be a c. c. A^*-algebra. A is finite-dimensional if and only if A has a unit.*

PROOF. It is well-known that if A is finite-dimensional, then A has a unit since it is semi-simple. Conversely suppose that A has a unit e. A c. c. B^*-algebra $\mathfrak{A}$ in which A is dense has e as a unit, and therefore $\mathfrak{A}$ is finite-dimensional. This implies that A is finite-dimensional, completing the proof.

§ 5. Dual A*-algebras

The following theorem will play an important rôle in our further study of dual A^*-algebras.

THEOREM 13. *Let A be a semi-simple dual Banach $*$-algebra in which $x^*x=0$ implies $x=0$.* If *A satisfies any of the following conditions:*

(1) *Every primitive ideal of A is a direct summand.*

(2) *$x \to x^*$ is continuous.*

Then A is an A^*-algebra and a dense subalgebra of a dual B^*-algebra $\mathfrak{A}$ which is uniquely determined up to $*$-isomorphism.

PROOF. Every minimal right ideal of A is generated by a unique self-adjoint primitive idempotent of A. Using a result of Kaplansky [**4**] concerning the structure of semi-simple dual ring, we see that A is the closure of its socle which is a direct sum of simple dual ideals S_α of the form $Ae_\alpha A$ where e_α is a self-adjoint primitive idempoent. Suppose first that (2) holds. The closure $\bar{S}_\alpha$ and its left annihilator M_α are closed self-adjoint ideals and M_α is a primitive ideal of A. Since $\bar{S}_\alpha + M_\alpha$ is dense in A [**6**], so is the image of S_α in A/M_α. Let $[x]_\alpha$ stand for the coset $x+M_\alpha$ and $\|[x]_\alpha\|$ the factor space norm. We can introduce in $[e_\alpha A]_\alpha$ an inner product $([x]_\alpha, [y]_\alpha)$ by the relation $([x]_\alpha, [y]_\alpha)[e_\alpha]_\alpha = [e_\alpha x y^* e_\alpha]_\alpha$ [**14**]. The operator $T_{[z]_\alpha}$: $[x]_\alpha \to [xz]_\alpha$; $[x]_\alpha \in [e_\alpha A]_\alpha$ is continuous in the norm defined by the above inner product [**14**]. Denote by $|[z]_\alpha|$ the operator norm $\|T_{[z]_\alpha}\|$. We may assume that $\|x\| = \|x^*\|$. A/M_α may be considered as an A^*-algebra with an auxiliary norm $|[x]_\alpha|$, and therefore $\|x\|^2 \geq \|x^*x\| \geq \|[x^*x]_\alpha\| \geq |[x^*x]_\alpha| = |[x]_\alpha|^2$. It is easily seen that we may regard A/M_α as a dense subalgebra of the C^*-algebra K_α of all completely continuous operators on the Hilbert space obtained by completing $[e_\alpha A]_\alpha$. Consider the $B^*(\infty)$-sum $\mathfrak{A}$ of all K_α's. $x \to [x]_\alpha$ is continuous and the image of the socle of A by this mapping is dense in $\mathfrak{A}$. Since $x \to \{[x]_\alpha\}$ is a $*$-isomorphism into, we may consider A as a dense subalgebra of $\mathfrak{A}$. Theorem 3 shows that $\mathfrak{A}$ is uniquely determined up to $*$-isomorphism.

Next we turn to the case (1). Then $A = \bar{S}_\alpha + M$ (direct sum), and M_α is self-adjoint since $M_\alpha = L(S_\alpha) = R(S_\alpha)$, S_α being self-adjoint. Similarly $\bar{S}_\alpha$ is self-adjoint. By the same argument as in the proof of Theorem 9, we can show that $x \to x^*$ is continuous, and therefore A satisfies the statement of the theorem to be proved.

