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On some mixed problems for fourth order hyperbolic equations

By

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§1. Introduction and statement of result

We consider some mixed problems for fourth order hyperbolic equations. Let S be a smooth and compact hypersurface in R^n $(n\geq 2)$ and Ω be the interior or exterior of S. Let

$$\text{(E)}\qquad Lu+Bu=\left(\frac{\partial^4}{\partial t^4}+(a_1+a_2+a_3)\frac{\partial^2}{\partial t^2}+a_3a_1\right)u+B\left(x, t, \frac{\partial}{\partial t}, D\right)u =f(x, t)$$

Here $a_k(k=1, 2, 3)$ are the following operators:

$$(1.1)\qquad a_k=-\sum_{i,j}^{n}\frac{\partial}{\partial x_i}\left(a_{k,ij}(x)\frac{\partial}{\partial x_j}\right)+b_k(x, D).$$
$$a_{k,ij}(x)=a_{k,ji}(x)$$

are real,

$$\sum_{ij}^{n}a_{k,ij}(x)\xi_i\xi_j\geq\delta|\xi|^2,\quad (\delta>0)$$

for every $(x, \xi)\in\Omega\times R^n$ $(k=1, 2, 3)$. B denotes an arbitrary third order differential operator. b_k are first order operators. Let us assume that all coefficients are sufficiently differentiable and bounded in $\overline{\Omega}$ or in $\overline{\Omega}\times(0, \infty)$. Recently S. Mizohata treated mixed problems for the equations of the form

$$L=\prod_{i=1}^{m}\left(\frac{\partial^2}{\partial t^2}+c_i(x)a(x, D)\right)+B_{2m-1},$$

$$c_i(x) > c_{i+1}(x),\ c_i(x) > 0 \quad (i=1, \cdots, m)$$

Let us consider the case $m=2$. The above equation has the form

$$\frac{\partial^4}{\partial t^4} + (c_1(x)+c_2(x))a\frac{\partial^2}{\partial t^2} + c_1c_2a^2 + (\text{operator of third order}).$$

Now it is not difficult to see that this operator can be considered as a special case of (E), by putting $a_1 = lc_1a$, $a_2 = (1-l)c_1a + \left(1-\frac{1}{l}\right)c_2a$ and $a_3 = \frac{1}{l}c_2a$, l being a constant less than 1 chosen closely to 1. In other words the operators a_1, a_2 and a_3 are obtained by the multiplication of some functions to the operator a.

We consider a generalization of this case. Roughly speaking we are going to assume some relations among the operators a_k only at the boundary. However we don't assume any relation among them in Ω. Moreover, as we shall see later, the hypothesis (H) imposed below is sufficient for the treatment of our problems. Our method is fairly different from that of [1]. Let us denote the Sobolev space $H^p(\Omega)$ simply by H^p, and its norm by $\|\ .\ \|_p$ and denote the closure of $\mathcal{D}(\Omega)$ in H^1 by $\mathcal{D}^1_{L^2}$. Let us consider the subspaces $D(a_k)$ of H^3 defined by

$$D(a_k) = \{u \in H^3 \cap \mathcal{D}^1_{L^2};\ a_ku \in \mathcal{D}^1_{L^2}\} \quad (k=1, 2, 3).$$

Namely, $u \in H^3$ belongs to $D(a_k)$ means that not only u itself but also a_ku vanish at the boundary. We assume that

$$\text{(H)} \qquad D(a_1) = D(a_2) = D(a_3).$$

Our boundary conditions are followings:

Case I. $\quad u|_S = 0 \qquad a_1u|_S = 0$

Case II. $\quad \left(\frac{\partial}{\partial n_1} + \sigma(s)\right)u|_S = 0,\ \left(\frac{\partial}{\partial n_1} + \sigma(s)\right)a_1u|_S = 0,$ where

$$(1.2) \qquad \frac{\partial}{\partial n_k} = \sum_{ij}^{n} a_{k,ij}(x)\cos(\nu, x_j)\frac{\partial}{\partial x_i}, \quad (\nu;\ \text{outer unit normal}),$$

and $\sigma(s)$ is a smooth complex-valued function defined on S.

At first we consider the case where $B=0$. Put

$$(1.3)\qquad u_0=u,\ u_1=\frac{\partial}{\partial t}u,\ u_2=\frac{\partial^2 u}{\partial t^2}+a_1u,\ u_3=\frac{\partial^3}{\partial t^3}u+(a_1+a_2)\frac{\partial}{\partial t}u.$$

Then the equation (E) with $B\equiv 0$ is reduced to

$$(1.4)\qquad \frac{d}{dt}U_t=AU_t+F(t),$$

where $U_t={}^t(u_0(t), u_1(t), u_2(t), u_3(t))$, $F(t)={}^t(0,0,0,f(t))$, and

$$(1\cdot 5)\qquad A=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -a_1 & 0 & 1 & 0 \\ 0 & -a_2 & 0 & 1 \\ 0 & 0 & -a_3 & 0 \end{pmatrix}.$$

Conversely if U_t satisfies (1.4), then the first component $u_0(x,t)$ of U_t satisfies (E) with $B\equiv 0$. Using the notation below:

$$N=\left\{u\in H^2;\ \left(\frac{\partial}{\partial n_1}+\sigma\right)u\,|_S=0\right\},$$

we introduce two Hilbert spaces according to Case I and Case II:

$$(1.6)\qquad \begin{aligned}\mathcal{H}_1&=D(a_1)\times H^2\cap\mathcal{D}^1_{L^2}\times\mathcal{D}^1_{L^2}\times L^2\\ \mathcal{H}_2&=H^3\cap N\times N\times H^1\times L^2.\end{aligned}$$

These spaces are closed subspaces of $H^3\times H^2\times H^1\times L^2$ equipped with the canonical norm

$$(1.7)\qquad \|U\|^2=\|u_0\|_3^2+\|u_1\|_2^2+\|u_2\|_1^2+\|u_3\|_0^2.$$

According to Cases I and II, we take the definition domains of A as follows:

$$(1.8)\qquad \begin{aligned}D(A)_1&=H^4\cap D(a_1)\times D(a_1)\times H^2\cap\mathcal{D}^1_{L^2}\times\mathcal{D}^1_{L^2}\\ D(A)_2&=N(a_1)\times H^3\cap N\times N\times H^1,\quad\text{where}\end{aligned}$$

$$(1.9)\qquad N(a_1)=\{u:\ u\in H^4\cap N,\ a_1u\in N\}.$$

For the convenience we prepare another norm defined below for $U\in D(A)_i$ $(i=1,2)$:

(1.10) $$\|U\|^2_{D(A)_i}=\|u_0\|^2_4+\|u_1\|^2_3+\|u_2\|^2_2+\|u_1\|^2_1.$$

Using these notations we can show the fact that $D(A)_1$ and $D(A)_2$ are dense in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. In fact $\mathcal{D}^1_{L^2}$, $H^2\cap\mathcal{D}^1_{L^2}$ and N are evidently dense in L^2, $\mathcal{D}^1_{L^2}$ and H^1 respectively. In view of the regularity theorem on elliptic boundary problems, a_1+sI is a bijection for a sufficiently large positive constant s, from $H^3\cap N$ onto H^1, or from $H^2\cap\mathcal{D}^1_{L^2}$ onto L^2. Remark that $D(a_1)=\{u\in H^3\cap\mathcal{D}^1_{L^2};\ (a_1+sI)u\in\mathcal{D}^1_{L^2}\}$ and that $N(a_1)=\{u\in H^4\cap N;\ (a_1+sI)u\in N\}$. Then it follows that $D(a_1)$ is dense in $H^2\cap\mathcal{D}^1_{L^2}$ and $N(a_1)$ is dense in $H^3\cap N$, from the fact that $\mathcal{D}^1_{L^2}$ and N are dense in L^2 and in H^1 respectively.

Therefore to the evolution equation (1.4) we can apply the Hille-Yosida's theorem. Then considering the energy inequality, we can use the successive approximation method to the equation (E). Thus we can arrive at the following result:

For any $f(t)$ *in* $\mathcal{E}^1_t(L^2)$[1] *and any initial data* $\left(u(x,0),\ \frac{\partial}{\partial t}u(x,0),\ \frac{\partial^2}{\partial t^2}u(x,0),\ \frac{\partial^3}{\partial t^3}u(x,0)\right)$ *in* $D(A)_i$ $(i=1$ or $2)$, *there exists a unique solution of the equation* (E), *satisfying the boundary conditions* (I) *or* (II). *The solution* $U(t)=\left(u(x,t),\ \frac{\partial}{\partial t}u(x,t),\ \frac{\partial^2}{\partial t^2}u(x,t),\ \frac{\partial^3}{\partial t^3}(x,t)\right)$ *is in* $\mathcal{E}^1_t(\mathcal{H}_i)\cap\mathcal{E}^0_t(D(A)_i)$, (*Theorem* 1). *Moreover when we assume the compatibility conditions on the initial data and the regularity of* $f(t)$, *then the solution has the same regularity as the initial data*, (*Theorem* 2).

