A problem in solid geometry

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§ 1. Introduction.

The problem which we will consider here in this paper has its origin in a talk among our colleagues some years ago. The talk was like this: "Suppose the earth be made of transparent glass and suppose there be a material body contained in it. Assume that the body is seen like a round disc (i.e., the set of all lines of sight to the body is a solid circular cone) from every point on the surface of the earth[1]. Is the body then a ball?[2] We may allow any discontinuous varying of the radius and the center of the disc as the seeing point varies."

In the following, we shall give an affirmative answer to this question in a slightly generalized form.

I thank here my friend Mr. K. M. Rao for his critical reading of the manuscript.

§ 2. Statement of the problem.

Since the statement of the problem in the above is too intuitive, let us restate it in more rigorous terms.

DEFINITION 1. Let p be a point and A be a set in space (Euclidean three space). A straight half line is called *a ray from the point p to the set A* if it starts at p and passes some point of A. We denote by $C_p(A)$ the set of all rays from the point p to the set A and call it *the sight cone at p for A.*

DEFINITION 2. In space, a set A is said to be *equivalent to a ball* if there exists a ball B such that

$$(1) \qquad\qquad B \supseteq A \supseteq \partial B$$

where ∂B denotes the bounding sphere of the ball B.

DEFINITION 3. Let A and B be two sets in space. We say that *A is strictly contained in B* if the condition

$$\bar{A} \subseteq \overset{\circ}{B}$$

holds, where $\bar{A}$ denotes the closure of A and $\overset{\circ}{B}$ the interior of B.

DEFINITION 4. In space, let S be a closed convex surface, i.e., the topological

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1) The surface of the earth is supposed to be a sphere.
2) A ball is, of course, a set of points whose distances from a fixed point are less than or equal to a certain positive number.

boundary of a compact convex set D with non-empty interior. We simply say that a set A *is strictly contained in* S, if A is strictly contained in D.

Using these terms, an affirmative answer to the problem is given, in a generalized form, by the following

THEOREM. *In space, let* S *be a closed convex surface and* A *be a set strictly contained in it. If the sight cone* $C_p(A)$ *at* p *for* A *is always a solid circular cone for any point* p *on* S, *then the set* A *is equivalent to a ball.*

Our problem is to prove the above theorem. We shall prove it as Theorem 5 at the end of the following paragraph. The notion of *independence* defined in Definition 5 below will play the central role.

§3. Solution to the problem.

By the very definition of sight cone, we see the following

PROPOSITION 1. *For any point* p *and for any set* $A, C_p(A)$ *contains* A. *For any two sets* A_1 *and* A_2 *with* $A_1 \subseteq A_2$, $C_p(A_1) \subseteq C_p(A_2)$ *for any point* p.

For this, we see

PROPOSITION 2. *For any set* A *and for any two points* p *and* q, *every ray from* p *to* A *passes the cone* $C_q(A)$ *and, in turn, every ray from* q *to* A *passes the cone* $C_p(A)$.

Proof. This is nothing but the relations: $C_p(A) \subseteq C_p(C_q(A))$ and $C_q(A) \subseteq C_q(C_p(A))$. And these relations are clear from the preceeding proposition.

Now we give

DEFINITION 5. Let A be a set and let p and q be two distinct points in space. p is called *independent of* q *with respect to* A, if the condition

$$(2) \qquad\qquad\qquad p \notin C_q(A)$$

holds. We say that p *and* q *are independent with respect to* A if each of them is independent of the other. We also say that a set V is an *independent set with respect to* A if any pair of points in V are independent.

DEFINITION 6. A surface[3] S is called *a locally independent surface with respect to* a set A if every point of S has a neighbourhood (relative to S) which is an independent set with respect to A.

We proceed hereafter by supposing that there are given a set A and a surface S. *We assume once for all that the sight cone* $C_p(A)$ *is always a solid circular cone* (naturally with its vertical angle less than π) *for any point* p *on* S. From now

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3) By a surface, we understand a topologically embedded two dimensional manifold in space. A closed convex surface is a surface in this sense since it is homeomorphic to a two dimensional sphere. (C.f. the foot-note 7).

on, we shall simply write C_p for $C_p(A)$ and we shall simply say "independent" instead of saying "independent with respect to A".

We denote by I_p the central axis of the cone C_p, I_p being a ray (from p) but not a whole line.

THEOREM 1. *If p and q are independent, then the axes I_p and I_q intersect. And in this case, there exists a common inscribed ball to both of the cones C_p and C_q with center at the intersecting point.*

Proof. Let g be the straight line passing both points p and q. Since these points are independent, we have the relations

$$(3) \qquad\qquad g \cap C_p = \{p\} \quad \text{and} \quad g \cap C_q = \{q\} \ .$$

In fact, if we deny, for instance, the first of these relations, we see that g should contain a ray from p to A. We should, therefore, have a point of A on the line g. In such a situation, however, it is impossible for the points p and q to be independent. Thus we should have (3).