In an A^*-algebra any $*$-subalgebra is semi-simple, $x \to x^*$ is continuous and $x^*x=0$ implies $x=0$ [**15**]. This together with Theorem 13 gives

THEOREM 14. *Any dual A^*-algebra has a unique auxiliary morm and is a dense subalgebra of a dual B^*-algebra which is uniquely determined up to $*$-isomorphism.*

Next we consider the conditions under which a dual A^*-algebra becomes c. c. To this end we need a lemma.

LEMMA 7. *Any dual A^*-algebra with a unit is finite-dimensional and a direct sum of full matrix algebra over the complex field.*

PROOF. We can prove this lemma in the same manner as in the proof of Theorem 12.

THEOREM 15. *Let A be a dual A^*-algebra and let $\mathfrak{A}$ be a B^*-algebra in which A is dense. Then the following conditions are equivalent:*

(1) *A is c. c..*

(2) *There exists in A a family of orthogonal self-adjoint central idempotents $\{e_\alpha\}$ such that no non zero elements are orthogonal to $\{e_\alpha\}$.*

(3) *$\mathfrak{A}$ is c. c..*

(4) *A is strongly semi-simple.*

PROOF. (1)→(2) follows from the proof of Theorem 9.

(2)→(3). A is generated by $\{e_\alpha\}$. If $z\in\mathfrak{A}$ is orthogonal to $\{e_\alpha\}$, then z is orthogonal to A and therefore to $\mathfrak{A}$. This implies $zz^*=0$, thet is, $z=0$. Hence Theorem 7 shows that (2) implies (3).

(3)→(4). Let M be any primitive ideal. From the proof of Theorem 13 we see that a completion of A/M by a certain auxiliary norm is a B^*-algebra and a closed simple ideal of $\mathfrak{A}$. Since $\mathfrak{A}$ is c. c., this completion has a unit and therefore A/M is finite-dimensional and has a unit. Therefore A/M is simple. Consequently M is a regular maximal ideal. Since this is true for every primitive ideal, A is strongly semi-simple.

(4)→(1). Let M denote any regular maximal ideal and S its left annihilator. Since A/M is simple and has a unit, S is isomorphic with A/M. Therefore S has a unit e. It is easy to see that e is a self-adjoint central idempotent. Lemma 7 shows that $S=Ae$ is finite-dimensional and therefore c. c.. The closure of the union of all such S coincides with A since A is strongly semi-simple. Clearly every element of S is c. c. in A. Hence A is c. c.. Thus the theorem is completely proved.

If A is a commutative A^*-algebra, then a dual B^*-algebra $\mathfrak{A}$ is commutative, and therefore c. c. by Theorem 7. This together with Theorem 15 gives

COROLLARY. *Any commutative dual A^*-algebra is c. c..*

§ 6. w. c. c. A*-algebras

We start with

LEMMA 8. *Let B be a Banach $*$-algebra which is a dense ideal in a dual*

A^*-algebra A. Let I denote any closed right ideal of B and $\bar{I}$ its closure in A. Then

(1) *$\bar{I}\cap B=R_B(L_B(I))$, the right annihilator of the left annihilator of I in B.*

(2) *$\bar{I}B\subset I$.*

(3) *B is dual if and only if, for every $x\in B$, the closure of xB in B contains x.*

PROOF. (1) $L(\bar{I})=L(I)\supset L_B(I)=L(\bar{I})\cap B\supset BL(\bar{I})$. Since $x\in\overline{Ax}$ for every $x\in A$ [**4**], it follows that $\overline{BL(\bar{I})}=\overline{AL(\bar{I})}=L(\bar{I})$, and therefore $\overline{L_B(I)}=L(\bar{I})$. This implies that $R_B(L_B(I))=R(\overline{L_B(I)})\cap B=R(L(\bar{I}))\cap B=\bar{I}\cap B$ since A is dual [**4**].

