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# On the differentiation of De Rham cohomology classes with respect to parameters

By

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## Introduction.

Let $X$ and $S$ be smooth schemes over a field $k$, and let $\pi: X \to S$ be a smooth $k$-morphism. We are concerned with constructing a canonical integrable connection, the "Gauss-Manin connection", on the relative De Rham cohomology sheaves $\mathcal{H}^q_{DR}(X/S)$.

In his 1966/67 Harvard Seminar, Mumford defined this connection by means of a certain connecting homomorphism. We noticed that this connecting homomorphism was the differential $d_1$ between certain $E_1$ terms of a spectral sequence. This observation implied immediately the *integrability* of the connection, and the existence, when $S$ is *affine*, of a "Leray spectral sequence" for the De Rham cohomology.

We begin by explaining the formalism of connections. We then recall the notion of relative De Rham cohomology sheaves, construct the Gauss-Manin connection, and prove its fundamental properties. Next, we "explicitly" calculate the connection, and show that it agrees with the original definition given by Manin [5], and later extended by Katz [4]. We conclude by giving the "Leray spectral sequence" when $S$ is affine.

## 1. Connections.

Let $S$ be a smooth scheme over the field $k$, and let $\mathcal{E}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-modules. A connection on $\mathcal{E}$ is a homomorphism $\rho$ of abelian sheaves

$$\rho\colon \mathcal{E}\to\Omega^1_{S/k}\otimes_{\mathcal{O}_S}\mathcal{E}$$

such that

$$\rho(fe)=f\rho(e)+df\otimes e, \tag{1}$$

where $f$ and $e$ are sections of $\mathcal{O}_S$ and $\mathcal{E}$ respectively over an open subset of $S$, and $df$ denotes the image of $f$ under the canonical exterior differentiation $d\colon \mathcal{O}_S\to\Omega^1_{S/k}$.

A connection $\rho$ may be extended to a homomorphism of abelian sheaves

$$\rho_i\colon \Omega^i_{S/k}\otimes_{\mathcal{O}_S}\mathcal{E}\to\Omega^{i+1}_{S/k}\otimes_{\mathcal{O}_S}\mathcal{E}$$

by

$$\rho_i(\omega\otimes e)=d\omega\otimes e+(-1)^i\omega\wedge\rho(e) \tag{2}$$

where $\omega$ and $e$ are sections of $\Omega^i_{S/k}$ and $\mathcal{E}$ respectively over an open subset of $S$, and where $\omega\wedge\rho(e)$ denotes the image of $\omega\otimes\rho(e)$ under the canonical map

$$\Omega^i_{S/k}\otimes_{\mathcal{O}_S}(\Omega^1_{S/k}\otimes_{\mathcal{O}_S}\mathcal{E})\to\Omega^{i+1}_{S/k}\otimes_{\mathcal{O}_S}\mathcal{E}$$

sending $\omega\otimes\tau\otimes e$ to $(\omega\wedge\tau)\otimes e$.

The curvature $K$ of the connection $\rho$ is the $\mathcal{O}_S$-linear map $K=\rho_1\circ\rho\colon \mathcal{E}\to\Omega^2_{S/k}\otimes\mathcal{O}_S\mathcal{E}$. One easily verifies that

$$\rho_{i+1}\circ\rho_i(\omega\otimes e)=\omega\wedge K(e),$$

where $\omega$ and $e$ are sections of $\Omega^i_{S/k}$ and $\mathcal{E}$ respectively over an open subset of $S$.

The connection $\rho$ is called *integrable* if $K=0$. An integrable connection $\rho$ on $\mathcal{E}$ thus gives rise to a *complex*

$$0\to\mathcal{E}\xrightarrow{\rho}\Omega^1_{S/k}\otimes_{\mathcal{O}_S}\mathcal{E}\xrightarrow{\rho_1}\Omega^2_{S/k}\otimes_{\mathcal{O}_S}\mathcal{E}\xrightarrow{\rho_2}\cdots \tag{3}$$

which we will denote simply by $\Omega^{\cdot}_{S/k}\otimes_{\mathcal{O}_S}\mathcal{E}$ when there is no confusion.

Let $\mathcal{D}er_k(\mathcal{O}_S)$ denote the sheaf of germs of $k$-derivations of $\mathcal{O}_S$

into itself. We note for later use that $\mathcal{D}er_k(\mathcal{O}_S)$ is naturally a sheaf of $k$-Lie algebras, while, as $\mathcal{O}_S$-module, it is isomorphic to $\mathcal{H}om_{\mathcal{O}_S}(\Omega^1_{S/k}, \mathcal{O}_S)$.

Let $\mathcal{E}nd_k(\mathcal{E})$ denote the sheaf of germs of $k$-linear endomorphisms of $\mathcal{E}$. We note that $\mathcal{E}nd_k(\mathcal{E})$ also carries the structure of sheaf of $k$-Lie algebras, as well as that of $\mathcal{O}_S$-module.