Take a plane H which is orthogonal to the line g. Denote by O the intersecting point of H and g. Now project all the figures on H along g. By the relations (3) above, we see that the cones C_p and C_q become certain *minor* angular regions on H with the common origin O which is the common image of the vertices p and q. Then, by Proposition 2, we see that these angular regions coincide. From this coincidence, going back to the original figures, we see that the axes I_p and I_q should lie on the plane L determined by the line g and the bisecting line of the angular region on H.

Now consider the sections of the cones C_p and C_q by the plane L. Then we get two angular regions D_p and D_q the origins of which are vertices p and q. Let us show that the axes I_p and I_q intersect in the plane L. According to the relations (3), we see that both the regions D_p and D_q lie on the same side of the line g in the plane L since, the section of the set A by L being in the intersection $D_p \cap D_q$, $D_p \cap D_q$ is not empty. Now take line g as horizontal and suppose that both regions D_p and D_q are in the upper half of the plane L. We suppose that the point p is on the left of the point q (both points lying on the horizontal line g). We denote by s_1 and s_2 the sides of D_p and by t_1 and t_2 those of D_q (from left to right). We see, by Proposition 2, that the line s_1 should cut t_1 and that t_2 should cut s_2. From these facts we can easily conclude that I_p and I_q should intersect.

Now consider the inscribed balls B_p and B_q to the cones C_p and C_q respectively with common center at the intersecting point of the axes I_p and I_q. We see that, in fact, these balls coincide because we knew already that the projections of both cones coincided on the plane H. This concludes the proof.

THEOREM 2. *If S is a locally independent surface, then, every point of S has a neighbourhood V (relative to S) such that there exists a common inscribed ball to all the cones C_p for p in V.*

Proof. S being locally independent, every point of S has a neighbourhood U (relative to S) which is an independent set. Since S is locally Euclidean[4], there exists in U another neighbourhood V which is homeomorphic to the interior of a square in plane. Let us show that this neighbourhood V has the property stated in the theorem.

To this end, we first show that it is impossible for all the axes I_p (p in V) to lie in a single plane. Suppose the contrary and assume that all the axes I_p (p in V) are lying in a plane L. Then, a fortiori, V is a subset of L. Since V is homeomorphic to a two dimensional open set, we can conclude, by *Brower's invariance theorem of domain*[5], that V is an open subset of L. Take a point p in V, then, since I_p lies in L and since V is open in L, it is easy to see that there exist other points in V which are contained in C_p. This contradicts the assumption that V is an independent set. Thus we have shown that there exist at least three points p_1, p_2, p_3 in V the axes $I_{p_1}, I_{p_2}, I_{p_3}$ of which do not lie in a single plane.

By Theorem 1, I_{p_1} and I_{p_2} intersect. Let us denote the intersecting point by c_{12}. Again by Theorem 1, I_{p_3} intersects I_{p_1} and I_{p_2}. But, since I_{p_3} does not lie in the plane determined by I_{p_1} and I_{p_2}, I_{p_3} should pass the common point c_{12} of I_{p_1} and I_{p_2}. Let us denote by c this common point to these three axes. Now, we shall show that, for any point p in V, I_p passes this point c. Let L be the plane determined by I_{p_1} and I_{p_2}. If I_p does not lie in L, then, by the same argument as above, we see that I_p should pass the common point of I_{p_1} and I_{p_2} which is the point c. If, on the contrary, I_p does lie in L, then I_p should intersect I_{p_3} at a point in the plane L. But since c is the only common point to I_{p_3} and L, I_p should pass the point c.

Now, consider the inscribed ball B_p to the cone C_p with center at the point c. These balls B_p for p in V should coincide since, according to Theorem 1, any pair of them coincide. This completes the proof.

THEOREM 3. *If S is a locally independent surface, then each connected component of S has a common inscribed ball to all the cones C_p for p in the component.*

Proof. Let p be a point on the surface S. By the previous theorem, we know that p has a neighbourhood V such that there exists a common inscribed ball to all the cones C_q for q in V. Denote by B_p this common inscribed ball.

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4) See the foot-note 3).
5) See [2] pp. 95–96.

Now, let B be a ball. We denote by S_B the set of those points p of S for which $B_p = B$ holds. Then, by Theorem 2, S_B is always an open set[6] in S. Since S is clearly the disjoint union of these S_B, each S_B is open and closed in S. Therefore, each component of S should be contained in some S_B. This concludes the proof.

For the proof of Theorem 4 below, we need the following elementary

PROPOSITION 3. *Let D be a convex set with non-empty interior. If p is an adherent point of D and if q is an interior point of D, then every point on the segment connecting p and q lies, with the only possible exception of p, in the interior of D.[7]*

For the proof of this proposition, see [1] p. 51.