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§2. Some lemmas

In this section we show some lemmas concerning uniformly elliptic operators of second order in Ω. Lemma 1, 2 and 4 are used in order

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1) $f(t)\in\mathcal{E}^p_t(H)$ means that $f(t)$ is p times continuously differentiable in t with values in H. $(p=0,1,2,\cdots)$

to show the positivity of the hermitian forms defined in the next section. Lemma 2, 5 and 6 are necessary for a priori estimates. At first we introduce the following local transformations near the boundary, attached to the uniformly elliptic operator

$$a=-\sum_{ij}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)+(\text{first order operator}).$$

Take an open finite covering $\{\Omega_p\}$ of S, satisfying the following conditions, where Ω_p are open sets in R^n. In each Ω_p, there exists an integer k $(1\leq k\leq n)$ such that $\cos(\nu, x_k)\neq 0$ and S is represented by $x_k=\psi_p(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_n)$. Then in each $\Omega_p\cap S$ we have

$$J(s)\equiv\frac{1}{\cos(\nu, x_k)}\frac{1}{|m|}\sum_{ij}a_{ij}(x)\cos(\nu, x_i)\cos(\nu, x_j)\geq\delta>0,$$

where $|m|$ is given below, and $s\in\Omega_p\cap S$. Consider the following transformation:

$$\begin{aligned}&x_j-x_j^0=x_j'-\frac{m_j(x_1'+x_1^0, \cdots, x_{k-1}'+x_{k-1}^0, \psi_p, x_{k+1}'+x_{k+1}^0, \cdots, x_n'+x_n^0)}{|m|}y, \qquad j\neq k\\ (2.1)\qquad &x_k=\psi_p(x_1'+x_1^0, \cdots, x_{k-1}'+x_{k-1}^0, x_{k+1}'+x_{k+1}^0, \cdots, x_n'+x_n^0)\\ &\qquad-\frac{m_k(x_1'+x_1^0, \cdots, x_{k-1}'+x_{k-1}^0, \psi_p, x_{k+1}'+x_{k+1}^0, \cdots, x_n'+x_n^0)}{|m|}y,\end{aligned}$$

where $m_j(x)=\sum_i a_{ij}(x)\cos(\nu, x_i)$, $|m|^2=\sum_{i=1}^{n}m_i^2$,

$$x^0=(x_1^0, \cdots, x_n^0)\in\Omega_p\cap S.$$

Jacobian of (2.1) is sufficiently close to $J(s)$ in the place where y is sufficiently small. Take a new finite covering $\{\omega_k\}$ of S which is a refinement of $\{\Omega_p\}$. For $x^0\in\omega_k\cap S$ the local transformation (2.1) maps $\omega_k\cap\bar{\Omega}$ one to one onto Σ. Σ denotes the intersection of some neighbourhood of the origin and the upper half space $\{(x', y): y\geq 0\}$. Then S is transformed to $y=0$ and the conormal directions of a correspond to the outer normal directions on $\{(x', y), y=0\}\cap\Sigma$. For every s on S, let $r(s)$ be the radius of maximum sphere with center s contained in one of $\{\omega_k\}$. Then S being compact we can choose a positive number δ satisfying $\delta \leq r(s)$ for every s on S. In the neighbourhood $\Gamma = \left\{x;\ \mathrm{dis}(S, x) < \frac{\delta}{2}\right\}$ of S, the sufficiently smooth function $y = \Phi(x)$ is determined uniquely independent of the choice of k. In fact the meaning of $\Phi(x)$ is the distance from x to S measured along the straight line issued from S with conormal direction.

Using $\Phi(x)$ we define the following smooth positive function in Γ attached to the uniformly elliptic operator a:

$$\alpha(x) = \sum a_{ij}(x) \frac{\partial \Phi}{\partial x_i} \frac{\partial \Phi}{\partial x_j}. \tag{2.2}$$

Now we state the lemma concerning the decomposition of a.

Lemma 1. *Assume that a satisfies* (1.1), *then a is written in $\bar{\Omega}$ in the following form*

$$a = n^*(x, D) n(x, D) - \sum_{j:\ \text{finite}} t_j(x, D) s_j(x, D) + (\text{first order term}). \tag{2.3}$$

Here t_j and s_j are first order operators and tangential on S. The operator $n(x, D)$ has the following form:

$$n(x, D) = \frac{\zeta(x)}{\sqrt{\alpha(x)}} \sum_{ij}^{n} a_{ij}(x) \left(-\frac{\partial \Phi}{\partial x_j}(x)\right) \frac{\partial}{\partial x_i}, \tag{2.4}$$

where $\zeta(x)$ is a C^∞-function taking the value 1 *in some neighbourhood of S in Γ, and vanishing outside of Γ. Therefore we can consider $n(x, D)$ as an operator with smooth coefficients defined in $\bar{\Omega}$. $n^*(x, D)$ is the formal adjoint operator of $n(x, D)$.*

Definition. We say that a first order differential operator $t(x, D)$ is tangential at the boundary S, if $t(x, D) = \sum_j^n c_j(x) \frac{\partial}{\partial x_j} + d(x)$ satisfies $\sum_j c_j(x) \cos(\nu, x_j) = 0$ for all $x \in S$. Then we have the following relation:

$$(t(x, D)u, v) = (u, t^*(x, D)v(x)) \quad \text{for all } u, v \in H^1. \tag{2.5}$$

Proof of Lemma 1. Consider the local transformations of type (2.1). Put

$$(2.5)\qquad \begin{cases} D={}^t\left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \cdots, \dfrac{\partial}{\partial x_n}\right), \\ D'={}^t\left(\dfrac{\partial}{\partial x_1'}, \cdots, \dfrac{\partial}{\partial x_{k-1}'}, \dfrac{\partial}{\partial x_{k+1}'}, \cdots, \dfrac{\partial}{\partial x_n'}, \dfrac{\partial}{\partial y}\right), \\ A=(a_{ij}(x)). \end{cases}$$

Let the inverse of (2.1) be as follows.

$$(2.6)\qquad \begin{cases} x_j'=\psi_j(x), & j\neq k \\ y=\Phi(x) \end{cases}$$

Then we have

$$(2.7)\qquad D=\left(\begin{array}{c|c} \psi_{ij} & \begin{matrix}\Phi_1\\ \vdots \\ \Phi_n\end{matrix}\end{array}\right)D'=TD',$$

where
$$\Psi_{ij}=\frac{\partial\psi_j}{\partial x_i}, \quad \Phi_i=\frac{\partial\Phi}{\partial x_i}, \quad j\neq k,\ i=1, \cdots, n.$$

Therefore ${}^tD={}^tD'{}^tT+$operator of smooth coefficient of zero order, and ${}^tDAD={}^tD'{}^tTATD'+$first order operator, hold. By local transformation (2.1), $-a$ takes the form

$$(2.8)\qquad -\tilde{a}=c_1(x', y)\frac{\partial^2}{\partial y^2}+2\sum c_{2,i}(x', y)\frac{\partial}{\partial x_k'}\frac{\partial}{\partial y}+\sum_{ij}c_{3ij}\frac{\partial}{\partial x_i'}\frac{\partial}{\partial x_j'}$$
$$+(\text{first order term}),$$

where
$$c_1(x', y)=\sum_{ij}a_{ij}(x)\frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j}=\alpha(x),$$
$$c_{2i}(x', y)=\sum_{jl}a_{jl}(x)\frac{\partial\Phi}{\partial x_j}\frac{\partial\Psi_i}{\partial x_l}\quad (i\neq k)$$
$$c_{3ij}(x', y)=\sum_{lm}a_{lm}(x)\frac{\partial\Psi_i}{\partial x_l}\frac{\partial\Psi_j}{\partial x_m}\quad (i, j\neq k)$$

And similarly $n(x, D)$ becomes the following form: (Let k be n.)

$$(2.9)\qquad \tilde{n}(x', y, D')=\frac{\zeta(x)}{\sqrt{\alpha(x)}}\left\{c_1(x', y)\frac{\partial}{\partial y}+\sum_i^{n-1}c_{2i}(x', y)\frac{\partial}{\partial x_i'}\right\}.$$

By (2.8) and (2.9), we can see that the conormal directions of a are transformed also to the conormal directions of $\tilde{a}$.

On the other hand, from (2.1) we have

$$\frac{\partial}{\partial y}=-\sum_{i}^{n}\frac{m_i}{|m|}\frac{\partial}{\partial x_i}.$$

Considering that

$$n(x, D)=\frac{\zeta(x)}{\sqrt{\alpha(\alpha)}}\left|\frac{\partial\Phi}{\partial x}\right|\sum_{i=1}^{n}m_i\frac{\partial}{\partial x_i},\quad \left(\left|\frac{\partial\Phi}{\partial x}\right|^2=\sum\left(\frac{\partial\Phi}{\partial x_i}\right)^2\right)$$

holds on S, we can see

$$c_{2i}(x', 0)\equiv 0,\quad i=1, \cdots, n-1.$$

By (2.8) and (2.9) we can write $\tilde{a}$ in the following form:

(2.10) $$\tilde{a}=\tilde{n}^*(x', y, D')\tilde{n}(x', y, D')-\sum t_j(x', y, D_{x'})s_j(x', y, D_{x'})$$
$$+(\text{first order term}).$$

Consider the family of local transformations of type (2.1) such that the union of the corresponding $\{\omega_k\}$ covers S, and take a suitable partition of unity $\sum\eta_j^2(x)=1$ on $\bar{\Omega}$. If the support of η_j contains a part of boundary, the local form of $a\eta_j^2$ is

(2.11) $$\widetilde{a\eta_j^2}=\eta_j^2 n^*(x', y, D')n(x', y, D')$$
$$-\sum_l(t_l(x', y, D_{x'})\eta_j)(s(x', y, D_{x'})\eta_j)+(\text{first order})$$

For $\eta_j(x)$ in $\mathcal{D}(\Omega)$, $a\eta_j^2(x)$ are products of tangential operators on S. Summation of (2.11) gives (2.3) in some neighbourhood of S. (q. e. d.)

