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The Spaces of Entire Function of Finite Order

Malyutin KG[1]* and Studenikina IG[2]

[1]*Department of Mathematics, Southwest State University, Russia*
[2]*Department of Mathematics, Sumy State University, Ukraine*

Abstract

This paper is a continuation of the research of the first author. We consider the linear topology space of entire functions of a proximate order and normal type with respect to the proximate order. We obtain the form of continuous linear functional on this space.

Keywords: Entire function; Proximate order; Normal type; Continuous linear functional

Introduction

This paper is a continuation of the research [1] where the linear topology space of entire functions of a proximate order and normal type, less than or equal σ, with respect to the proximate order were considered. We introduce the necessary definitions. A function $\rho(r)$, defined on the ray $(0,\infty)$ and satisfying the Lipschitz condition on any segment $[a, b]\subset(0,\infty)$ that satisfies the conditions

$$\lim_{r\to\infty}\rho(r)=\rho\geq 0, \text{and} \lim_{r\to\infty} r\,\rho'(r)\ln r=0$$

This is called *a proximate order*.

A detailed exposition of the properties of proximate order can be found [2,3]. In this paper we use the notation $V(r)=r^{\rho(r)}$. We will assume that $V(r)$ is an increasing function on $(0,\infty)$ and $\lim\limits_{r\to+0}V(r)=0$.

We now formulate some simple property of proximate order that we shall need frequently [2].

For $r\to\infty$ and $0<a\leq k\leq b<\infty$ asymptotic inequality holds uniformly in k.

$$(1-\varepsilon)k^{\rho}V(r)<V(kr)<(1+\varepsilon)k^{\rho}V(r) \quad (1)$$

Let $M_f(r)=\max\limits_{|z|=r}|f(z)|$. If for the entire function $f(z)$ the quantity

$$\sigma_f=\limsup_{r\to\infty}\frac{\log M_f(r)}{V(r)}$$

Is different from zero and infinity, then $\rho(r)$ is called of *a proximate order of the given entire function f(z) and σf is called the type of the function f(z) with respect to the proximate order* $\rho(r)$.Let $\rho(r)$ be a proximate order, $\lim\limits_{x\to\infty}\rho(r)=\rho\geq 0$. A single valued function $f(z)$ of the complex variable z is said to belong to the space $[\rho(r),¥)$ if $f(z)$ has the order less than $\rho(r)$ or equal $\rho(r)$ but in this case type less than ¥. A sequence of functions $\{f_n(z)\}$ from $[\rho(r),¥)$ converges in the sense of $[\rho(r),¥)$ if

(i) It converges uniformly on compacts, (ii) there exists $\beta<1$ such that

$$|f_n(z)|<C(\beta)\exp[\beta V|z|)],\ |z|>r_0(\beta)(n\geq 1),$$

where $r_0(\beta)$ does not depend on $(n\geq 1)$. For a suitable $C(\beta)$, which does not depend on n, for all z

$$|f_n(z)|<C(\beta)\exp[\beta V|z|)] \quad (n\geq 1) \quad (2)$$

The space $[\rho(r),¥)$ is the linear topology space with sequence topology. Furthermore, a single valued function $f(z)$ of the complex variable z is said to belong to the space $[\rho(r), p]$ if $f(z)$ has the order less than $\rho(r)$ or equal $\rho(r)$ but in this case type less than or equal p. A sequence of functions $\{f_n(z)\}$ from $[\rho(r), p]$ converges in the sense of $[\rho(r), p]$ if (i) it converges uniformly on compacts, (ii) for all $\varepsilon>0$ there exists $r_0(\varepsilon)$ does not depend on n such that

$$|f_n(z)|<\exp[(p+\varepsilon)V|z|)],\ |z|>r_0(\varepsilon)(n\geq 1).$$

The space $[\rho(r), p]$ is also the linear topology space with sequence topology. We introduce the function $\varphi(t)$ defined to be the unique solution of the equation $t=V(r)$. So

$\varphi(V(t))=t$. (3)

Theorem 1.1 ([2, Theorem 2', p.42])