(2) Let z be any element of $\bar{I}$. Choose a sequence $\{z_n\}$ from I in such a way that $\|z_n-z\|_A\to 0$. By Lemma 4, $\|z_nb-zb\|_B\leq c\|z_n-z\|_A\|b\|_B$ for every $b\in B$, where $\|\ \|_A$ and $\|\ \|_B$ stand for norms in A and B respectively. Hence $\|z_nb-zb\|_B\to 0$, that is, $zb\in I$.

(3) If B is dual, then the closure of xB in B contains x [**4**]. Therefore it is sufficient to prove the converse. Let I be any colsed right ideal of B. Take any $x\in\bar{I}\cap B$. Since the closure of xB in B contains x and I is closed, it follows from (2) that $x\in I$, that is, $\bar{I}\cap B\supset I$, and therefore $\bar{I}\cap B=I$. (1) implies thet $I=R_B(L_B(I))$. By means of the $*$-involution we see that, for every closed left ideal J, we obtain $J=L_B(R_B(J))$. Therefore B is dual. The proof is completed.

LEMMA 9. *Let B be a Banach $*$-algebra which is a dense ideal in an A^*-algebra A. Then*

(1) *A is w. c. c. if B is w. c. c. and A^2 is dense in A.*

(2) *B is w. c. c. if A is w. c. c. and B^2 is dense in B.*

PROOF. (1) Let b, b' be any element of B. Let $\{x_n\}$ be any sequence from A such that $\|x_n\|_A=1$ $(n=1, 2, \cdots)$. By Lemma 4, $\{b'x_n\}$ is bounded in B, and therefore there exists a subsequence $\{x_{n'}\}$ such that $\{bb'x_{n'}\}$ converges weakly to an element in B. Since the mapping $x\to x$ from B into A is continuous [**15**], $\{bb'x_{n'}\}$ converges weakly to an element in A, that is, bb' is a w. c. c. element of A. It is easy to see that B^2 is dense in A. This completes the proof of (1) since the set of w. c. c. elements is closed [**12**].

(2) Let b, b' be any elements of B. Let $\{x_n\}$ be any bounded sequence from B, which is also bounded in A [**15**]. There exists a subsequence $\{x_{n'}\}$ such that $\{b'x_{n'}\}$ converges weakly to an element z of A in A. Let ϕ be any continuous linear functional on B. If we put $\psi(x)=\phi(bx)$, $x\in A$, then ψ is a continuous linear function on A. Since B is an ideal of A, $bz\in B$. It follows from $\phi(bb'x_{n'}-bz)=\psi(b'x_{n'}-z)\to 0$ that bb' is a w. c. c. element of B. But B^2 is dense in B. Therefore B is w. c. c., completing the proof.

It is to be noted that the lemma is valid with c. c. instead of w. c. c. in the statement of the lemma.

LEMMA 10. *If A is a Banach $*$-algebra satisfying (β_k), then $A=A^2$.*

PROOF. Let x be a self-adjoint element with non-negative spectrum. Using the Gelfand representation we can find $y\in A$ such that $x=y^2$, that is, $x\in A^2$. Since any element of A is a linear combination of such x's, we have our conclusion, completing the proof.

Now we shall show

THEOREM 16. *Let A be a w. c. c. A^*-algebra with an auxiliary norm $|x|$ such that $\|xy\|\leq c|x|\,\|y\|$ for every $x,y\in A$, where c is a constant. Let one of the following conditions be satisfied:*

(1) *The closure of xA contains x for every $x\in A$.*

(2) *A is reflexive.*

Then A is dual and is a dense ideal of the completion $\mathfrak{A}$ of A by $|x|$. $\mathfrak{A}$ is equivalent to a w. c. c. B^*-algebra. For any family of orthogonal self-adjoint idempotents $\{e_\alpha\}$ of A, $\sum e_\alpha x$ is summable in the norm of A, and especially when $\{e_\alpha\}$ is a maximal family, $x=\sum e_\alpha x$ holds for every $x\in A$.