Remark. Let $n_k(x, D)$ be the operator $n(x, D)$ corresponding to the operator a_k, then at the boundary S, using $\cos(\nu,x_i)=\frac{-\partial\Phi}{\partial x_i}\Big/\left|\frac{\partial\Phi}{\partial x}\right|$ we have the following relation:

(2.12) $$\frac{\partial}{\partial n_k}=(\sum_{ij}a_{k,ij}(x)\cos(\nu, x_i)\cos(\nu, x_j))^{1/2}n_k(x, D),\quad \text{on } S.$$

The following lemma concerning the relation between a_k is just a characterization of hypothesis (H).

Lemma 2. *Assume* (H), *then we have*

1) $\dfrac{\partial}{\partial n_i}=\beta_i(x)\dfrac{\partial}{\partial n_1}$ *on* S, *where* $\beta_i(x)=\dfrac{\alpha_i(x)}{\alpha_1(x)}$. $\alpha_i(x)$ $(i=1,2,3)$ *are* $\alpha(x)$ *corresponding to the operator* a_i. $\beta_i(x)$ *are defined in* Γ.

2) *If* u *belongs to* $H^3\cap\mathcal{D}^1_{L^2}$, $(a_i-\beta_i(x)a_1)u$ *belong to* $\mathcal{D}^1_{L^2}$ $(i=2,3)$.

Proof. We fix the local transformation (2.1) corresponding to a_1. After this transformation, a_1 and a_2 take the following forms

$$(2.13)\quad \begin{cases} -\tilde{a}_1=\alpha_1(x',y)\dfrac{\partial^2}{\partial y^2}+2\sum\limits_{i=1}^{n-1}b_{1i}(x',y)\dfrac{\partial}{\partial x'_i}\dfrac{\partial}{\partial y}+c_1(x',y)\dfrac{\partial}{\partial y} \\ \qquad +t_1(x',y,D') \\ -\tilde{a}_2=\alpha'_2(x',y)\dfrac{\partial^2}{\partial y^2}+2\sum\limits_{i=1}^{n-1}b_{2i}(x',y)\dfrac{\partial}{\partial x'_i}\dfrac{\partial}{\partial y}+c_2(x',y)\dfrac{\partial}{\partial y} \\ \qquad +t_2(x',y,D'_x), \end{cases}$$

where $b_{1i}(x',0)\equiv 0$, and $t_k(x',y,D_{x'})$ $(k=1,2)$ do not contain $\dfrac{\partial}{\partial y}$. $\alpha'_2(x,y)$ means $\sum\limits_{ij}a_{2ij}\dfrac{\partial\Phi_1}{\partial x_i}\dfrac{\partial\Phi_1}{\partial x_j}$. Here Φ_i is Φ which comes from a_i $(i=1$ or $2)$. Then let us prove the following facts:

$$(2.14)\qquad b_{2i}(x',0)\equiv 0$$

$$(2.15)\qquad c_2(x',0)=\frac{\alpha'_2}{\alpha_1}c_1(x',0).$$

(2.14) means that the conormal directions of a_1 and a_2 are same. Therefore the local transformations (2.1) corresponding to a_1 and a_2 are same ones. So we have $\Phi_1=\Phi_2$ and $\alpha'_2=\alpha_2$. Thus we obtain 1). By (2.14) and (2.15), we can see 2). Now let's prove (2.14) and (2.15) in the following two steps.

(1) First step (localization). Assume that $u\in H^3$ satisfies $\gamma u=\gamma a_1 u=0$ in $\omega\cap S$. (ω is one of $\{\omega_k\}$ and γ is the trace operator.) Then we are going to prove that for any compact set K in $\omega\cap S$, there exists a function u_K defined in $\bar{\Omega}$ such that $u_K=u$ in some neighbourhood of K, and u_K, a_1u_K belong to $\mathcal{D}^1_{L^2}$. Consider the map (2.1)

and we can assume that S is a hyperplane $y=0$ and a_1 has the form (2.13).

Take the following C^∞-function $\psi(x')$ in x'-space defined as follows:

$\psi(x')=1$ in some neighbourhood of K, $\psi(x')=0$ in some neighbourhood of $C(\omega\cap S)$. Put $g=a_1\psi(x')u$, then the support of γg is in $\omega\cap S-K$.

$$v_K=\frac{1}{2\alpha_1(x',y)}y^2\gamma g+\psi(x')u$$

satisfies $\gamma a_1 v_K=\gamma v_K=0$, and the support of v_K is in $\omega\times R^1(y)$. Take a C^∞ function $\varphi(y)$ taking value 1 in a small neighbourhood of 0 and vanishing outside of some neighbourhood of 0. By inverse transformation of $\varphi(y)v_K$, one can yield a function u_K which satisfies the desired conditions.

(2) Second step. For a_k in (2.13), let us put

$$d=\frac{\alpha_2'(x',y)}{\alpha_1(x',y)}a_1-a_2=\left(\sum_i^{n-1}d_{1j}(x',y)\frac{\partial}{\partial x_i'}+d_2(x',y)\right)\frac{\partial}{\partial y}$$
$$+d_3(x',y,D_{x'}).$$

$d_3(x',y,D_{x'})$ does not contain $\dfrac{\partial}{\partial y}$. From the assumption (H) and First step, $\gamma du=0$ must hold for the function u satisfying $\gamma a_1 u=\gamma u=0$.

Now consider

$$u=-\frac{c_1(x',0)}{2\alpha_1(x',0)}y^2+y,$$

Then $\gamma du=d_2(x',0)\equiv 0$ follows from $\gamma u=\gamma a_1 u=0$. Take

$$u=x_i'y-\frac{b_{2i}(x',0)}{\alpha_1(x',0)}y^2,$$

then we have $\gamma du=d_{1i}(x',0)\equiv 0$. Thus (2.14) and (2.15) follow.

(q. e. d.)

Now we explain the common method in the proofs of Lemmas 3～6. We use the local transformations of type (2.1) and a suitable partition of unity $\sum\eta_j^2=1$ on $\overline{\Omega}$ corresponding to the covering $\{\omega_k\}$ of S. Then the proofs of Lemma 3~6 are reduced to those of in the domain $\sum$ and for function u with small support satisfying some conditions on $y=0$. Let us rewrite (x', y) by $(x_1, \cdots, x_{n-1}, y)$. In the proofs $(\cdot, \cdot)$ and $\|\cdot\|$ mean $(\cdot, \cdot)_{L^2(\Sigma)}$ and $\|\cdot\|_{L^2(\Sigma)}$, respectively and $\|\cdot\|_p$ means $\|\cdot\|_{p,L^2(\Sigma)}$ $(p=1, 2, 3)$.

Lemma 3. *Let a_1, a_2 be uniformly elliptic operators. Then there exist positive constants δ and r such that*

$$\mathrm{Re}(a_1u, a_2u)+r\|u\|^2\geq\delta\|u\|_2^2 \tag{2·16}$$

for $u\in H^2\cap\mathcal{D}_{L^2}^1$ or for $u\in\left\{u\in H^2:\left(\frac{\partial}{\partial n_2}+\sigma(s)\right)u|_s=0\right\}$.

Proof. Consider the local transformations of type (2.1) corresponding to a_2. It suffices to prove the following inequality for the functions u satisfying $u|_{y=0}=0$ or $\left(\frac{\partial}{\partial y}+c(x)\right)u|_{y=0}=0$, $c(x)$ being a smooth function determined by $\sigma(s)$ and $a_{2ij}(x)$:

$$\begin{aligned} I=\mathrm{Re}((D_y^2+2\sum_i^{n-1}b_{1i}D_{x_i}D_y+\sum_{ij}^{n-1}c_{1ij}D_{x_i}D_{x_j})u,\\ (D_y^2+\sum_{ij}^{n-1}c_{2ij}D_{x_i}D_{x_j})u)_{L^2(\Sigma)}\geq\delta\|u\|_{2,L^2(\Sigma)}^2-r\|u\|_0^2, \quad \text{where}\\ D_y=\frac{1}{i}\left(\frac{\partial}{\partial y}+c(x)\right) \quad D_{x_j}=\frac{1}{2\pi i}\frac{\partial}{\partial x_j} \quad (j=1, 2, \cdots, n-1).\end{aligned} \tag{2.17}$$

We can assume that the coefficients are constants in (2.15), taking account of the fact that the oscillations of the coefficients are small. Then by Green's formula and each boundary condition we have

$$\mathrm{Re}(\sum_{ij}^{n-1}c_{1ij}D_{x_i}D_{x_j}u, D_y^2u)\equiv\sum_{ij}^{n-1}\mathrm{Re}(c_{1ij}D_{x_i}D_yu, D_{x_j}D_yu) \tag{2.18}$$

$$\mathrm{Re}(D_y^2u, \sum_{ij}^{n-1}c_{2ij}D_{x_i}D_{x_j}u)\equiv\mathrm{Re}\sum_{ij}^{n-1}(c_{2ij}D_{x_i}D_yu, D_{x_j}D_yu), \tag{2.19}$$

where we have used the following notation; for bi-linear forms $A[u, v]$, $B[u, v]$ defined on some subspaces of H^1, $A\equiv B$ means that