THEOREM 4. *A closed convex surface which strictly contains the set A is a connected locally independent surface.*

Proof. Let S be a closed convex surface which strictly contains the set A. Since S is homeomorphic to a sphere, S is clearly connected. Let us show that S is locally independent. Denote by D the compact convex set with non-empty interior whose boundary is the surface S. Since D contains strictly the set A, S and $\bar{A}$ are disjoint (see Definitions 3 and 4). Now take any point p on S. Since p is not adherent to A, there exists an open ball E with center at p such that $E \cap A$ is empty. Put

$$V = E \cap S .$$

We shall show that this V is an independent set.

To this end, it is sufficient to show that the condition

$$(4) \qquad q_1 \notin C_{q_2}$$

holds for any two distinct points q_1 and q_2 in V. To prove the relation (4), take any ray R from q_2 to the set A. We are to show that

$$(5) \qquad q_1 \notin R .$$

Let F be the complement of the open ball E in space, and divide R into two parts $R_1 = R \cap E$ and $R_2 = R \cap F$. Since q_1 is in E, (5) is equivalent to

$$(6) \qquad q_1 \notin R_1 .$$

R being a ray, there exists a point q in $A \cap R$. Denote by R_0 the segment connecting q_2 and q. Since $A \subseteq \overset{\circ}{D}$, we see, by Proposition 3, that q_2 is the only point of R_0 on the boundary S of D. It is clear that $R_1 \subseteq R_0$, since q is a point in F. Therefore q_2 is the only point of R_1 on the boundary S of D. But q_1 is on $S = \partial D$ and distinct from q_2. Hence (6) holds. This completes the proof.

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6) Some of these sets S_B might be empty.
7) From this proposition, one can also easily see that a closed convex surface is homeomorphic to a sphere.

Now we are in position to answer the question posed in Introduction.

THEOREM 5. *In space, let S be a closed convex surface and A be a set strictly contained in it. If the sight cone $C_p(A)$ at p for A is always a solid circular cone for any point p on S, then the set A is equivalent to a ball.*

Proof. By Theorem 3 and Theorem 4, we know already that there exists a common inscribed ball B to all the cones $C_p(A)$ for p on S. Let us show that the set A is equivalent to this ball B.

First we prove that

(7) $A \subseteq B$.

Suppose that there be a point q of A which does not belong to B. Then, it is evident that we can draw a straight line g which does not touch the ball B. Since A is strictly contained in S and since S is a closed convex surface, g should intersect S at two points. Let p be any one of these two points and consider the cone $C_p(A)$. The ray R from p to q should be contained in $C_p(A)$ but R does not touch the ball B. This contradicts the fact that B is an inscribed ball to $C_p(A)$. Hence we should have (7).

Second we prove that

(8) $\partial B \subseteq A$.

Suppose that there be a point r of the sphere ∂B which does not belong to the set A. Consider a straight line h tangent to ∂B at r. By a similar argument to what was used above, h intersects S at two points. Let p be any one of these points and consider the cone $C_p(A)$. Denote again by R the ray from p to r. By the construction above and by the relation (7), R is not a ray from p to A. But since B is an inscribed ball to $C_p(A)$, R should be contained in $C_p(A)$. Thus we arrived at a contradiction. Hence we should have the relation (8).

From (7) and (8), we conclude that A is equivalent to the ball B. This completes the proof.

§4. Remarks.

1. Higher dimensional analogue of the problem.

Our problem was posed and has been studied so far in the ordinary three space. It has, however, its n-dimensional analogue for general $n > 3$, though it does then lose its intuitive meaning. This generalized problem can also be solved in a way quite analogous to that we have used for the case $n = 3$. A slight modification may be required only in the proof of Theorem 1 for general $n > 3$. We shall indicate it here.

In that proof, we should replace "*plane H*" by "*hyperplane H*", "*minor angular region with origin O*" by "*$(n-1)$-dimensional solid circular cone with vertex O*",

"*bisecting line of the angular region*" by "*axis of the $(n-1)$-dimensional solid cone in H*", and "*plane L*" by "*two dimensional linear variety L determined by the line g and the axis of the $(n-1)$-dimensional solid circular cone in H*".

2. Convexity assumption on the surface S.

We posed convexity assumption on the surface S in our problem. This assumption was utilized to deduce the local independence property of S with respect to the set A. And the independence property was crucial in our argument. It would seem, however, very natural that the problem could be solved for general closed surface S containing the set A. This remains to be solved.

References

[1] Bourbaki, N., Espaces vectoriels topologiques, (Chap. I, II.), Paris, (1953).
[2] Hurewicz, W. and Wallman, H., Dimension theory, Princeton, (1941).