PROOF. $\mathfrak{A}$ is a Banach $*$-algebra satisfying (β_k) and therefore, by Lemma 10, $\mathfrak{A}=\mathfrak{A}^2$. Lemma 9 shows that $\mathfrak{A}$ is w. c. c.. It follows from Theorem 8 that we may assume that $\mathfrak{A}$ is a B^*-algebra with norm $|x|$. Let (1) be satisfied. A is dual by Lemma 8. We shall show the summability of $\sum e_\alpha x$ for every $x\in A$. $\sum e_\alpha x$ is summable in the norm of $\mathfrak{A}$ by Corollary 1 to Theorem 6. Moreover, when $\{e_\alpha\}$ is maximal in A, it is also maximal in $\mathfrak{A}$ and $x=\sum e_\alpha x$ by the same theorem. Since the mapping $z\rightarrow za$, $a\in A$ being fixed, from $\mathfrak{A}$ into A is continuous, it follows that, for every $y\in A$, $\sum e_\alpha xy$ is summable in the norm of A. Using (1) we can take y_n such that $\|x-xy_n\|<\dfrac{1}{n}$ $(n=1, 2, \cdots)$. Except for a set of at most countable e_{α_k}, $e_\alpha xy_n=0$ $(n=1, 2, \cdots)$. $\|(e_{\alpha_1}+\cdots+e_{\alpha_n})x\|\leq\|(e_{\alpha_1}+\cdots+e_{\alpha_n})xy_n\|+c\|x-xy_n\|$. It is easy to see that $\sum e_{\alpha_k}x$ is summable in the norm of A. This implies that $\sum e_\alpha x$ is summable in the norm of A. Next suppose that (2) is satisfied. It follows from Lemma 8 that A is dual since it is locally weakly compact. Except for a set of at most countable e_{α_k}, $e_\alpha x=0$. $\|(e_{\alpha_1}+\cdots+e_{\alpha_n})x\|\leq c\|x\|$, that is, $\{(e_{\alpha_1}+\cdots+e_{\alpha_n})x\}$ is bounded. Since A is locally weakly compact, $\{(e_{\alpha_1}+\cdots+e_{\alpha_n})x\}$ has a weakly convergent subsequence in A. It is easy to see that $\sum e_\alpha x$ is summable in the norm of A. Therefore $\sum e_\alpha x$ is summable in A. Moreover, if $\{e_\alpha\}$ is maximal, we have $x=\sum e_\alpha x$. Thus the proof is completed.

In a similar manner we have

THEOREM 17. *Let $\mathfrak{A}$ be a w. c. c. B^*-algebra and A be a Banach $*$-algebra which is a dense ideal of $\mathfrak{A}$. Let A satisfy* (1) *or* (2) *of the preceding theorem. Then A is dual and w. c. c.*.

Let A be a dual A^*-algebra with an auxiliary norm $|x|$. Let $\mathfrak{A}$ be the completion of A by $|x|$. If A is an ideal of $\mathfrak{A}$, we say that A is *of the 1st kind.* Otherwise we say that A is *of the 2nd kind.* Now we give a necessary and sufficient condition for a dual A^*-algebra to be of the 1st kind.

THEOREM 18. *For a dual A^*-algebra to be of the 1st kind it is necessary and sufficient that $\|x\|_1=\underset{\|y\|=1}{\text{l. u. b.}}\|xy\|$ satisfies (β_k).*