$|A[u,u]-B[u,u]|\leq\varepsilon\|u\|_2^2+\gamma\|u\|_0^2$ holds for arbitrally small positive ε when γ is sufficiently large. Putting

$$I_1=\iint|D_y^2u|^2dxdy+\iint2\sum_i^{n-1}b_{1i}D_{x_i}D_yu\,\overline{D_y^2u}\,dxdy \tag{2.20}$$
$$+\sum_{ij}^{n-1}\iint c_{1ij}D_{x_j}D_yu\,\overline{D_{x_j}D_yu}\,dxdy,$$

$$I_2=\sum_{kl}^{n-1}\Big\{\iint c_{2kl}D_{x_k}D_yu\overline{D_{x_l}D_yu}\,dxdy \tag{2.21}$$
$$+\iint2\sum_i^{n-1}b_{1i}D_{x_i}D_yu\,\overline{c_{2kl}D_{x_k}D_{x_l}u}\,dxdy$$
$$+\iint\sum_{ij}^{n-1}c_{1ij}D_{x_i}D_{x_l}u\,\overline{c_{2kl}D_{x_k}D_{x_l}u}\,dxdy,$$

then we have $I\equiv\mathrm{Re}\,I_1+\mathrm{Re}\,I_2$. Consider the Fourier transformation with respect to $(x_1,\cdots,x_{n-1})$ and Plancherel's equality, then we can see

$$I_1=\iint\{|\mathcal{F}_x(D_y^2u)|^2+2\sum_i^{n-1}\xi_i\mathcal{F}_x(D_yu)\overline{\mathcal{F}_x(D_y^2u)} \tag{2.22}$$
$$+\sum_{ij}^{n-1}c_{1ij}(\xi_i\mathcal{F}_x(D_yu))\overline{(\xi_j\mathcal{F}_x(D_yu))}\}\,d\xi dy$$

From the ellipticity of a_1, the following inequality holds

$$\mathrm{Re}\{|\tau|^2+2\sum_i^{n-1}b_{1i}\lambda_i\bar{\tau}-\sum_{ij}^{n-1}c_{1ij}\lambda_i\bar{\lambda}_j\}\geq\delta\{|\tau|^2+\sum|\lambda_j|^2\} \tag{2.23}$$

for complex numbers τ, λ_i $(i=1,\cdots,n-1)$.

By (2.22), (2.23) we have

$$\mathrm{Re}\,I_1\geq\delta\Big\{\iint|\mathcal{F}_x(D_y^2u)|^2d\xi dy+\iint\sum_i^{n-1}|\xi_i\mathcal{F}_x(D_yu)|^2d\xi\,dy\Big\} \tag{2.24}$$
$$\geq\frac{1}{4\pi}\delta\left\|\frac{\partial}{\partial y}u\right\|_1^2-r\|u\|_1^2.$$

In the similar way we have

$$\mathrm{Re}\,I_2\geq\delta\iint(\sum_{k,l}^{n-1}c_{2kl}\xi_k\xi_l)\{|\mathcal{F}_x(D_yu)|^2+\sum_i^{n-1}|\xi_i\mathcal{F}_x(u)|^2\}\,d\xi dy \tag{2.25}$$
$$\geq\left(\frac{\delta}{4\pi}\right)^2\sum_i^{n-1}\left\|\frac{\partial}{\partial x_i}u\right\|_1^2-r\|u\|_1^2.$$

From (2.24) and (2.25) we obtain (2.17) for another $\delta>0$. (q.e.d.)

Remark. Lemma 3 holds for $u\in\left\{u\in H^2;\ \left(\frac{\partial}{\partial n}+\frac{\partial}{\partial\tau}(s)+\sigma(s)\right)u\Big|_s=0\right\}$, where $\frac{\partial}{\partial\tau}(s)$ means a tangential derivative smoothly depending on $s\in S$, and $\frac{\partial}{\partial n}$ is the normal derivative.

Because (2.18) and (2.19) hold for the functions u satisfying $\left(\frac{\partial}{\partial y}+\sum_i^{n-1}\sigma_i(x)\frac{\partial}{\partial x_i}+c(x)\right)u\Big|_{y=0}=0$. In fact for u satisfying $(D_y+t(x, D))u\Big|_{y=0}=0$ we have the following relations:

$$
\begin{aligned}
&\mathrm{Re}(c_{1ij}D_{x_i}D_{x_j}u, D_y^2u)=\mathrm{Re}(c_{1ij}D_{x_i}D_{x_j}u, D_y(D_y+t(x, D)u)\\
&\qquad-\mathrm{Re}(c_{1ij}D_{x_i}D_{x_j}u, D_yt(x, D)u)\\
&\equiv\mathrm{Re}(c_{1ij}D_{x_j}D_yu, D_{x_j}D_yu)+\mathrm{Re}(c_{1ij}D_{x_j}D_yu, D_{x_j}t(x, D)u)\\
&\qquad-\mathrm{Re}(c_{1ij}D_{x_i}D_{x_j}u, D_yt(x, D)u)\\
&\equiv\mathrm{Re}(c_{1ij}D_{x_i}D_yu, D_{x_j}D_yu)+\{\mathrm{Re}(D_yt(x, D)u, c_{1ij}D_{x_i}D_{x_j}u)\\
&\qquad-\mathrm{Re}(c_{1ij}D_{x_i}D_{x_j}u, D_yt(x, D)u)\}\\
&=\mathrm{Re}(c_{1ij}D_{x_i}D_yu, D_{x_j}D_yu).
\end{aligned}
$$

With respect to (2.19), the same argument can be used.

Such a type of inequality as in Lemma 3 has never been mentioned before. It would be interesting to use that inequality in another case without the assumption (H).

Lemma 4. *Under the assumption* (H) *we have the following inequality*:

$$
\mathrm{Re}\sum_{ij}^{n}\left(a_{2ij}(x)\frac{\partial}{\partial x_j}a_1u,\ \frac{\partial}{\partial x_i}a_3u\right)\geq\delta\|u\|_3^2-r\|u\|_2^2 \tag{2.26}
$$

for $u\in D(a_1)$ *or for* $u\in H^3\cap N$.

Proof. Since $H^4\cap D(a_1)$ and $N(a_1)$ are dense in $D(a_1)$ and $H^3\cap N$ respectively, it suffices to prove (2.26) for $H^4\cap D(a_1)$ or $N(a_1)$. Now let us introduce the similar notation to that of Lemma 3:

$$
A\equiv B \text{ means } |A[u, u]-B[u, u]|\leq\varepsilon\|u\|_3^2+r\|u\|_0^2. \tag{2.27}
$$

$a_k(k=1, 2, 3)$ have the following forms in Σ:

$$(2.28)\qquad -a_k=\alpha_k(x, y)\frac{\partial^2}{\partial y^2}+2\sum_{i=1}^{n-1}b_{ki}(x, y)\frac{\partial}{\partial x_i}\frac{\partial}{\partial y}+\sum_{i,j}^{n-1}c_{k,ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}+d_k(x)\frac{\partial}{\partial y}+\cdots,$$

where $b_{ki}(x, 0)\equiv 0$ because of the assumption (H).

Boundary conditions $u|_{y=0}=a_1u|_{y=0}=0$ are equivalent to $u|_{y=0}=0$ and $u|_{y=0}=\left(\frac{\partial}{\partial y}+d(x)\right)\frac{\partial}{\partial y}u|_{y=0}=0$, $d(x)$ being a smooth function. Another boundary condition is $\left(\frac{\partial}{\partial y}+c(x)\right)u|_{y=0}=0$. In each case, we have the following, assuming $\alpha_2(x, y)\equiv 1$ and using the above notation;

$$(2.29)\qquad \mathrm{Re}(a_2a_1u, a_3u)_{L^2(\Sigma)}\equiv \mathrm{Re}\left(\frac{\partial}{\partial y}a_1u, \frac{\partial}{\partial y}a_3u\right)+\mathrm{Re}\sum_{ij}\left(c_{2;j}(x, y)\frac{\partial}{\partial x_i}a_1u, \frac{\partial}{\partial x_j}a_2u\right)=J_1+J_2.$$

Here we can assume that the coefficients are constants as before.