Now fix a connection $\rho$ on $\mathcal{E}$; $\rho$ gives rise to an $\mathcal{O}_S$-linear mapping

$$\mathcal{D}er_k(\mathcal{O}_S) \longrightarrow \mathcal{E}nd_k(\mathcal{E})$$

sending $D$ to $\widetilde{D}$, where $\widetilde{D}$ is the composite

$$\mathcal{E} \xrightarrow{\rho} \Omega^1_{S/k} \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{D \otimes 1} \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{E} \cong \mathcal{E}.$$

Notice that

$$\widetilde{D}(fe) = D(f)e + f\widetilde{D}(e) \tag{4}$$

whenever $D, f$ and $e$ are sections of $\mathcal{D}er_k(\mathcal{O}_S)$, $\mathcal{O}_S$ and $\mathcal{E}$ respectively over an open subset of $S$. Conversely, because $S$ is smooth over $k$, any $\mathcal{O}_S$-linear mapping $\mathcal{D}er_k(\mathcal{O}_S) \to \mathcal{E}nd_k(\mathcal{E})$ satisfying (4) arises from a unique connection $\rho$.

The connection $\rho$ is *integrable* precisely when the mapping $\mathcal{D}er_k(\mathcal{O}_S) \to \mathcal{E}nd_k(\mathcal{E})$ is also a Lie-algebra homomorphism. This can be seen by using the well known fact that for $D_1$ and $D_2$ in $\mathcal{D}er_k(\mathcal{O}_S)$, we have $[\widetilde{D}_1, \widetilde{D}_2] - \widetilde{[D_1, D_2]} = (D_1 \wedge D_2)(K)$, where the right hand side is the composite map

$$\mathcal{E} \xrightarrow{K} \Omega^2_{S/k} \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{(D_1 \wedge D_2) \otimes 1} \mathcal{O}_S \otimes_{\mathcal{O}_S} \mathcal{E} \cong \mathcal{E}.$$

## 2. Relative De Rham cohomology.

Let $\pi: X \to S$ be a smooth $k$-morphism of smooth $k$-schemes. The relative De Rham cohomology sheaf $\mathcal{H}^{\cdot}_{DR}(X/S)$ is, by definition, the quasi-coherent sheaf of graded anticommutative algebras on $S$ defined by

$$\mathcal{H}^q_{DR}(X/S) = \mathbf{R}^q\pi_*(\Omega^\cdot_{X/S})$$

where $\Omega^\cdot_{X/S}$ denotes the complex of $S$-differentials on $X$, and $\mathbf{R}^q\pi_*$ is the $q$-th hyperderived functor of $\pi_*$.

We now describe a canonical integrable connection $\nabla = \nabla(X/S, q)$ on each cohomology sheaf $\mathcal{H}^q_{DR}(X/S)$, the "Gauss-Manin connection."

We recall that, because $\pi$ is smooth, the sequence

$$0 \to \pi^*(\Omega^1_{S/k}) \to \Omega^1_{X/k} \to \Omega^1_{X/S} \to 0 \tag{5}$$

is exact. The complex $\Omega^\cdot_{X/k}$ admits a canonical filtration

$$\Omega^\cdot_{X/k} = F^0(\Omega^\cdot_{X/k}) \supset F^1(\Omega^\cdot_{X/k}) \supset F^2(\Omega^\cdot_{X/k}) \supset \cdots, \tag{6}$$

where

$$F^i = F^i(\Omega^\cdot_{X/k}) = \text{image}\,[\Omega^{\cdot-i}_{X/k} \otimes_{\mathcal{O}_X} \pi^*(\Omega^i_{S/k}) \to \Omega^\cdot_{X/k}].$$

Because the sheaves $\Omega^i_{X/k}$ and $\Omega^i_{S/k}$ on $X$ and $S$ respectively are locally free, the exactness of (5) allows us to conclude that the associated graded objects of this filtration are given by

$$gr^i = gr^i(\Omega^\cdot_{X/k}) = F^i/F^{i+1} = \pi^*(\Omega^i_{S/k}) \otimes_{\mathcal{O}_X} \Omega^{\cdot-i}_{X/S}$$

Consider the functor $\mathbf{R}^0\pi_*$ from the category of complexes of abelian sheaves on $X$ to the category of abelian sheaves on $S$. The derived functors of $\mathbf{R}^0\pi_*$ are $\mathbf{R}^q\pi_*$. Applying the spectral sequence of a finitely filtered object [$EGA$, $O_{\text{III}}$, 13. 6. 4] to $\Omega^\cdot_{X/k}$, we obtain a spectral sequence abutting to (the associated graded object with respect to the filtration of) $\mathbf{R}^q\pi_*(\Omega^\cdot_{X/k})$, while

$$\begin{aligned} E_1^{p,q} &= \mathbf{R}^{p+q}\pi_*(gr^p) = \mathbf{R}^{p+q}\pi_*(\pi^*(\Omega^p_{S/k}) \otimes_{\mathcal{O}_X} \Omega^{\cdot-p}_{X/S}) \\ &= \mathbf{R}^q\pi_*(\pi^*(\Omega^p_{S/k}) \otimes_{\mathcal{O}_X} \Omega^\cdot_{X/S}) \cong \Omega^p_{S/k} \otimes_{\mathcal{O}_S} \mathbf{R}^q\pi_*(\Omega^\cdot_{X/S}) \\ &= \Omega^p_{S/k} \otimes_{\mathcal{O}_S} \mathcal{H}^q_{DR}(X/S) \end{aligned} \tag{7}$$

We get the isomorphism in the equality above, because $\Omega^p_{S/k}$ is locally free and because the differential in the complex $\pi^*(\Omega^p_{S/k}) \otimes_{\mathcal{O}_X} \Omega^\cdot_{X/S}$ is $\pi^{-1}(\mathcal{O}_S)$-linear.