The type σf of the entire function $f(z)=\sum\limits_{n=0}^{\infty}c_nz^n$ with the proximate order $\rho(r)$ ($\rho>0$) is given by the equation

$$\limsup_{n\to\infty}\varphi(n)^n\sqrt[n]{|c_n|}=(e\sigma f\rho)^{1\backslash\rho} \quad (4)$$

Let $\rho>0$

$$d_n=\frac{(e\sigma\rho)^{n\backslash\rho}}{(\varphi(n))^n}\ n\geq 1,\ d_0=1$$

For a function $f(z)=\sum\limits_{n=0}^{\infty}c_nz^n\in[\rho(r),p]$ we associate the function

$$f(z)=\sum_{n=0}^{\infty}b_nz^n,\ b_n=\frac{c_n}{d_n}(n\geq 0) \quad (5)$$

It is regular, in any case in the circle $|z|<1$ [1]. Fact mapping function $f(z)$ of $[\rho(r), p]$ to the function $F(z)$ as indicated above will be celebrating a record $f(z)\sim F(z)$.

In [1] it is proved the following two theorems.

Theorem 1.2

In order to be a sequence $\{f_n(z)\}$ of functions from $[\rho(r), p]$ to converge in the sense of $[\rho(r), p]$ necessary and sufficient that the sequence $\{f_n(z)\}$ $(fn(z)\sim Fn(z))$ converges uniformly inside the disk $|z|<1$.

***Corresponding author:** Malyutin KG, Professor, Department of Mathematics, Southwest State University, Russia, Tel: 800-642-0684; E-mail: malyutinkg@gmail.com

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Theorem 1.3

Continuous linear functional l on the space $[\rho(r), p]$ *has the form*

$$l(f)=\sum_{n=0}^{\infty}a_n c_n,\ f(z)=\sum_{n=0}^{\infty}c_n z^n, \tag{6}$$

Where the quantities an satisfy

$$\limsup_{n\to\infty}\varphi^{-1}(n)\sqrt[n]{|a_n|}=0 \tag{7}$$

The following is our main result.

Theorem 1.4

Continuous linear functional l on the space $[\rho(r)$,¥ *has the form*

$$l(f)=\sum_{n=0}^{\infty}a_n c_n,\ f(z)=\sum_{n=0}^{\infty}c_n z^n, \tag{8}$$

Where the quantities an satisfy

$$\limsup_{n\to\infty}\varphi^{-1}(n)\sqrt[n]{|a_n|}=0 \tag{9}$$

The Space of Entire Functions [ρ(r),¥)

We now prove the theorem 1.4. Let $l(f)$ be a continuous linear functional on the space [$\rho(r)$,¥). Set $l(z^n)=a_n(n\geq 0)$ Let $f(z)=\sum_{n=0}^{\infty}c_n z^n$ be a function in [$\rho(r)$,¥). Since the series converges in the sense of [$\rho(r)$,¥) then, by continuous of the functional,

$$l(f)=\sum_{n=0}^{\infty}c_n l(z^n)=\sum_{n=0}^{\infty}c_n a_n$$

Hence

$$l(f)=\sum_{n=0}^{\infty}a_n c_n,\ f(z)=\sum_{n=0}^{\infty}c_n z^n \tag{10}$$

Take an arbitrary finite $p>0$. Functional $l(f)$ is, in particular, continuous linear functional on the space $[\rho(r), p]$. By theorem 1.3, the condition

$$\limsup_{n\to\infty}\varphi^{-1}(n)\sqrt[n]{|a_n|}<(epp)^{-1/\rho}$$

But p is arbitrary, hence,

$$\limsup_{n\to\infty}\varphi^{-1}(n)\sqrt[n]{|a_n|}=0$$

We now verify that if the condition (9) then the functional (10) is continuous linear functional on the space [$\rho(r)$,¥). Let $f(z)=\sum_{n=0}^{\infty}c_n z^n\in[\rho(r),\infty]$ By theorem 1.1, $\limsup_{n\to\infty}\varphi(n)\sqrt[n]{|c_n|}=(e\sigma f\rho)^{-1/\rho}<\infty$. Then $\limsup_{n\to\infty}\sqrt[n]{|a_n c_n|}=\limsup_{n\to\infty}\varphi(n)\sqrt[n]{|c_n|}\limsup_{n\to\infty}\varphi^{-1}(n)\sqrt[n]{|a_n|}=0$

And then the series (10) converges.