Then the proof of

$$(2.30)\qquad \mathrm{Re}(a_2a_1u, a_3u)\geq\delta\|u\|_3^2-r\|u\|_0^2$$

is reduced to that of Lemma 3. In fact

$$(2.31)\qquad J_1\equiv\mathrm{Re}\left(a_1\frac{\partial}{\partial y}u, a_3\frac{\partial}{\partial y}u\right)\geq\left\|\frac{\partial}{\partial y}u\right\|_2^2-r\left\|\frac{\partial}{\partial y}u\right\|_0^2$$

follows in the first case: $\left(\frac{\partial}{\partial y}+d(x)\right)\frac{\partial}{\partial y}u|_{y=0}=0$. In the second case

$$(2.32)\qquad J_1\equiv\mathrm{Re}\left(a_1\left(\frac{\partial}{\partial y}+c(x)\right)u, a_3\left(\frac{\partial}{\partial y}+c(x)\right)u\right)\geq\delta\left\|\left(\frac{\partial}{\partial y}+c(x)\right)u\right\|_2^2-r\left\|\left(\frac{\partial}{\partial y}+c(x)\right)u\right\|_0^2$$

holds. For J_2, following the same process of argument as in Lemma 3, we have

$$(2.34)\qquad J_2\geq\delta\sum_i^{n-1}\left\|\frac{\partial}{\partial x_i}u\right\|_2^2-r\|u\|_0^2$$

From (2. 31), (2. 32) and (2. 34), (2. 30) follows. Thus we have

$$\text{(2. 35)} \qquad \mathrm{Re}(a_2 a_1 u, a_3 u)_{L^2(\Omega)} \geq \delta \|u\|^2_{3L^2(\Omega)} - r\|u\|^2_{L^2(\Omega)}.$$

This inequality means (2. 26) for $u \in H^4 \cap D(a_1)$ or $u \in N(a_1)$. (q.e.d.)

Lemma 5. *Under the assumption* (H) *there exists a positive constant* C *such that*

$$\text{(2. 36)} \qquad |(a_1 u_1, a_2 u_2) - (a_2 u_1, a_1 u_2)| \leq C\|u_1\|_2 \|u_2\|_1$$

for $u_1, u_2 \in N$ *or* $u_1, u_2 \in H^2 \cap \mathcal{D}^1_{L^2}$.

Proof. Here we use the following notation:

$$A \equiv B \text{ means } |A[u_1, u_2] - B[u_1, u_2]| \leq C\|u_1\|_2 \|u_2\|_1.$$

(1) For $u_1 \in H^4 \cap D(a_1)$ and $u_2 \in H^2 \cap \mathcal{D}^1_{L^2}$, we have

$$(a_1 u_1, a_2 u_2) = (a_2^* a_1 u_1, u_2) \equiv (a_1^* a_2 u_1, u_2) = (a_2 u_1, a_1 u_2).$$

Considering the fact that $H^4 \cap D(a_1)$ is dense in $H^2 \cap \mathcal{D}^1_{L^2}$, we have (2. 36) for $u_1, u_2 \in H^2 \cap \mathcal{D}^1_{L^2}$.

(2) Let us prove (2. 36) for $u_1 \in H^3 \cap N$ and $u_2 \in N$. For the functions u_1 and u_2 satisfying $\left(\frac{\partial}{\partial y} + c(x)\right) u_i \Big|_{y=0} = 0$ $(i = 1, 2)$, we have

$$\text{(2. 37)} \qquad \begin{aligned} &\left(\frac{\partial^2}{\partial y^2} u_1, \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} u_2\right)_{L^2(\Sigma)} \\ &\equiv \left(\left(\frac{\partial}{\partial y} + c(x)\right)\left(\frac{\partial}{\partial y} + c(x)\right) u_1, \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} u_2\right) \\ &\equiv -\left(\left(\frac{\partial}{\partial y} + c(x)\right)\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial y} + c(x)\right) u_1, \frac{\partial}{\partial x_k} u_2\right) \\ &\equiv \left(\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial y} + c(x)\right) u_1, \frac{\partial}{\partial x_k}\left(\frac{\partial}{\partial y} + c(x)\right) u_2\right) \\ &\equiv -\left(\left(\frac{\partial}{\partial y} + c(x)\right)\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} u_1, \left(\frac{\partial}{\partial y} + c(x)\right) u_2\right) \\ &\equiv \left(\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} u_1, \frac{\partial^2}{\partial y^2} u_2\right), \end{aligned}$$

and in the same way,

$$(2.38)\qquad \left(\frac{\partial^2}{\partial y^2}u_1,\ \frac{\partial}{\partial x_i}\frac{\partial}{\partial y}u_2\right)\equiv\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial y}u_1,\ \frac{\partial^2}{\partial y^2}u_2\right).$$

From (2.37) and (2.38), (2.36) follows for $u_1\in H^3\cap N$, $u_2\in N$. This completes the proof of Lemma 5.

Lemma 6. *Under the assumption* (H) *there exists a positive constant C such that*

$$(2.39)\qquad |(a_1u_1,\ a_2a_3u_0)-(a_2u_1,\ a_1a_3u_0)|\leq C\|u_1\|_2\|u_0\|_3$$

for $u_0\in H^4\cap D(a_1)$, $u_1\in D(a_1)$, or $u_0\in N(a_1)$, $u_1\in H^3\cap N$.

Proof. For $u_0\in H^4\cap D(a_1)$, $u_1\in D(a_1)$, (2.39) follows immediately from Lemma 5. Here we use the notation; $A\equiv B$ if $|A[u_1, u_2]-B[u_1, u_2]|\leq C\|u_1\|_2\|u_0\|_3$ holds. In order to apply Lemma 2 (ii), we decompose a_3 as follows:

$$(2.40)\qquad a_3=\beta_3(x)a_1+(a_3-\beta_3(x)a_1),$$

where $\beta_3(x)$ is a smooth function obtained by the prolongation of $\dfrac{\alpha_3(x)}{\alpha_1(x)}$ defined in Γ. For $u_0\in N(a_1)$ and $u_1\in H^3\cap N$, using Lemma 5 we have

$$\begin{aligned}(2.41)\qquad &(a_1u_1,\ a_2(\beta_3(x)a_1u_0))-(a_2u_1,\ a_1(\beta_3(x)a_1u_0))\\ &\equiv(a_1u_1,\ \beta_3(x)a_2(a_1u_0))-(a_2u_1,\ \beta_3(x)a_1(a_1u_0))\equiv0,\end{aligned}$$

and by Lemma 2 (ii), the following relations hold:

$$\begin{aligned}(2.42)\qquad &\left(\frac{\partial^2}{\partial y^2}u_1,\ \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}(a_3-\beta_3(x)a_1)u_0\right)\\ &\equiv-\left(\frac{\partial}{\partial y}\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial y}+c(x)\right)u_1,\ \frac{\partial}{\partial x_k}(a_3-\beta_3(x)a_1)u_0\right)\\ &\equiv\left(\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial y}+c(x)\right)u_1,\ \frac{\partial}{\partial x_k}(a_3-\beta_3(x)a_1)\left(\frac{\partial}{\partial y}+c(x)\right)u_0\right)\\ &\equiv\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}u_1,\ \frac{\partial}{\partial y}(a_3-\beta_3(x)a_1)\left(\frac{\partial}{\partial y}+c(x)\right)u_0\right)\\ &\equiv\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}u_1,\ \frac{\partial^2}{\partial y^2}(a_3-\beta_3(x)a_1)u_0\right),\end{aligned}$$

$$(2.43)\qquad \left(\frac{\partial^2}{\partial y^2}u_1,\ \frac{\partial}{\partial x_i}\frac{\partial}{\partial y}(a_3-\beta_3(x)a_1)u_0\right)$$
$$\equiv\left(\frac{\partial}{\partial x_i}\frac{\partial}{\partial y}u_1,\ \frac{\partial^2}{\partial y^2}(a_3-\beta_3(x)a_1)u_0\right)$$

for $i, k=1, \cdots, n-1$.

From (2.41), (2.42) and (2.43), we obtain (2.39). (q. e. d.)

§3. Evolution equation and a priori estimate

We introduce the following hermitian form in $\mathcal{H}_1$ defined by

$$(3.1)\qquad (U, V)_{\mathcal{H}_1}=\sum_{ij}^{n}\left\{\left(a_{2,ij}(x)\frac{\partial}{\partial x_j}a_1u_0,\ \frac{\partial}{\partial x_i}a_3v_0\right)\right.$$
$$\left.+\left(a_{2,ij}(x)\frac{\partial}{\partial x_j}a_3u_0,\ \frac{\partial}{\partial x_i}a_1v_0\right)\right\}+r(u_0, v_0)$$
$$+\{(a_2u_1, a_3v_1)+(a_3u_1, a_2v_1)+r(u_1, v_1)\}$$
$$+\left\{2\sum_{ij}^{n}\left(a_{3,ij}(x)\frac{\partial}{\partial x_i}u_2,\ \frac{\partial}{\partial x_j}v_2\right)+r(u_2, v_2)\right\}+2(u_3, v_3)$$

for $U=(u_0, u_1, u_2, u_3),\ V=(v_0, v_1, v_2, v_3)\in\mathcal{H}_1$,

where r is a positive constant.