Since the filtration on $\Omega^\cdot_{X/k}$ is compatible with the exterior product, i.e. $F^i \wedge F^j \subset F^{i+j}$, and since the sequence of functors $\mathbf{R}^q\pi_*$ is multiplicative, it follows that this spectral sequence has a product structure. Explicitly there are pairings, for each $p, q, p', q'$ and $r$

$$E_r^{p,q}\times E_r^{p',q'}\rightarrow E_r^{p+p',q+q'}$$

sending $(e, e')$ to $e\cdot e'$ where $e$ and $e'$ are sections of $E_r^{p,q}$ and $E_r^{p',q'}$ respectively, over an open subset of $S$. This pairing satisfies

$$e\cdot e'=(-1)^{(p+q)(p'+q')}e'\cdot e$$

and

$$d_r(e\cdot e')=d_r(e)\cdot e'+(-1)^{p+q}e\cdot d_r(e').$$

(This product is most easily constructed by means of the canonical flasque resolution, generalizing the procedure for the construction of cup product in Godement [2]).

In particular, let us consider the $E_1$ terms. Since $d_1$ has bidegree (1, 0), we obtain, for every $q$, the complex $E_1^{\cdot,q}$, which is explicitly

$$0\rightarrow \mathcal{H}^q_{DR}(X/S)\xrightarrow{d_1^{0,q}}\Omega^1_{S/k}\otimes_{\mathcal{O}_S}\mathcal{H}^q_{DR}(X/S)\xrightarrow{d_1^{1,q}}\Omega^2_{S/k}\otimes_{\mathcal{O}_S}\mathcal{H}^q_{DR}(X/S)\cdots$$

For $q=0$, the complex $E_1^{\cdot,0}$ is $\Omega^{\cdot}_{S/k}\otimes_{\mathcal{O}_S}\mathcal{H}^0_{DR}(X/S)$, with the differential $d_{S/k}\otimes 1$, where $d_{S/k}$ denotes the exterior differentiation in $\Omega^{\cdot}_{S/k}$, and so we may regard $\Omega^{\cdot}_{S/k}$ as a subcomplex of $E_1^{\cdot,0}$. Thus if $\omega$ and $e$ are sections of $\Omega^i_{S/k}$ (which is contained in $E_1^{i,0}$) and of $E_1^{0,q}=\mathcal{H}^q_{DR}(X/S)$ respectively over an open subset of $S$, we have

(8) $$d_1^{i,q}(\omega\cdot e)=d\omega\cdot e+(-1)^i\omega\cdot d_1^{0,q}e.$$

This shows that $d_1^{0,q}\colon \mathcal{H}^q_{DR}(X/S)\rightarrow\Omega^1_{S/k}\otimes_{\mathcal{O}_S}\mathcal{H}^q_{DR}(X/S)$ is a *connection* on $\mathcal{H}^q_{DR}(X/S)$, and that the $d_1^{i,q}$ are deduced from $d_1^{0,q}$ canonically according to the rule (2). The curvature is thus $d_1^{1,q}\cdot d_1^{0,q}=0$, and so $d_1^{0,q}$ is an *integrable connection.*

Further, letting $e_q$ and $e_{q'}$ be sections of $\mathcal{H}^q_{DR}(X/S)$ and $\mathcal{H}^{q'}_{DR}(X/S)$ respectively, over an open subset of $S$, we have

(9) $$d_1^{0,q+q'}(e_q\cdot e_{q'})=d_1^{0,q}(e_q)\cdot e_{q'}+(-1)^q e_q\cdot d_1^{0,q'}(e_{q'}).$$

We may now define the Gauss-Manin connection $\Delta$ on the relative De Rham cohomology sheaf $\mathcal{H}^q_{DR}(X/S)$ to be $d_1^{0,q}$.

**Theorem 1.** *Let $\pi$: $X \rightarrow S$ be a smooth k-morphism of smooth k-schemes. There exists a canonical integrable connection* $Д = Д(X/S, q)$ *on the relative De Rham cohomology group* $\mathcal{H}^q_{DR}(X/S)$. $Д$ *is compatible with the cup product in the sense that*

$$Д(e \cdot e') = Д(e) \cdot e' + (-1)^q e \cdot Д(e'), \tag{10}$$

*where e and e' are sections of* $\mathcal{H}^q_{DR}(X/S)$ *and* $\mathcal{H}^{q'}_{DR}(X/S)$ *respectively over an open subset of S.*

As explained earlier, $Д$ gives a homomorphism of sheaves of $k$-Lie algebras

$$\mathcal{D}er_k(\mathcal{O}_S) \rightarrow End_k(\mathcal{H}^q_{DR}(X/S))$$

sending $D$ to $\widetilde{D}$, such that

$$\widetilde{D}(e \cdot e') = \widetilde{D}(e) \cdot e' + e \cdot \widetilde{D}(e') \tag{11}$$

$$\widetilde{D}(f) = D(f), \tag{12}$$

where $D$, $e$, $e'$ and $f$ are sections of $\mathcal{D}er_k(\mathcal{O}_S)$, $\mathcal{H}^q_{DR}(X/S)$, $\mathcal{H}^{q'}_{DR}(X/S)$ and $\mathcal{O}_S$ (which is contained in $\mathcal{H}^0_{DR}(X/S)$) respectively, over an open subset of $S$.