PROOF. Let A be a dual A^*-algebra of the 1st kind. A is a dense ideal of a B^*-algebra $\mathfrak{A}$ with norm $|x|$. Lemma 4 shows that $\|xy\|\leq c|x|\ \|y\|$ for every $x,y\in A$ and for some constant c. This implies $\|x\|_1\leq c|x|$. $\|x\|_1$ has the usual norm properties (except for the completeness) and the multiplicative property. Let $\mathfrak{A}_1$ stand for the completion of A by $\|x\|_1$. The mapping $x\rightarrow x$ from A with norm $\|x\|$ onto A with norm $\|x\|_1$ can be extended to a continuous homomorphism ϕ of $\mathfrak{A}$ into $\mathfrak{A}_1$. The kernel J of ϕ is a closed ideal of $\mathfrak{A}$ with the property $J\frown A=0$. From the proof of Lemma 1 J must be a zero ideal, that is, ϕ is an isomorphism. Using the minimal character of the usual norm in the algebra $C(\Omega)$, we have $|x|^2=|x^*x|\leq\|x^*x\|_1\leq\|x^*\|_1\|x\|_1\leq c|x|\ \|x\|_1$. Thus two norms $|x|$ and $\|x\|_1$ are equivalent, and therefore $\|x\|_1$ satisfies (β_k). Conversely let us suppose that $\|x\|_1$ satisfies (β_k). A is a dual A^*-algebra with an auxiliary norm $\|x\|_1$. This implies that A is a dual A^*-algebra of the 1st kind. Thus the proof is completed.

Let A be a proper H^*-algebra as indicated in §**2.** A is reflexive and an A^*-algebra with an auxiliary norm $|x|=\underset{\|y\|=1}{\text{l. u. b.}}\|xy\|$. Hence A is a dual A^*-algebra of the 1st kind.

Any closed $*$-subalgebra of a dual B^*-algebra is dual. We shall show that the same is true for a dual A^*-algebra of the 1st kind. Namely

THEOREM 19. *Let A be a dual A^*-algebra of the 1st kind. Then any closed $*$-subalgebra B of A is a dual A^*-algebra of the 1st kind.*

PROOF. Let $\mathfrak{A}$ be a dual B^*-algebra with norm $|x|$ of which A is a dense ideal. Let $\mathfrak{B}$ denote the closure of B in $\mathfrak{A}$. $\mathfrak{B}$ is a closed $*$-subalgebra of $\mathfrak{A}$, and therefore a dual B^*-algebra. Using the inequality $\|xy\|\leq c|x|\ \|y\|$ it is easy to see that B is a dense ideal of $\mathfrak{B}$. The proof is completed if we show that the closure of xB in B contains x for every $x\in B$ (Lemma 8). Let $\{e_\alpha\}$ be a maximal family of orthogonal self-adjoint idempotents e_α in $\mathfrak{B}$. Since B is a dense ideal of $\mathfrak{B}$, we can easily see that $e_\alpha\in B$. By Theorem 16 $\sum xe_\alpha$ is summable in A for every $x\in B$, and therefore in B.

Since $\{e_\alpha\}$ is maximal in $\mathfrak{B}$, it follows that $x=\sum xe_\alpha$, and therefore the closure of xB in B contains x, completing the proof.

Corollary. *Let A be an A^*-algebra which is a dense ideal of a dual A^*-algebra A_1 of the 1st kind. A closed $*$-subalgebra B of A is dual if the closure of xB contains x for every $x\in B$.*

Proof. Theorem 19 shows that the closure B_1 of B in A_1 is dual. B is a dense ideal of B_1. It follows from Lemma 8 that B is dual.

§ 7. Some properties of dual A*-algebras

Let G be a compact group and $L_p(1\leq p<\infty)$ be the algebra of complex-valued measurable functions on G, whose p^{th} powers are integrable with respect to the Haar measure on G, and with convolution as multiplication. We shall also write C for the algebra of continuous functions on G, normed by the maximum of the absolute value. C is a dense ideal of L_2 and is c. c.. L_2 is dual since it is a proper H^*-algebra. It follows from Lemma 8 (3) that C is dual. Moreover C is of the 2nd kind unless G is finite. For, $\|x\|_1=\underset{\|y\|=1}{\text{l. u. b.}}\ \underset{g\in G}{\text{l. u. b.}}\ \left|\int x(gh^{-1})y(h)dh\right|=\underset{\|y\|=1}{\text{l. u. b.}}\ \left|\int x(h)y(h)dh\right|=\int|x(h)|dh$, so that the completion of C by the norm $\|x\|_1$ is L_1. L_1 is weakly complete. Hence L_1 satisfies (β_k) if and only if L_1 is finite-dimensional [**12**], or equivalently G is finite. In case of an infinite G we can even prove that L_1 is of the 2nd kind. To this end a notion due to Segal [**17**] is useful: an approximate identity of a normed algebra A is a Moore-Smith directed system $\{u_\alpha\}$ of elements of A such that $\|u_\alpha\|\leq 1$ and $\lim_\alpha xu_\alpha=x$ for every $x\in A$.