In the case II we use the hermitian form of the following type;

$$(3.2)\qquad (U, V)_{\mathcal{H}_2}=[u_0, v_0]+\{(a_2u_1, a_3v_1)+(a_3u_1, a_2v_1)+r(u_2, v_1)\}$$
$$+\left\{2\sum_{ij}^{n}\left(a_{3,ij}(x)\left(\frac{\partial}{\partial x_j}+\sigma_j\right)u_2,\ \left(\frac{\partial}{\partial x_i}+\sigma_i\right)v_2\right)+r(u_2, v_2)\right\}$$
$$+2(u_3, v_3).$$

By the analogy to Case I it would be natural to take the following hermitian form for $[u_0, v_0]$, using the decomposition of a_2 in view of Lemma 1:

$$((n_2+\rho)a_1u_0, (n_2+\rho)a_3v_0)+((n_2+\rho)a_3u_0, (n_2+\rho)a_1v_0)$$
$$+\sum_j(t_{2j}a_1u_0, s_{2j}a_3v_0)+\sum_j(s_{2j}a_3u_0, t_{2j}a_1v_0)+r(u_0, v_0).$$

However for this form the calculus by integration by parts concerning $(AU, U)_{\mathcal{H}_2}+(U, AU)_{\mathcal{H}_2}$ does not work well. For boundary integrals can not be estimated by $C\|U\|^2_{\mathcal{H}_2}$. Taking account of the fact that

$(a_3-\beta_3 a_1)(n_2+\rho)u_0$ and $(n_2+\rho)a_1 u_0$ vanish at the boundary for $u_0 \in N(a_1)$ in view of Lemma 2, we introduce the following hermitian form:

$$(3.3)\qquad [u_0, v_0]=((n_2+\rho)a_1u_0, \gamma_3(x, D)v_0)+(\gamma_3(x, D)u_0, (n_2+\rho)a_1v_0)$$
$$+\sum_j\{(t_{2j}a_1u_0, s_{2j}a_3v_0)+(s_{2j}a_3u_0, t_{2j}a_1v_0)\}+r(u_0, v_0),$$

where

$$(3.4)\qquad \gamma_3(x, D)=(a_3-\beta_3a_1)(n_2+\rho)+\beta_3(n_2+\rho)a_1\,.$$

Here $\sigma_i(x)$ $(i=1, 2, \cdots, n)$ and $\rho(x)$ appearing (3.2) and (3.3), are arbitrary sufficiently smooth functions satisfying on S the following conditions:

$$(3.5)\qquad \sum_{ij} a_{1,ij}\cos(\nu x_i)\sigma_j(x)=\sigma(s) \qquad \text{on} \quad S.$$

$$(\sum_{ij} a_{2,ij}\cos(\nu, x_i)\cos(\nu, x_j))^{1/2}(\sum_{ij} a_{1,ij}\cos(\nu, x_i)\cos(\nu, x_j))^{-1}$$
$$\times\sigma(s)=\rho(s), \quad \text{on } S. \quad \text{(in view of (2.12) and Lemma 2, 1))}$$

Then the following relations hold on S:

$$(3.6)\qquad \frac{\partial}{\partial n_3}+\sum a_{3,ij}\cos(\nu, x_i)\sigma_j=\beta_3\left(\frac{\partial}{\partial n_1}+\sum a_{1,ij}\cos(\nu, x_i)\sigma_j\right)$$
$$=\beta_3\left(\frac{\partial}{\partial n_1}+\sigma\right)$$

$$(3.7)\qquad n_2+\rho(s)=(\sum_{ij} a_{2,ij}(x)\cos(\nu, x_i)\cos(\nu, x_j))^{-1/2}\frac{\partial}{\partial n_2}+\rho(s)$$
$$=(\sum_{ij} a_{2,ij}(x)\cos(\nu, x_i)\cos(\nu, x_j))^{1/2}$$
$$\times(\sum_{ij} a_{1,ij}(x)\cos(\nu, x_i)\cos(\nu, x_j))^{-1}\left(\frac{\partial}{\partial n_1}+\sigma\right).$$

By virtue of Lemma 3 and Lemma 4, there exists a positive constant C such that

$$(3.8)\qquad \frac{1}{C}\|U\|^2\leq(U, U)_{\mathcal{H}_i}\leq C\|U\|^2 \quad (i=1, 2) \quad \text{for } U\in\mathcal{H}_i.$$

In fact, from Lemma 4 we have

Corollary. Under the same assumption as in Lemma 4, we have

(3.9) $C\|u_0\|_3^2 \geq \operatorname{Re}(a_2 a_1 u_0, a_3 u_0) + r\|u_0\|^2 \geq \delta\|u_0\|_3^2$, for $u_0 \in N(a_1)$.

Therefore considering that $N(a_1)$ is dense in $H^3 \cap N$ and using the decomposition of a_2, we have

$$\frac{1}{C}\|u_0\|_3^2 \leq [u_0, u_0] \leq C\|u_0\|_3^2 \quad \text{for } u_0 \in H^3 \cap N.$$

Hence $(U, V)_{\mathscr{H}_i}$ $(i=1,2)$ are positive hermitian forms.

Now we show the following estimate using Lemmas 5 and 6.

$$|(AU, U)_{\mathscr{H}_i} + (U, AU)_{\mathscr{H}_i}| \leq C\|U\|^2 \quad \text{for all } U \in D(A)_i \quad (i=1,2) \tag{3.10}$$

Case I. Substitute $U=(u_0, u_1, u_2, u_3)$ and $AU=(u_1, -a_1 u_0 + u_2, -a_2 u_1 + u_3, -a_3 u_2)$ to (3.1). Let us note $A \equiv B$ if $|A[U, U] - B[U, U]| \leq C\|U\|_{\mathscr{H}_i}^2$. $(i=1,2)$

$$\begin{aligned}(3.11)\quad (U, AU)_{\mathscr{H}_1} + (AU, U)_{\mathscr{H}_1} \equiv & \sum_{ij}\Big\{\Big(a_{2,ij}\frac{\partial}{\partial x_j}a_1 u_0, \frac{\partial}{\partial x_i}a_3 u_1\Big) \\ & + \Big(a_{2,ij}(x)\frac{\partial}{\partial x_j}a_3 u_0, \frac{\partial}{\partial x_i}a_1 u_1\Big) + \Big(a_{2,ij}(x)\frac{\partial}{\partial x_j}a_1 u_1, \frac{\partial}{\partial x_i}a_3 u_0\Big) \\ & + \Big(a_{2,ij}(x)\frac{\partial}{\partial x_j}a_3 u_1, \frac{\partial}{\partial x_i}a_1 u_0\Big)\Big\} \\ & + \{(a_2 u_1, a_3(-a_1 u_0)) + (a_3 u_1, a_2(-a_1 u_0)) \\ & + (a_2(-a_1 u_0), a_3 u_1) + (a_3(-a_1 u_0), a_2 u_1)\} \\ & + \{(a_2 u_1, a_3 u_2) + (a_3 u_1, a_2 u_2) + (a_2 u_2, a_3 u_1) + (a_3 u_2, a_2 u_1)\} \\ & + 2\sum_{i,j}\Big\{\Big(a_{3,ij}(x)\frac{\partial}{\partial x_j}u_2, \frac{\partial}{\partial x_i}(-a_2 u_1)\Big) \\ & + \Big(a_{3,ij}(x)\frac{\partial}{\partial x_j}(-a_2 u_1), \frac{\partial}{\partial x_i}u_2\Big)\Big\} \\ & + 2\sum_{ij}\Big\{\Big(a_{3,ij}(x)\frac{\partial}{\partial x_j}u_2, \frac{\partial}{\partial x_i}u_3\Big) + \Big(a_{3,ij}(x)\frac{\partial}{\partial x_j}u_3, \frac{\partial}{\partial x_i}u_2\Big)\Big\} \\ & + 2\{(u_3, -a_3 u_2) + (-a_3 u_2, u_3)\}\end{aligned}$$

holds. Since U belongs to $D(A)_1$,

(3.12) $\quad a_k u_0,\ a_k u_1\ (k=1,2,3)$ and $u_n\ (n=0,1,2,3)$

vanish at the boundary. Considering these properties we apply the

Green's formula, then the boundary integrals do not appear. Now remark the following (1), (2)

$$(1)\qquad \sum_{ij}\left(a_{3,ij}(x)\frac{\partial}{\partial x_j}u_2,\ \frac{\partial}{\partial x_i}(-a_2u_1)\right)+(a_2u_2,\ a_3u_1)$$

$$\equiv(a_2u_2,\ a_3u_1)-(a_3u_2,\ a_2u_1)\equiv 0\quad \text{(in view of Lemma 5)},$$

$$(2)\qquad \sum_{ij}\left(a_{2,ij}(x)\frac{\partial}{\partial x_j}a_3u_0,\ \frac{\partial}{\partial x_i}a_1u_1\right)+(a_3(-a_1u_0),\ a_2u_1)$$

$$\equiv(a_2a_3u_0,\ a_1u_1)-(a_3a_1u_0,\ a_2u_1)$$

$$\equiv(a_2a_3u_0,\ a_1u_1)-(a_1a_3u_0,\ a_2u_1)\equiv 0.\quad \text{(in view of Lemma 6)}$$

As the other couples can be estimated more easily, we obtain (3.10).