The formula (11) expresses that each $\widetilde{D}$ is a *k-derivation* of the sheaf of $\mathcal{O}_S$-algebras $\mathcal{H}^{\cdot}_{DR}(X/S)$. (The formula (11) differs from (10) by a sign, because, in defining $\widetilde{D}$, the term $\Omega^1_{S/k}$ appears on the extreme left.)

## 3. "Explicit" calculation of the connection.

*Reduction.*

The calculation rests on the general fact that, in the spectral sequence of a filtered object, the differential

$$d_1^{p,q}: E_1^{p,q} = \mathbf{R}^{p+q}\pi_*(gr^p) \rightarrow E_1^{p+1,q} = \mathbf{R}^{p+q+1}\pi_*(gr^{p+1})$$

is the *connecting homomorphism* of the functors $\mathbf{R}^q\pi_*$ for the exact sequence

(13) $$0 \to gr^{p+1} \to F^p/F^{p+2} \to gr^p \to 0.$$

Because the sheaves $\boldsymbol{R}^q\pi_*(F^i/F^j)$ are the sheaves associated to the *presheaves* on $S$

$$V \mapsto \boldsymbol{H}^q(\pi^{-1}(V),\ F^i/F^j|\pi^{-1}(V)),$$

it suffices to explicate the connecting homomorphism on these *presheaves*, indeed on the sections of the presheaves over arbitrarily small affine open subsets of $S$ (since we are ultimately concerned with the mapping induced on the *associated sheaves*).

For the remainder of this section, then, we will assume that $S$ is affine, and that $\Omega^1_{S/k}$ is free, and explicate the connection on global sections:

$$\nabla\colon \Gamma_S(\mathcal{H}^q_{DR}(X/S)) = \boldsymbol{H}^q(X, gr^0) \to \Omega^1_{S/k} \otimes \Gamma_S(\mathcal{H}^q_{DR}(X/S)) = \boldsymbol{H}^{q+1}(X, gr^1).$$

The problem is thus reduced to computing the connecting homomorphism of the functors $\boldsymbol{H}^q(X, ?)$ for the exact sequence (13).

*Čech calculation of the* $\boldsymbol{H}^q(X, F^i/F^j)$

Let $(\mathcal{L}^{\cdot}, d)$ be any complex of abelian sheaves on $X$, such that each $\mathcal{L}^p$ is quasi-coherent (such as $F^i/F^j$). Fix an affine open covering $\boldsymbol{U} = \{U_i\}$ of $X$; we define a double complex

$$C^{\cdot}(\boldsymbol{U}, \mathcal{L}^{\cdot}) = \sum_{p,q \geq 0} C^q(\boldsymbol{U}, \mathcal{L}^p)$$

as follows: $C^q(\boldsymbol{U}, \mathcal{L}^p)$ is the set of alternating $q$-cochains $\beta$ with values in $\mathcal{L}^q$, i.e. to each $(q+1)$-tuple, $i_0 < i_1 \cdots < i_q$, $\beta$ assigns a section $\beta(i_0, \cdots, i_q)$ of $\mathcal{L}^p$ over $U_{i_0} \cap \cdots \cap U_{i_q}$. The two differentials are

$$d\colon C^q(\boldsymbol{U}, \mathcal{L}^p) \to C^q(\boldsymbol{U}, \mathcal{L}^{p+1})$$

defined by

$$(d\beta)(i_0, \cdots, i_q) = d(\beta(i_0, \cdots, i_q))$$

and

$$\delta\colon C^q(\boldsymbol{U}, \mathcal{L}^p) \to C^{q+1}(\boldsymbol{U}, \mathcal{L}^p)$$

defined by

$$(\delta\beta)(i_0, \cdots, i_{q+1}) = (-1)^p \sum_{j=0}^{q+1} (-1)^j \beta(i_0, \cdots, \hat{i}_j, \cdots, i_{q+1}).$$

These satisfy the relations

$$d^2=0,\ \delta^2=0,\ d\delta+\delta d=0.$$

We define the associated single complex

$$K^\cdot(\mathcal{L}^\cdot)=\sum_{n\geq 0} K^n(\mathcal{L}^\cdot),$$

where $K^n(\mathcal{L}^\cdot)=\sum_{p+q=n} C^q(\boldsymbol{U}, \mathcal{L}^p)$, whose differential is $d+\delta$. Then the hypercohomology group $\boldsymbol{H}^q(X, \mathcal{L}^\cdot)$ is the $q$-th cohomology group of the complex $K^\cdot(\mathcal{L}^\cdot)$. (*EGA* III, 6.2.2) (Remark: the statement in *EGA* III, 6.2.2. remains valid even if $d$ is not $\mathcal{O}_X$-linear.)