Lemma 11. *Let B be a dual Banach $*$-algebra which is a dense ideal of an A^*-algebra A. If B has an approximate identity $\{u_\alpha\}$, then A is dual.*

Proof. Let a be any element of A. Consider $b\in B$ such that $\|a-b\|_A<\varepsilon$. Then $\|a-au_\alpha\|_A\leq\|a-b\|_A+\|b-u_\alpha\|+\|bu_\alpha-au_\alpha\|\leq\varepsilon+c\|b-bu_\alpha\|+c'\|b-a\|_A\|u_\alpha\|_B$, where c, c' are constants. This follows from the continuity of the mapping $x\to x$ from B into A [**15**], and the assumption that B is a dense ideal of A (Lemma 4). $\overline{\lim_\alpha}\|a-au_\alpha\|\leq\varepsilon(1+c')$ and therefore $\lim_\alpha\|a-au_\alpha\|=0$. This implies that $a\in\overline{aA}$ for every $a\in A$. It follows from a theorem of Kaplansky [**4**] that A is dual, since B is, a fortiori, still dual in the relative topology on B induced by A.

Lemma 12. *Let A be a dual A^*-algebra with an approximate identity and B a Banach $*$-algebra which is a dense ideal of A. Then, if B is reflexive, B is dual.*

Proof. B is locally weakly compact, and for any $x\in B$ $\{xu_\alpha\}$ is bounded in B (Lemma 4). Hence $\{xu_\alpha\}$ contains a cofinal set converging weakly to an element $z\in B$.

Since $\|x-xu_\alpha\|_A\to 0$, and every continuous linear functional on A is continuous on B [**15**], we have $z=x$. Hence x is contained in the closure of xB. Lemma 8 shows that B is dual, completing the proof.

By making use of these lemmas we shall show the following

THEOREM 20. *Let A be a dual A^*-algebra of the 2^{nd} kind with norm $\|x\|$. Put $\|x\|_1=\underset{\|y\|=1}{\text{l. u. b.}}\|xy\|$ for every $x\in A$. If $\|x^*\|_1=\|x\|_1$ and A has an approximate identity $\{u_\alpha\}$ with respect to the new norm $\|x\|_1$, then the completion A_1 of A by the norm $\|x\|_1$ is a dual A^*-algebra of the 2^{nd} kind. Moreover;*

(1) *Any dual A^*-algebra with A as a dense ideal is considered as a subalgebra of A_1.*