Case II.

$$(3.13)\qquad (AU,\ U)_{\mathcal{H}_2}+(U,\ AU)_{\mathcal{H}_2}\equiv[u_1,\ u_0]+[u_0,\ u_1]$$

$$+\{(a_2u_1,\ a_3(-a_1u_0))+(a_3u_1,\ a_2(-a_1u_0))$$

$$+(a_2(-a_1u_0),\ a_3u_1)+(a_3(-a_1u_0),\ a_2u_1)\}$$

$$+\{(a_2u_1,\ a_3u_2)+(a_3u_1,\ a_2u_2)+(a_2u_2,\ a_3u_1)+(a_3u_2,\ a_2u_1)\}$$

$$+2\sum_{ij}\left\{\left(a_{3,ij}\left(\frac{\partial}{\partial x_j}+\sigma_j\right)u_2,\ \left(\frac{\partial}{\partial x_i}+\sigma_i\right)(-a_2u_1)\right)\right.$$

$$\left.+\left(a_{3,ij}\left(\frac{\partial}{\partial x_j}+\sigma_j\right)(-a_2u_1),\ \left(\frac{\partial}{\partial x_i}+\sigma_i\right)u_2\right)\right\}$$

$$+2\sum_{ij}\left\{\left(a_{3,ij}\left(\frac{\partial}{\partial x_j}+\sigma_j\right)u_2,\ \left(\frac{\partial}{\partial x_i}+\sigma_i\right)u_3\right)\right.$$

$$\left.+\left(a_{3,ij}\left(\frac{\partial}{\partial x_j}+\sigma_j\right)u_3,\ \left(\frac{\partial}{\partial x_i}+\sigma_i\right)u_2\right)\right\}$$

$$+2(u_3,\ -a_3u_2)+2(-a_3u_2,\ u_3).$$

From $U\in D(A)_2$, $\left(\frac{\partial}{\partial n_1}+\sigma\right)u_k$ $(k=0,\ 1,\ 2)$, $(n_2(x,\ D)+\rho)a_1u_0$ $\gamma_3(x,\ D)u_0$ vanish at the boundary. Let us remark the followings:

$$(1)\qquad \left(a_{3,ij}\left(\frac{\partial}{\partial x_j}+\sigma_i\right)u_2,\ \left(\frac{\partial}{\partial x_i}+\sigma_i\right)u_3\right)$$

$$\equiv\left(-\frac{\partial}{\partial x_i}\left(a_{3,ij}\frac{\partial}{\partial x_j}\right)u_2,\ u_3\right)+\int\left(\frac{\partial}{\partial n_3}+\beta_3\sigma\right)u_2\bar{u}_3\,ds$$

$$\equiv(a_3u_2,\ u_3)+\int\beta_3\left(\frac{\partial}{\partial n_1}+\sigma\right)u_2\bar{u}_3\,ds=(a_3u_2,\ u_3)\quad \text{(see (3.5))}$$

(2) $\left(a_{3,ij}\left(\dfrac{\partial}{\partial x_j}+\sigma_j\right)u_2,\ \left(\dfrac{\partial}{\partial x_i}+\sigma_i\right)(-a_2u_1)\right)\equiv(a_3u_2,\ -a_2u_1)$

$\equiv-(a_2u_2,\ a_3u_1)$ (in view of Lemma 5)

(3) $[u_0, u_1]\equiv((n_2+\rho)a_1u_0, \gamma_3u_1)+(\gamma_3u_0, (n_2+\rho)a_1u_1)$

$+\sum_j\{(t_{2j}a_1u_0, s_{2j}a_3u_1)+(s_{2j}a_3u_0, t_{2j}a_1u_1)\}$

$\equiv((n_2^*n_2+\sum s_{2j}^*t_{2j})a_1u_0, a_3u_1)+((n_2^*n_2+\sum t_{2j}^*s_{2j})a_3u_0, a_1u_1)$

$\equiv(a_2a_1u_0, a_3u_1)+(a_2a_3u_0, a_1u_1)$

$\equiv(a_2a_1u_0, a_3u_1)+(a_3a_1u_0, a_2u_1)$ (in view of Lemma 6).

Other terms can be estimated in the same way. Hence we have the following a priori estimate.

Proposition 1. *For any $U\in D(A)_i$, there exists a positive number β such that*

$$\|(\lambda I-A)U\|_{\mathcal{H}_i}\geq(|\lambda|-\beta)\|U\|_{\mathcal{H}_i} \quad \textit{for } |\lambda|>\beta,\ \lambda \textit{ real}, \ (i=1, 2).$$

Let us show that there exists $U\in D(A)_i$ such that $(\lambda I-A)U=F$ holds for any $F=(f_1, f_2, f_3, f_4)$ in $\mathcal{H}_i$. For that purpose it suffices to prove that there exists $u\in H^4\cap D(a_1)$ or $u\in N(a_1)$ such that

(3.14) $$Au=(\lambda^4+(a_1+a_2+a_3)\lambda^2+a_3a_1)u=g$$

holds for any g in L^2. In fact if u be the solution of (3.14) for

(3.15) $$g=\lambda^3f_1+\lambda^2f_2+\lambda(a_2+a_3)f_1+\lambda f_3+f_4,$$

then putting

(3.16) $$u_0=u,\ u_1=\lambda u,\ u_2=\lambda^2u-a_1u,\ u_3=\lambda^3u-\lambda(a_1+a_2)u,$$

we can see that $U=(u_0, u_1, u_2, u_3)$ is in $D(A)_i$ and satisfies $(\lambda I-A)U=F$ for $F=(f_1, f_2, f_3, f_4)$ in $\mathcal{H}_i$ $(i=1, 2)$. This is reduced to the theory of the elliptic boundary value problems containing a real parameter (c.f. S. Mizohata [1]).

Let z_1 and z_2 be the roots with positive imaginary parts of $A_0(x_0, i\eta+iz\nu, \lambda)$. Here $A_0(x, D, \lambda)$ is the principal part of A and x_0 is on S. ν and η are the conormal and the tangential vector on

S respectively. Now we only remark that after local transformation (2.1) the Lopatinski's determinants are given by the following forms;

If $z_1 \neq z_2$

$$\begin{vmatrix} 1 & 1 \\ z_1^2 & z_2^2 \end{vmatrix} = z_2^2 - z_1^2 \neq 0 \qquad \text{in Case I}$$

$$\begin{vmatrix} z_1 & z_2 \\ z_1^3 & z_2^3 \end{vmatrix} = z_1 z_2 (z_2^2 - z_1^2) \neq 0 \qquad \text{in Case II.}$$

If $z_1 = z_2$

$$\begin{vmatrix} 1 & 0 \\ z_1^2 & 2z_2 \end{vmatrix} = 2z_1 \neq 0 \qquad \text{in Case I}$$

$$\begin{vmatrix} z_1 & 1 \\ z_1^3 & 3z_2^2 \end{vmatrix} = 2z_1^3 \neq 0 \qquad \text{in Case II.}$$

Proposition 2. *There exists a positive constant β such that $(\lambda I - A)^{-1}$ exists for $|\lambda| > \beta$ and satisfies*

$$\|(\lambda I - A)^{-1}\|_{\mathcal{H}_i} \leq \frac{1}{|\lambda| - \beta} \qquad (i = 1, 2).$$

§4. Existence of the solution

By virtue of proposition 2, we can apply Hille-Yosida's theorem to (E) with $B \equiv 0$. For given $F(t)$ such that $F(t)$ and $AF(t)$ are in $\mathcal{E}_t^0(\mathcal{H}_i)$ and for initial value U_0 in $D(A)_i$, the unique solution of $\frac{d}{dt} U = AU + F$ in $\mathcal{E}_t^0(D(A)_i) \cap \mathcal{E}_t^1(\mathcal{H}_i)$ is given by

$$U_t = S_t U_0 + \int_0^t S_{t-s} F(s) ds, \tag{4.1}$$

where S_t is the semi-group with the infinitesimal generator A. But in this situation we must assume that $AF(t)$ belongs to $\mathcal{E}_t^0(\mathcal{H}_i)$. In Case I, this assumption means that $f(t)$ is continuous in $\mathcal{D}_{L^2}^1$. To remove this restriction we need the energy inequality.

Between $U(t) = \left(u(t), \frac{\partial}{\partial t} u(t), \frac{\partial^2}{\partial t^2} u(t), \frac{\partial^3}{\partial t^3} u(t)\right)$ and $U_t = (u_0(t), u_1(t), u_2(t), u_3(t))$, we have the following relation form (1.2)

$$
(4.2)\qquad \begin{cases} u(t)=u_0(t) \\ \dfrac{\partial}{\partial t}u(t)=u_1(t) \\ \dfrac{\partial^2}{\partial t^2}u(t)=-a_1u_0(t)+u_2(t) \\ \dfrac{\partial^3}{\partial t^3}u(t)=-(a_1+a_2)u_1(t)+u_3(t) \end{cases}
$$

$U(t)$ belongs to $D(A)_i$, if and only if U_t belongs to $D(A)_i$ and there exists a constant C such that

$$
(4.3)\qquad \frac{1}{C}\|U_t\|_{D(A)_i}\leq\|U(t)\|_{D(A)_i}<C\|U_t\|_{D(A)_i}.
$$

For $\mathcal{H}_i$ and $\|\cdot\|_{\mathcal{H}_i}$, the same relations hold.