Since the covering $\boldsymbol{U}$ is affine, we obtain *exact* sequences of complexes of abelian groups

$$(14)\qquad 0\to K^\cdot(F^1)\to K^\cdot(F^0)\to K^\cdot(gr^0)\to 0$$

$$(15)\qquad 0\to K^\cdot(gr^1)\to K^\cdot(F^0/F^2)\to K^\cdot(gr^0)\to 0.$$

The connecting homomorphism of the functors $\boldsymbol{H}^q(X, ?)$ for (13) is that arising from (15).

*Local calculations.*

Fix a basis $\{ds_1, \cdots, ds_r\}$ of $\Omega^1_{S/k}$, and cover $X$ by affine open sets $U_\alpha$ such that $\Omega^1_{U_\alpha/k}$ admits a basis of the form $\{ds_1, \cdots, ds_r, dx_1^\alpha, \cdots, dx_n^\alpha\}$. The canonical filtration on $\Omega^\cdot_{X/k}$ is given by

$$F^j(\Omega^\cdot_{X/k})=\sum_{i_1<\cdots<i_j} ds_{i_1}\wedge\cdots\wedge ds_{i_j}\wedge\Omega^{\cdot-j}_{X/k}.$$

Denote by $\left\{\psi_\alpha\left(\frac{\partial}{\partial s_1}\right), \cdots, \psi_\alpha\left(\frac{\partial}{\partial s_r}\right), \frac{\partial}{\partial x_1^\alpha}, \cdots, \frac{\partial}{\partial x_n^\alpha}\right\}$ the basis of $\mathcal{D}er_k(\mathcal{O}_{U_\alpha})$ *dual* to $\{ds_1, \cdots, ds_r, dx_1^\alpha, \cdots, dx_n^\alpha\}$ i.e.

$$\begin{cases} \psi_\alpha\left(\dfrac{\partial}{\partial s_i}\right)(s_j)=\delta_{ij} \\ \psi_\alpha\left(\dfrac{\partial}{\partial s_i}\right)(x_j^\alpha)=0 \end{cases}$$

and

$$\begin{cases} \dfrac{\partial}{\partial x_i^\alpha}(s_j)=0 \\ \dfrac{\partial}{\partial x_i^\alpha}(x_j^\alpha)=\delta_{ij}. \end{cases}$$

This determines a decomposition of the exterior differentiation $d_X$ in $\Omega^{\cdot}_{U_\alpha/k}$

(16) $$d_X=d_S^\alpha+d_{X/S}^\alpha$$

defined by

$$d_S^\alpha(h\omega)=\sum_{i=1}^r \psi_\alpha\left(\frac{\partial}{\partial s_i}\right)(h)ds_j\wedge\omega$$

and

$$d_{X/S}^\alpha(h\omega)=\sum_{i=1}^n \frac{\partial}{\partial x_i^\alpha}(h)dx_i^\alpha\wedge\omega,$$

where $h$ is in $\mathcal{O}_{U_\alpha}$, and $\omega$ represents a monomial in the $ds_i$ and $dx_j^\alpha$. Notice that $d_S^\alpha$, $d_{X/S}^\alpha$ and $d_X$ mutually commute.

Define $\varphi_\alpha$: $\Omega^{\cdot}_{U_\alpha/S}\to\Omega^{\cdot}_{U_\alpha/k}$

by

$$\varphi_\alpha(\mu d_{X/S}(g_1)\wedge\cdots\wedge d_{X/S}(g_p))=\mu d_{X/S}^\alpha(g_1)\wedge\cdots\wedge d_{X/S}^\alpha(g_p).$$

We omit the proofs of Lemmas 1 through 5.

**Lemma 1.** *$\varphi_\alpha$ splits the exact sequence of $\mathcal{O}_{U_\alpha}$-modules*

$$0\to F^1(\Omega^{\cdot}_{U_\alpha/k})\to\Omega^{\cdot}_{U_\alpha/k}\to\Omega^{\cdot}_{U_\alpha/S}\to 0$$

*and* $\varphi_\alpha\circ d_{X/S}=d_{X/S}^\alpha\circ\varphi_\alpha$.

Define $\varphi$: $C^q(\boldsymbol{U}, \Omega^p_{X/S})\to C^q(\boldsymbol{U}, \Omega^p_{X/k})$

by

$$(\varphi\beta)(i_0, \cdots, i_q)=\varphi_{i_0}(\beta(i_0, \cdots, i_q)),$$

where $i_0<\cdots<i_q$.

**Lemma 2.** *$\varphi$ splits the exact sequence of abelian groups*

$$0\to K^{\cdot}(F^1)\to K^{\cdot}(F^0)\to K^{\cdot}(gr^0)\to 0.$$

Define $J$: $C^q(\boldsymbol{U}, \Omega^p_{X/S})\to C^{q+1}(\boldsymbol{U}, \Omega^p_{X/k})$

by

$$(J\beta)(i_0, \cdots, i_{q+1}) = (-1)^{p+1}(\varphi_{i_0} - \varphi_{i_1})(\beta(i_1, \cdots, i_{q+1})),$$

where $i_0 < \cdots < i_{q+1}$.

