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Non-immersion theorems for lens spaces

By

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§1. Introduction

The purpose of this paper is to show some non-immersion theorems for lens spaces. For the proof we shall use the theorem of T. Kambe which determines the structure of K_Λ-rings of the lens space [6] and the theorem of T. Kambe, H. Matsunaga and H. Toda on stunted lens spaces [7].

Throughout this note p is always an odd prime. Let S^{2n+1} be the unit $(2n+1)$-sphere. A point of S^{2n+1} is represented by a sequence $(z_0, \cdots, z_n)$ of complex numbers z_i $(i=0, \cdots, n)$ with $\sum |z_i|^2=1$. Let γ be the rotation of S^{2n+1} defined by

$$\gamma(z_0, \cdots, z_n) = (\lambda z_0, \cdots, \lambda z_n), \qquad \text{where} \quad \lambda = e^{2\pi i/p},$$

and let Γ be the topological transformation group of S^{2n+1} of order p generated by γ. Then

$$L^n(p) = S^{2n+1}/\Gamma$$

is the lens space mod p. It is an $(2n+1)$-dimensional compact, connected differentiable manifold without boundary. Let $\{z_0, \cdots, z_n\} \in L^n(p)$ denote the equivalence class of $(z_0, \cdots, z_n) \in S^{2n+1}$. The space $L^{k-1}(p)$ is naturally embedded in $L^k(p)$ by identifying $\{z_0, \cdots, z_{k-1}\}$ with $\{z_0, \cdots, z_{k-1}, 0\}$. Let $L_0^k(p) = \{\{z_0, \cdots, z_k\} \in L^k(p) \mid z_k$ is real and $z_k \geqq 0\}$. Then $L^k(p) - L_0^k(p) = e^{2k+1}$($(2k+1)$-cell) and $L_0^k(p) - L^{k-1}(p) = e^{2k}$ ($2k$-cell), $k \leqq n$. Thus $L^n(p)$ has a cell structure given by

$$L^n(p) = S^1 \cup e^2 \cup e^3 \cup \cdots \cup e^{2n} \cup e^{2n+1}.$$

(cf. [6] and [7]).

Let M^n be an n-dimensional differentiable manifold and R^k be the k-dimensional Euclidean space. By $M^n \subseteq R^k$ (respectively $M^n \not\subseteq R^k$) we mean that M^n can be immersed (respectively cannot be immersed) in R^k.

After some preparations in §2, we shall give in §3 a necessary condition for immersibility of certain lens spaces (Theorem 3). As applications, in §4 we shall prove some non-immersion theorems for lens spaces. For example, we obtain the following result (Theorem 4. (*II*)).

Let p be a prime with $p \geqq 5$. Let α and β be odd integers such that $0<\alpha \leqq (2p-1)/3$ and $0<\beta \leqq (p-2)/3$, and let l and k be integers such that $l>k \geqq 0$ and $l>1$ if $\alpha>1$, or $l>k \geqq 0$ and $l>2$ if $\alpha=1$.

Then $L^n(p) \not\subseteq R^{3n+1}$ for $n=\alpha p^l+\beta p^k$.

The method of the proof is similar to that of J. Adem and S. Gitler [1] with which they have given a simple proof for the James' non-immersion theorem for real projective spaces ([5], Theorem 1. 1). In [1], they uses the twisted normal bundle and the S-reducibility, while we shall use the ordinary normal bundle and the Steenrod reduced power operations.

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§2. Preliminaries

Let X be a finite connected CW-complex. Let $\varepsilon(X)$ denote the set of equivalence classes of real vector bundles over X and let

$$\theta:\ \varepsilon(X) \rightarrow KO(X)$$

be a natural map, where $KO(X)$ is the associated Grothendieck group. When we consider the complex vector bundles, the associated Grothendieck group is denoted by $K(X)$.

An element $\alpha \in KO(X)$ is said to be *positive* if there is an element $\beta \in \varepsilon(X)$ such that $\theta(\beta)=\alpha$. We shall drop the symbol θ and regard (an equivalence class of) a vector bundle as an element of both $\varepsilon(X)$ and $KO(X)$.

A *geometrical dimension* of an element $\alpha \in KO(X)$ (written $g\text{-}\dim \alpha$) is the least integer k such that $\alpha + k$ is positive, where k is the k-dimensional trivial bundle over X.

Let CP^n be the complex projective space of complex n-dimension. Let $\xi (\in K(CP^n))$ denote the canonical line bundle over CP^n and $r(\xi)(\in KO(CP^n))$ denote the real restriction of ξ. Let

$$\pi : L^n(p) \to CP^n$$

be the natural projection. Define

$$\eta = \pi^* r(\xi) = r(\pi^* \xi) \in KO(L^n(p)),$$

that is, η is the induced bundle of $r(\xi)$ by π.

Let τ be the tangent bundle of $L^n(p)$. Then the following equality holds (cf. [6], Lemma (4, 7)), where $\oplus$ denotes the Whitney sum.

$$\tau \oplus 1 = (n+1)\eta . \tag{1}$$

Define $\sigma = \eta - 2 (\in \widetilde{KO}(L^n(p))$, the stable class of η. The main theorem of T. Kambe ([6], Theorem 2) is as follows.