Proposition 3. *Assume that $f(t)$ is in $\mathcal{E}_t^1(L^2)$, then we have*

$$
(4.4)\qquad \begin{aligned} &\|U(t)\|_{D(A)_i}+\left\|\frac{\partial^4}{\partial t^4}u(t)\right\|_0 \\ &\leq C(T)\left\{\|U(0)\|_{D(A)_i}+\|f(0)\|_0+\int_0^t\|f'(t)\|_0\,dt\right\} \\ &\quad 0\leq t\leq T \quad (i=1,2), \end{aligned}
$$

for the solutions $U(t)\in\mathcal{E}_t^0(D(A)_i)\cap\mathcal{E}_t^1(\mathcal{H}_i)$ of the eqution (E) *with $B\equiv0$.*

Proof. Consider the estimate (3.10) then we can see

$$
\frac{d}{dt}\|U(t)\|_{\mathcal{H}_i}\leq C\left\{\|U(t)\|_{\mathcal{H}_i}+\|F(t)\|_{\mathcal{H}_i}\right\}.
$$

By integration of this inequality it follows that

$$
(4.5)\qquad \|U(t)\|_{\mathcal{H}_i}\leq C(T)\left\{\|U(0)\|_{\mathcal{H}_i}+\int_0^t\|f(s)\|_0\,ds\right\}.
$$

At first let us assume that $U(t)$ belongs to $\mathcal{E}_t^2(\mathcal{H}_i)\cap\mathcal{E}_t^1(D(A)_i)$ and that $F(t)$ is in $\mathcal{E}_t^2(\mathcal{H}_i)$. Put

$$
(4.6)\qquad \begin{aligned} &\frac{\partial}{\partial t}u_j(t)=v_j(t) \qquad (j=0,1,2,3) \\ &V_t=(v_0(t),v_1(t),v_2(t),v_3(t)) \end{aligned}
$$

then we have

$$(4.7)\qquad \frac{\partial}{\partial t}V_t=AV_t+\frac{\partial F}{\partial t}$$

$$(4.8)\qquad \|V(t)\|_{\mathcal{H}_i}\leq C(T)\left\{\|V(0)\|_{\mathcal{H}_i}+\int_0^t\left\|\frac{\partial f}{\partial t}(s)\right\|_0 ds\right\}.$$

Consider the regularity theorem of elliptic equations, then we have (4.4) from (4.5) and (4.8). In order to remove the above assumption we use Friedrichs' mollifier with respect to t. $U_\varepsilon(t)=\varphi_\varepsilon(t)*U(t)$ and $F_\varepsilon(t)=\varphi_\varepsilon*F(t)$ are in $\mathcal{E}_t^2(\mathcal{H}_i)\cap\mathcal{E}_t^1(D(A)_i)$ and in $\mathcal{E}_t^2(\mathcal{H}_i)$ respectively. So we have (4.4) for $U_\varepsilon(t)$ and $f_\varepsilon(t)$. Let ε tend to zero, then we obtain (4.4) for $U(t)$ and $f(t)$.

Theorem 1. *For any $f(t)$ in $\mathcal{E}_t^1(L^2)$ and any initial data $U(0)$ in $D(A)_i$,* (E) *has the unique solution $U(t)$ satisfying the boundary conditions* (I) *or* (II). *$U(t)$ are in $\mathcal{E}_t^1(\mathcal{H}_i)\cap\mathcal{E}_t^0(D(A)_i)$ and energy inequality* (4.4) *holds.*

Proof. At first we consider the equation (E) with $B\equiv 0$. For given $f(t)\in\mathcal{E}_t^1(L^2)$, we can choose the sequence of functions $\{f_n(t)\}$ such that

1) $F_n(t)$ and $AF_n(t)$ are in $\mathcal{E}_t^0(\mathcal{H}_i)$, $(F_n(t)=(0,0,0,f_n(t))$,

2) $\|f_n(0)-f(0)\|$ and $\int_0^T\|f_n'(t)-f'(t)\|_0 ds$ tend to zero when n tends to ∞.

By virtue of (4.4), the limit of the solutions $U_n(t)$ of the equation $\frac{d}{dt}U=AU+F_n$ exists independently of the choice of $\{f_n(t)\}$. Denote it by $U(t)$, then $U(t)$ satisfies the boundary condition I or II. Next consider the case $B\not\equiv 0$. Now define the sequence of functions $\{U_n(t)\}$ successively as follows. $U_n(t)$ is the solution of

$$\frac{d}{dt}U=AU+F_n,\ \text{where}\ F_n=(0,0,0,-Bu_{n-1}+f),\ u_{-1}\equiv 0.$$

By (4.4) $\{U_n\}$ converges and the limit $U(t)$ satisfies (E) and (4.4).

The solution that we have constructed by successive approximation method, is the unique solution of the equation (E). In fact for two solutions $U(t)$ and $V(t)$ of (E) belonging to $\mathcal{E}_t^1(\mathcal{H}_i)\cap\mathcal{E}_t^0(D(A)_i)$, we can apply (4.4), then

$$\|U(t)-V(t)\|_{D(A)_i}\leq C(T)\int_0^t\|B(u(s)-v(s)\|_0\,ds \tag{4.9}$$
$$\leq C'(T)\int_0^t\|U(s)-V(s)\|_{D(A)_i}\,ds$$

holds. Therefore the following inequality holds

$$\max_{0\leq s\leq t}\|U(s)-V(s)\|_{D(A)_i}\leq C'(T)t\max_{0\leq s\leq t}\|U(s)-V(s)\|_{D(A)_i}.$$

This means $U(s)=V(s)$ for small s, so that $U(s)=V(s)$ for every $s\geq 0$. (q. e. d.)

Now we introduce some notations for the convenience of discussing the regularity of each solution. Corresponding to Case I and Case II, we denote

$$D(a_1)_n=\begin{cases}D(a_1) & n\geq 2\\ \mathcal{D}_{L^2}^1 & n=0,\ 1\end{cases} \tag{4.10}$$

$$N(a_1)_n=\begin{cases}N(a_1) & n\geq 3\\ N & n=1,\ 2\\ H^1 & n=0.\end{cases}$$

$$\begin{aligned}&\text{Case I}\quad D(A)_{1,p}=H^{p+4}\cap D(a_1)_{p+3}\times H^{p+3}\cap D(a_1)_{p+2}\\ &\qquad\qquad\times H^{p+2}\cap D(a_1)_{p+1}\times H^{p+1}\cap D(a_1)_p\\ &\text{Case II}\quad D(A)_{2,p}=H^{p+4}\cap N(a_1)_{p+3}\times H^{p+3}\cap N(a_1)_{p+2}\\ &\qquad\qquad\times H^{p+2}\cap N(a_1)_{p+1}\times H^{p+1}\cap N(a_1)_p,\\ &\qquad\qquad\text{for } p\geq 0.\end{aligned} \tag{4.11}$$

$$D(A)_{i,0}=D(A)_i\quad (i=1,2).$$

We define the norm of $D(A)_{i,p}$ as follows

$$\|U\|_{D(A)_{i\,p}}^2=\|u_0\|_{p+4}^2+\|u_1\|_{p+3}^2+\|u_2\|_{p+2}^2+\|u_3\|_{p+1}^2. \tag{4.12}$$

Using such a notation we can state the regularity theorem.

Theorem 2. *Assume the following* 1), 2), 3),

1) $U(0)$ *is in* $D(A)_{i,p}$
2) $(f(t), f'(t), \cdots, f^{(p)}(t))$ *is in* $\mathcal{E}_t^1(H^p \times H^{p-1} \times \cdots \times L^2)$
3) *(compatibility conditions) Put* $(\varphi_0, \varphi_1, \varphi_2, \varphi_3) = (u_0, u_1, u_2, u_3)$ *and we assume*

$$\varphi_{q+3} = f^{(q)}(0) - (a_1 + a_2 + a_3)\varphi_{q+2} - a_3 a_1 \varphi_q - B^{(q)}(\varphi_i\ i \leq q+3), \tag{4.13}$$

φ_{q+3} *belongs to* $D(a_1)_{p-q+2}$ *or* $N(a_1)_{p-q+2}$, $p \geq q \geq 0$.

Then the solution $U(t)$ *is in* $\mathcal{E}_t^0(D(A)_{i,p}) \cap \mathcal{E}_t^1(D(A)_{i,p-1})$ $(i=1, 2)$.

Proof. Suppose that $p \geq 1$ and consider the equation

$$Lv = f'(t) - B'(u), \text{ with initial data } (u_1, u_2, u_3, \varphi_4),$$

where u means the solution of $Lu = f(t) - B(u)$. Since B is a third order operator, $f'(t) - B'(u)$ is in $\mathcal{E}_t^1(L^2)$. Put

$$u(t) = u_0 + \int_0^t v(s)ds.$$

Then $u(t)$ satisfies

$$Lu + Bu = f(t). \tag{4.14}$$

From the elliptic regularity theorem it follows that $U(t)$ is continuous in $D(A)_{i,1}$. Step by step consider the equation

$$Lw = f^{(n)}(t) - B^{(n)}(u) \tag{4.15}$$

with initial data $(\varphi_n, \varphi_{n+1}, \varphi_{n+2}, \varphi_{n+3})$, $(n = 2, 3, \cdots, p)$.

Using the solution of (4.15) in the case $n = p$, we put

$$u(t) = \varphi_0 + t\varphi_1 + \frac{t^2}{2}\varphi_2 + \cdots + \frac{t^{p-1}}{(p-1)!}\varphi_{p-1} + \int_0^t \frac{(t-s)^{p-1}}{(p-1)!} w(s)ds. \tag{4.16}$$

Then we can see that $u(t)$ satisfies (4.14) and $U(t)$ is in $\mathcal{E}_t^0(D(A)_{i,p}) \cap \mathcal{E}_t^1(D(A)_{i,p-1})$.

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