By Lemma 2, we have $J(K^{\cdot}(gr^0)) \subset K^{\cdot}(F^1)$.

**Lemma 3.** $\delta\varphi - \varphi\delta = J\cdot$

Define the *total Lie derivative with respect to* $S$

$$L_S\colon C^q(\boldsymbol{U}, \Omega^p_{X/k}) \to C^q(\boldsymbol{U}, \Omega^{p+1}_{X/k})$$

by

$L_S(\beta)(i_0, \cdots, i_q) = d_S^{i_0}(\beta(i_0, \cdots, i_q))$, where $i_0 < \cdots < i_q$. Notice that $L_S(K^{\cdot}(F^i)) \subset K^{\cdot}(F^{i+1})$.

Combining Lemma 1 and (16), we obtain

**Lemma 4.** $d_X \circ \varphi = L_S \circ \varphi + \varphi \circ d_{X/S}$.

Combining Lemmas 3 and 4, we find

**Lemma 5.** $(d_X + \delta) \circ \varphi = L_S \circ \varphi + J + \varphi \circ (d_{X/S} + \delta)$

$$\begin{array}{ccccc}
 & & K^r(F^0) & \underset{\varphi}{\overset{\text{mod. } F^1}{\rightleftarrows}} & K^r(gr^0) \\
L_S\circ\varphi + J \downarrow & & \downarrow d_X + \delta & & \downarrow d_{X/S} + \delta \\
K^{r+1}(F^1) & \to & K^{r+1}(F^0) & \overset{\text{mod. } F^1}{\longrightarrow} & K^{r+1}(gr^0)
\end{array}$$

*Thus the connecting homomorphism for the exact sequence* (14) *is induced by the map* (*of abelian groups*)

$$L_S \circ \varphi + J\colon \quad K^{\cdot}(gr^0) \to K^0(F^1).$$

Define, for each $U_\alpha$, the *total interior product with respect to* $S$ $\qquad I^\alpha\colon \ \Omega^p_{U_\alpha/k} \to \Omega^p_{U_\alpha/k}$

by

$$I^\alpha(\mu dg_1 \wedge \cdots \wedge dg_p) = \mu \sum_{i=1}^{p} dg_1 \wedge \cdots \wedge dg_{i-1} \wedge d_S^\alpha(g_i) \wedge dg_{i+1} \wedge \cdots \wedge dg_p$$

$$= \mu \sum_{i=1}^{p} (-1)^{i-1} \sum_j \psi_\alpha\left(\frac{\partial}{\partial s_j}\right)(g_i) ds_j \wedge dg_1 \wedge \cdots \wedge \widehat{dg_i} \wedge \cdots \wedge dg_p.$$

When $p=0$, we put $I^\alpha = 0$. Notice $I^\alpha(F^0) \subset F^1$.

Define $\lambda: C^q(\boldsymbol{U}, \Omega^p_{X/k}) \to C^{q+1}(\boldsymbol{U}, \Omega^p_{X/k})$

by

$$(\lambda\beta)(i_0, \cdots, i_{q+1}) = (-1)^p (I^{i_0} - I^{i_1})\beta(i_1, \cdots, i_{q+1}).$$

Notice that $\lambda(K^\cdot(F^i)) \subset K^\cdot(F^{i+1})$.

**Lemma 6.** $\lambda \circ \varphi \equiv J \mod. K^\cdot(F^2)$.

*Proof.* Let $\beta \in C^q(\boldsymbol{U}, \Omega^p_{X/S})$. Fix $(i_0, \cdots, i_{q+1})$ and let $\omega = \beta(i_1, \cdots, i_{q+1})$. We must show that $(-1)^p(I^{i_0} - I^{i_1})(\varphi_{i_1}(\omega)) \equiv (-1)^{p+1}(\varphi_{i_0} - \varphi_{i_1})(\omega) \mod. F^2$.

By linearity, we may suppose $\varphi_{i_1}(\omega) = \mu\, dg_1 \wedge \cdots \wedge dg_p$. Then

$$\begin{aligned}\varphi_{i_0}(\omega) &= \mu d^{i_0}_{X/S}(g_1) \wedge \cdots \wedge d^{i_0}_{X/S}(g_p)\\ &= \mu(dg_1 - d^{i_0}_S(g_1)) \wedge \cdots \wedge (dg_p - d^{i_0}_S(g_p))\\ &= \mu dg_1 \wedge \cdots \wedge dg_p - \sum_{j=1}^{p} \mu dg_1 \wedge \cdots \wedge dg_{j-1} d^{i_0}_S(g_j) \wedge dg_{j+1} \wedge \cdots\\ &\qquad \wedge dg_p + \text{terms in } F^2.\end{aligned}$$

Thus $\varphi_{i_0}(\omega) \equiv \varphi_{i_1}(\omega) - I^{i_0}\varphi_{i_1}(\omega) \mod. F^2$, and $I^{i_1}\varphi_{i_1} = 0$. QED.