(2) *There exists no A^*-algebra with A_1 as a dense proper ideal.*

(3) *Any reflexive Banach $*$-algebra which is a dense ideal of A_1 is dual.*

PROOF. It follows from the definition of A_1 that $\{u_\alpha\}$ is an approximate identity of A_1 as well. First we show that A_1 is semi-simple, that is, the radical R of A_1 is a zero ideal. For otherwise there would exist a self-adjoint primitive idempotent $e\in A$ such that $Re\neq 0$, since $Re=0$ for every e implies that $RA=0$ and therefore $Ru_\alpha=0$ and $R=0$. Then $eRe=(\text{the complex field})\times e$. This implies that $e\in R$, which is a contradiction since $-e$ has no quasi-inverse. And it is easy to see that $x^*x=0$ implies $x=0$ for every $x\in A_1$. Therefore Theorem 13 shows that A_1 is a dual A^*-algebra with a unique (to within equivalence) auxiliary norm $|x|_1$. $|x|_1$ is also an auxiliary norm of A. Now we show that A_1 is of the 2^{nd} kind. If it were of the 1^{st} kind, then, for every $x, y\in A$, $\|xu_\alpha y\|\leq\|xu_\alpha\|_1\|y\|\leq c|x|_1\|u_\alpha\|_1\|y\|\leq c|x|_1\|y\|$, c being a constant, and therefore $\|xy\|\leq c|x|_1\|y\|$ which implies that A is of the 1^{st} kind. This is a contradiction. Ad(1): Let B be any dual A^*-algebra such that A is a dense ideal of B. Then $\|xy\|_A\leq c\|x\|_B\|y\|_A$ for every $x, y\in A$, c being a constant. This implies that $\|x\|_1\leq c\|x\|_B$. Hence we have a continuous $*$-homomorphism ϕ of B into A_1 as a result of the extension of the mapping $x\to x$, $x\in A$. It is easy to see that the kernel of ϕ is a zero ideal, completing the proof of (1). Ad (2): Let x be any element of A_1. As $\|xu_\alpha\|_1\to\|x\|_1$ for every $x\in A_1$, we have $\|x\|_1=\underset{\|y\|_1=1}{\text{l. u. b.}}\|xy\|_1$. Let A_2 denote any A^*-algebra with A_1 as a dense ideal. It follows from Lemma 11 that A_2 is dual. Hence case (1) shows that A_2 is mapped $*$-isomorphically onto A_1 in such a way that $x\leftrightarrow x$ for every $x\in A_1$, that is, $A_1=A_2$. This completes the proof of (2). Ad (3): This is a simple corollary of Lemma 12.

As C satisfies the conditions indicated in the preceding Theorem, L_1 is of the 2^{nd} kind unless G is finite. Moreover it is c. c. (Lemma 9). Any element $x\in L_1$ is associated with an operator on L_2 defined as a left multiplication by x. The closure of the algebra of all such operators on L_2 forms a dual B^*-algebra associated with L_1. L_p is dual by (3).

Kaplansky [4] has shown that a Banach $*$-algebra A $(C \subset A \subset L)$ with certain properties (such as to assure that A is an ideal of L and C is dense in A) is dual. This follows also from

Theorem 21. *Let B be a dual Banach $*$-algebra which is a dense ideal of a dual A^*-algebra A. Let B have an approximate identity $\{u_\alpha\}$ with bounded $\{\|u_\alpha\|_A\}$ (Here we do not assume that $\{\|u_\alpha\|_B\}$ is bounded.). If A' be a Banach $*$-algebra such that $B \subset A' \subset A$ where B is dense in A' and A' is an ideal of A, then A' is dual.*

Proof. Let x be any element of A'. Choose $b \in B$ such that $\|x-b\|_{A'} < \varepsilon$. Then $\|x - xu_\alpha\|_{A'} \leq \|x-b\|_{A'} + \|b - bu_\alpha\|_{A'} + \|bu_\alpha - xu_\alpha\|_{A'} \leq \varepsilon + c'\|b - bu_\alpha\|_B + c''\|b-x\|_{A'}\|u_\alpha\|_A$ where c', c'' are constants. This inequality shows that $\|x - xu_\alpha\| \to 0$, and therefore, by Lemma 8, A' is dual, completing the proof.

The fundamental theorems on the theory of almost periodic functions on a group [9, p. 47] are read as follows: (1) The algebra is a dual A^*-algebra. (2) Every closed right ideal of the algebra is the closure of the union of minimal right ideals contained in it. But duality translates (2) into the statement (3) Any closed right ideal is the intersection of maximal regular right ideals containing it. In a c. c. dual A^*-algebra the statement (3) is always true (Theorem 11). Therefore various group algebras of a compact group have the property (2).