Let $\{f_k(z)=\sum_{n=0}^{\infty}c_n^{(k)}z^n\subset[\rho(r),\infty)\ \ f_k(z)\to f(z)=\sum_{n=0}^{\infty}c_n z^n$ if $k\to$¥ and let l satisfies (9). By (2), there exists $\beta>0$ such that $\{\{f_k(z)\}, f(z)\}$Ì$[r(r), \beta]$ in the sense $[\rho(r), \beta]$. By (9) and (8) l is continuous linear functional on the space $[\rho(r), \beta]$. Then $l(f_k)\to l\ l(f)$ if $k\to$¥. Therefore l is continuous linear functional on the space [$\rho(r)$,¥).

Space of Entire Functions $E_\rho(r)$

We now consider the space of entire functions $E_{\rho(r)}$ which have a proximate orders less then $\rho(r)$. A proximate order $\rho_1(r)$ less then $\rho(r)$ if $0<\lim_{r\to\infty}\rho(r)=\rho_1<\lim_{r\to\infty}\rho(r)=\rho A$ single valued function $f(z)$ of the complex variable z is said to belong to the space $E_{\rho(r)}$ if $f(z)$ has the order less than $\rho(r)$.

A sequence of functions $\{f_n(z)\}$ from $E_{\rho(r)}$ converges in the sense of $E_{\rho(r)}$ if (i) it converges uniformly on compacts, (ii) there exists proximate order $\rho_1(r)$, $0<\lim_{r\to\infty}\rho_1(r)=\rho_1<\lim_{r\to\infty}\rho(r)=\rho$ such that

$$|f_n(z)|<\exp[V_1|z|)],\ |z|>r_0(\beta)(n\geq 1) \tag{11}$$

where $r_0(\beta)$ does not depend on $(n\geq 1)$, $V_1(r)=r^{r1(r)}$.The space $E_{\rho(r)}$ is the linear topology space with sequence topology. A continuous linear functional $l(f)$ on the space $E_{\rho(r)}$ has the form (8). Let us find the conditions that satisfy the values a_n. The functional $l(f)$ is in particular continuous linear functional on the space [$\rho1(r)$,¥) for all proximate order $\rho_1(r)$, $0<\lim_{r\to\infty}\rho_1(r)=\rho_1<\lim_{r\to\infty}\rho(r)=\rho$. Therefore, by theorem

$$\limsup_{n\to\infty}\varphi_1^{-1}(n)\sqrt[n]{|a_n|}=0, \tag{12}$$

where $\varphi_1(t)$ defined to be the unique solution of the equation $t=V_1(r)$. From this

$$\frac{\log|a_n|}{\varphi_1(n)n}<1,\ n>n_0$$

So $\rho_1(r)$ is arbitrary less then $\rho(r)$ that

$$\limsup_{n\to\infty}\frac{\log|a_n|}{\varphi(n)n}\leq 1 \tag{13}$$

Contrary, let the condition (13) is true and $\rho1(r)$ is arbitrary less than $\rho(r)$. So $\varphi_1(n)>\varphi(n)$, $n>n_0$, that

$$\frac{\log|a_n|}{\varphi_1(n)n}<1,\ n>n_0$$

Therefor the condition (12) is true and $l(f)$ is continuous linear functional on the space [$\rho(r)$,¥). So $\rho_1(r)$ is arbitrary less then, $\rho(r)$ that $l(f)$ is continuous linear functional on the space $E_{r(r)}$.

Theorem 3.1

Continuous linear functional l on the space $E_{\rho(r)}$ *has the form*

$$l(f)=\sum_{n=0}^{\infty}a_n c_n,\ f(z)=\sum_{n=0}^{\infty}c_n z^n$$

Where the quantities an satisfy

$$\limsup_{n\to\infty}\frac{\log|a_n|}{\varphi(n)n}\leq 1$$

Remark: The case of the spaces [ρ,¥) and E_r, where $\rho(r)=\rho>0$, considered A.F Leont'ev [4].

Conclusion

The linear topology space of entire functions of a proximate order and normal type with respect to the proximate order is considered. We obtain the form of continuous linear functional on this space through our work.

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