Thus the connecting homomorphism of (15) is induced by the map of abelian groups

$$K^\cdot(gr^0) \xrightarrow{\varphi} K^\cdot(F^0) \xrightarrow{L_S+\lambda} K^\cdot(F^1) \xrightarrow{\text{mod. } F^2} K^\cdot(gr^1).$$

Because $\varphi$ is a section of $K^\cdot(F^0) \xrightarrow{\text{mod. } F^1} K^\cdot(gr^0)$, and $(L_S + \lambda)(K^\cdot(F^1)) \subset K^\cdot(F^2)$, this connecting homomorphism is deduced from $L_S + \lambda$ by passage to quotients, i.e.

$$K^\cdot(gr^0) = K^\cdot(F^0)/K^\cdot(F^1) \xrightarrow{L_S+\lambda} K^\cdot(F^1)/K^\cdot(F^2) = K^\cdot(gr^1).$$

An elementary computation shows

**Lemma 7.** *$L_S + \lambda$ commutes with the total differential $d_X + \delta$ of $K^\cdot(F^0) = K^\cdot(\Omega^\cdot_{X/k})$.*

**Theorem 2.** *When $S$ is affine, with $\Omega^1_{S/k}$ free, there exists a map of complexes of degree* 1

$$K^{\cdot}(gr^0) \xrightarrow{L_s+\lambda} K^{\cdot}(gr^1)$$

*which yields, upon passage to cohomology, the Gauss-Manin connection* $\nabla(X/S)$: $\mathcal{H}^{\cdot}_{DR}(X/S) \to \Omega^1_{S/k} \otimes_{\mathcal{O}_S} \mathcal{H}^{\cdot}_{DR}(X/S)$.

**Remark.** This was the original *definition* of the connection. (cf. Manin [5] and Katz [4]).

## 4. The Leray spectral sequence for *De* Rham cohomology.

As before let $\pi: X \to S$ be a smooth $k$-morphism of smooth $k$-schemes. It was conjectured by Grothendieck ([3], Footnote (13)) that there is a "Leray spectral sequence"

$$E_2^{p,q} = \boldsymbol{H}^p(S, \Omega^{\cdot}_{S/k} \otimes_{\mathcal{O}_S} \mathcal{H}^q_{DR}(X/S)) \Longrightarrow \mathrm{H}^{p+q}_{DR}(X/k).$$

Here $\Omega^{\cdot}_{S/k} \otimes_{\mathcal{O}_S} \mathcal{H}^q_{DR}(X/S)$ is the *complex* of sheaves on $S$ deduced from the Gauss-Manin connection on $\mathcal{H}^q_{DR}(X/S)$ as in (3). $\boldsymbol{H}^p(S, ?)$ is the $p$-th hyperderived functor $\boldsymbol{R}^p\Gamma_S$ of the global section functor $\Gamma_S$, and finally $H^{p+q}_{DR}(X/k)$ is the De Rham cohomology group of $X/k$, i.e.

$$H^{p+q}_{DR}(X/k) = \boldsymbol{H}^{p+q}(X, \Omega^{\cdot}_{X/k}) = \boldsymbol{R}^{p+q}\Gamma_X(\Omega^{\cdot}_{X/k}).$$

In this section, we prove the existence of such a spectral sequence in the special case when $S$ is affine. The technique is similar to that used in the previous section.

The desired spectral sequence is that of the finitely filtered object $\Omega^{\cdot}_{X/k}$ (filtered as in (6)), but now with respect to the derived functors of $\boldsymbol{R}^0\Gamma_X$. This abuts to (the associated graded object with respect to the filtration) of $\boldsymbol{R}^q\Gamma_X(\Omega^{\cdot}_{X/k}) = H^q_{DR}(X/k)$, so it remains to compute the $E_2$ term.

The $E_1$ term is

$$\begin{aligned} E_1^{p,q} &= \boldsymbol{R}^{p+q}\Gamma_X(gr^p) = \boldsymbol{R}^{p+q}\Gamma_X(\pi^*\Omega^p_{S/k} \otimes_{\mathcal{O}_X} \Omega^{\cdot -p}_{X/S}) \\ &= \boldsymbol{R}^q\Gamma_X(\pi^*\Omega^p_{S/k} \otimes_{\mathcal{O}_X} \Omega^{\cdot}_{X/S}). \end{aligned}$$

**Lemma 8.** $\boldsymbol{R}^q\Gamma_X(\pi^*\Omega^p_{S/k} \otimes_{\mathcal{O}_X} \Omega^{\cdot}_{X/S}) = \Gamma_S(\boldsymbol{R}^q\pi_*(\pi^*\Omega^p_{S/k} \otimes_{\mathcal{O}_X} \Omega^{\cdot}_{X/S}))$.