Theorem 22. *Let A be a dual A^*-algebra with the property (3) indicated above. Any dual Banach $*$-algebra B which is a dense ideal of A has the property (3) as well.*

Proof. Let I be any closed right ideal of B. Let $\bar{I}$ stand for the closure of I in A. From the fact that B is dual we can see as in the proof of Theorem 11 that $I = \bar{I} \cap A$. Since A has the property (3) we can write $I = \bigcap M_\alpha$, where M_α is a regular maximal right ideal of A. It follows from the duality of A that $L(M_\alpha) = Ae_\alpha$, where e_α is a self-adjoint primitive idempotent of A, and $M_\alpha = \{z\,;\ e_\alpha z = 0,\ z \in A\}$. Put $N_\alpha = M_\alpha \cap A$. Since B is a dense ideal of A it is easy to see that $e_\alpha \in B$ and $N_\alpha = \{x;\ e_\alpha x = 0, x \in B$ which is a regular maximal right ideal of B. Hence we obtain $I = \bigcap N_\alpha$. The proof is completed.

We remark that in the proof of Theorem 22 the fact that B is an ideal of A is only used to assure that $e_\alpha \in B$. Lemma 6 shows that a dual B^*-algebra has the property (3) [7]. Therefore by Theorem 22 any dual A^*-algebra of the 1st kind and any dual A^*-algebra contained in it as a dense ideal have the property (3) as well.

As to symmetry we remark that if A is a symmetric Banach $*$-algebra, every ideal of A is symmetric. Using this we see that any dual A^*-algebra of the 1st kind and its dense ideal are symmetric. For example a proper H^*-algebra is symmetric. Moreover from the proof of Theorem 9 every c. c. A^*-algebra is symmetric. Therefore the group algebras considered by Kaplansky [**4**] are symmetric.

§ 8. Commutative case

We conclude this section with a short discussion on a commutative dual A^*-algebra. We saw that any commutative dual A^*-algebra is c. c. (Corollary to Theorem 15). Now we show

THEOREM 23. *Let A be a weakly complete commutative dual A^*-algebra. Let $\{e_\alpha\}$ be a maximal family of orthogonal self-adjoint idempotents of A. Then A is of the 1st kind if and only if, for every $x\in A$, $\sum e_\alpha x$ is summable to x in the norm of A.*

PROOF. It is sufficient to prove "if" part. Let $\mathfrak{A}$ be a commutative dual B^*-algebra in which A is dense. For any $z\in\mathfrak{A}$ we have $z=\sum\mu_\alpha e_\alpha$, where the number of μ_α's such that $|\mu_\alpha|\geq\varepsilon>0$ is finite for every positive number ε. Let ϕ be any continuous linear functional on A. Then $\phi(x)=\sum\lambda_\alpha\phi(e_\alpha)$, where $x=\sum e_\alpha x=\sum\lambda_\alpha e_\alpha$ and $x\in A$. Since $|\phi(zxe_\alpha)|=|\mu_\alpha|\ |\phi(xe_\alpha)|$, $\sum|\phi(zxe_\alpha)|$ is convergent. This implies by a theorem of Orlicz [**2**, p. 240] that $\sum zxe_\alpha$ is summable to an element $u\in A$ in A. On the other hand $\sum zxe_\alpha$ is summable to zx in $\mathfrak{A}$. Hence $u=z.x$. This shows that A is an ideal of $\mathfrak{A}$, which is to be proved.

Consider the compact group of real numbers mod 1 and its group algebras L_p, $p>1$. The functions $e_n=\exp(in\pi t)$ $(n=1, 2, \cdots)$ form a maximal family of orthogonal self-adjoint idempotents of each L_p. By a well-known result of M. Riesz [**19**, p. 153] $\{e_0, e_1, e_{-1}, \cdots\}$ is a basis of L_p provided $p>1$. But S. Karlin [**8**] has shown that if $p\neq 2$, this basis is not unconditional for L_p. Therefore it follows from a result of the preceding § and Theorem 23 that L_p is of the 2nd kind unless $p=2$.

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