*Proof.* The factorization $\boldsymbol{R}^0\Gamma_X=\Gamma_S\circ\boldsymbol{R}^0\pi_*$ yields a spectral sequence of composition

$$E_2^{a,b}=R^a\Gamma_S\circ\boldsymbol{R}^b\pi_*\Longrightarrow\boldsymbol{R}^{a+b}\Gamma_X.$$

Because the complex $\pi^*\Omega^p_{S/k}\otimes_{\mathcal{O}_X}\Omega^\cdot_{X/S}$ consists of quasi-coherent $\mathcal{O}_X$-modules, and its differential is $\pi^{-1}(\mathcal{O}_S)$-linear, the $\mathcal{O}_S$-modules $\boldsymbol{R}^b\pi_*(\pi^*\Omega^p_{S/k}\otimes_{\mathcal{O}_X}\Omega^\cdot_{X/S})$ are quasi-coherent, and hence, $S$ being *affine*, $E_2^{a,b}=0$ for $a\neq0$, and $\boldsymbol{R}^b\Gamma_X(\pi^*\Omega^p_{S/k}\otimes_{\mathcal{O}_X}\Omega^\cdot_{X/S})\cong E_2^{0,b}=\Gamma_S\boldsymbol{R}^b\pi_*(\pi^*\Omega^p_{S/k}\otimes$ $\mathcal{O}_X\Omega^\cdot_{X/S})$. QED.

Thus we get

$$\begin{aligned}E_1^{p,q}&=\Gamma_S(\boldsymbol{R}^q\pi_*(\pi^*\Omega^p_{S/k}\otimes_{\mathcal{O}_X}\Omega^\cdot_{X/S}))\\&\cong\Gamma_S(\Omega^p_{S/k}\otimes_{\mathcal{O}_S}\mathcal{H}^q_{DR}(X/S))\end{aligned}$$

the global sections of the $E_1$ term in (7). Further the $d_1$ of this spectral sequence

$$d_1^{p,q}\colon \Gamma_S(\Omega^p_{S/k}\otimes_{\mathcal{O}_S}\mathcal{H}^q_{DR}(X/S))\rightarrow\Gamma_S(\Omega^{p+1}_{S/k}\otimes_{\mathcal{O}_S}\mathcal{H}^q_{DR}(X/S))$$

is obtained by applying $\Gamma_S$ to the $d_1$ of the spectral sequence (7), i.e. $d_1=\Gamma_S(Д)$.

Thus we get

$$E_2^{p,q}=H^p\ (\text{the complex }\Gamma_S(\Omega^\cdot_{S/k}\otimes_{\mathcal{O}_S}\mathcal{H}^q_{DR}(X/S)),\ \Gamma_S(Д)).$$

**Lemma 9.** $E_2^{p,q}=\boldsymbol{R}^p\Gamma_S(\Omega^\cdot_{S/k}\otimes\mathcal{O}_S\mathcal{H}^q_{DR}(X/S))$.

*Proof.* The factorization $\boldsymbol{R}^0\Gamma_S=H^0\circ\Gamma_S$ yields a spectral sequence of composition

$$E_2^{a,b}=H^a\circ R^b\Gamma_S\Longrightarrow\boldsymbol{R}^{a+b}\Gamma_S.$$

Because $S$ is *affine* and $\Omega^\cdot_{S/k}\otimes_{\mathcal{O}_S}\mathcal{H}^q_{DR}(X/S)$ is a complex of quasi-coherent $\mathcal{O}_S$-modules, $E_2^{a,b}=0$ for $b\neq0$, and so $\boldsymbol{R}^a\Gamma_S(\Omega^\cdot_{S/k}\otimes_{\mathcal{O}_S}\mathcal{H}^q_{DR}(X/$ $S))=H^a(\Gamma_S(\Omega^\cdot_{S'k}\otimes\mathcal{O}_S\mathcal{H}^q_{DR}(X/S))$.

Thus we have proven

**Theorem 3.** *There exist a Leray spectral sequence of De Rham cohomology when S is affine.*

**Corollary.** *When S is an affine curve, the Leray spectral sequence reduces to the long exact sequence*

$$\xrightarrow{\nabla} \Omega^1_{S/k} \otimes_{\mathcal{O}_S} H^{q-1}_{DR}(X/S) \to H^q_{DR}(X/k) \to H^q_{DR}(X/S) \xrightarrow{\nabla}$$
$$\to \Omega^1_{S/k} \otimes_{\mathcal{O}_S} H^q_{DR}(X/S) \to$$

In particular, if $X$ is so small that $\Omega^1_{S/k} = \mathcal{O}_S ds$, and $\frac{\partial}{\partial s} \in \mathcal{D}er_k(\mathcal{O}_S)$ is the derivation dual to $ds$, we have short exact sequences

$$(17) \qquad 0 \to H^{q-1}_{DR}(X/S)/\frac{\partial}{\partial s}(H^{q-1}_{DR}(X/S)) \to H^q_{DR}(X/k) \to$$
$$\to H^q_{DR}(X/S)^{\partial/\partial s} \to 0,$$

where $H^q_{DR}(X/S)^{\partial/\partial s}$ is the subset of elements killed by $\frac{\partial}{\partial s}$.

**Remarks.**

(i) In the Leray spectral sequence, the term $E_2^{0,q}$ is the module of *rational* solutions of the Picard-Fuchs equations in $H^q_{DR}(X/S)$.

(ii) Recent investigations by Dwork of one-parameter families of hypersurfaces employ the $p$-adic analytic analogue of (17). (Dwork [1]).

(*Added in proof.*) P. Deligne has pointed out that Theorem 3 is valid without assuming the base $S$ to be affine. To prove this fact we have to use *filtered double complexes.*

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