Theorem. *Let p be an odd prime, $q = (p-1)/2$ and $n = s(p-1) + r$ $(0 \leqq r < p-1)$. Then*

$$\widetilde{KO}(L^n(p)) \approx \begin{cases} (Z_{p^{s+1}})^{[r/2]} + (Z_{p^s})^{q-[r/2]} & (\text{if } n \not\equiv 0 \bmod 4) \\ Z_2 + (Z_{p^{s+1}})^{[r/2]} + (Z_{p^s})^{q-[r/2]} & (\text{if } n \equiv 0 \bmod 4) \end{cases}$$

and the direct summand $(Z_{p^{s+1}})^{[r/2]}$ and $(Z_{p^s})^{q-[r/2]}$ are generated additively by $\sigma, \cdots, \sigma^{[r/2]}$ and $\sigma^{[r/2]+1}, \cdots, \sigma^q$ respectively. Moreover its ring structure is given by

$$\sigma^{q+1} = \sum_{i=1}^{q} \frac{-(2q+1)}{2i-1} \binom{q+i-1}{2i-2} \sigma^i, \quad \sigma^{[n/2]+1} = 0 .$$

In the theorem, $(Z_a)^b$ indicates the direct sum of b-copies of a cyclic group Z_a of order a and $[c]$ denotes the integral part of c. Note that

$$p^{s+\varepsilon} \sigma = 0, \tag{2}$$

where $\varepsilon = 0$ or 1 according as $[r/2] = 0$ or $[r/2] \geqq 1$.

Then we can prove the following theorem.

Theorem 1. *Let n and k be positive integers with $0<k\leqq 2n+1$ and let a be a positive integer such that $2ap^{s+\varepsilon}>4n+3$. The necessary and sufficient condition for $L^n(p)\subseteq R^{2n+1+k}$ is that the bundle $\{ap^{s+\varepsilon}-(n+1)\}\eta$ has $\{2ap^{s+\varepsilon}-(2n+k+2)\}$ independent non-zero cross-sections.*

Proof. If $L^n(p)$ is immersed in R^{2n+1+k}, then we have a normal bundle ν of dimension k and

$$(3)\qquad \tau\oplus\nu = 2n+k+1\,.$$

Hence, by (1), (2) and (3), we have

$$\nu-k = \{ap^{s+\varepsilon}-(n+1)\}\eta-2\{ap^{s+\varepsilon}-(n+1)\}\in\widetilde{KO}(L^n(p))\,.$$

Thus we see

$$(4)\qquad \nu\oplus\{2ap^{s+\varepsilon}-(2n+k+2)\}=\{ap^{s+\varepsilon}-(n+1)\}\eta\,,$$

since the dimension of the bundles of both sides is greater than $2n+1$ (cf. [10], Lemma (1.2)). The formula (4) implies that the bundle $\{ap^{s+\varepsilon}-(n+1)\}\eta$ has $2ap^{s+\varepsilon}-(2n+k+2)$ inpependent non-zero cross-sections.

Assume that there exists a vector bundle α of dimension k such that

$$(5)\qquad \{ap^{s+\varepsilon}-(n+1)\}\eta = \alpha\oplus\{2ap^{s+\varepsilon}-(2n+k+2)\}\,.$$

Denote by τ_0 the stable class of τ. From (1), (2) and (5) we have $-\tau_0=\alpha-k$, and so $g\text{-}\dim(-\tau_0)\leqq k$. Therefore, by the theorem of Hirsch ([4], Theorem 6.4 and [3], Proposition 3.2) we have

$$L^n(p)\subseteq R^{2n+1+k}\,.$$

The cohomology algebra over Z_p of $L^n(p)$ is given as follows (cf. [11], p. 68).

$H^*(L^n(p)\,;Z_p)$ is the tensor product of the exterior algebra on a generator $y\in H^1(L^n(p)\,;Z_p)$ and the truncated polynomial algebra on a generator $x\in H^2(L^n(p)\,;Z_p)$ with relations $y^2=0$, $\Delta y=-x$ and $x^{n+1}=0$, where Δ is the Bockstein coboundary operator associated with the exact coefficient sequence: $0\to Z_p\to Z_{p^2}\to Z_p\to 0$.

§ 3. Stunted lens spaces

Let α be a vector bundle over X and let X^{α} denote its Thom complex. For a positive integer t and a space Y, denote by S^tY the t-fold suspension of Y. The following result is shown by M. F. Atiyah ([2], Lemma (2.4)).

There is a natural homeomorphism:

$$S^t(X^{\alpha}) \approx X^{\alpha\oplus t}.$$

Recently, T. Kambe, H. Matsunaga and H. Toda have proved the following theorem on stunted lens spaces ([7], Theorem 1).

There exists a natural homeomorphism:

$$L^m(p)/L^{m-n-1}(p) \approx (L^n(p))^{(m-n)\eta}.$$

By making use of these theorems we have the following result.

Theorem 2. *Let n and k be positive integers with $0<k\leqq 2n+1$, and let $n=s(p-1)+r(0\leqq r<p-1)$. Assume that a is a positive integer such that $2ap^{s+\varepsilon}>4n+3$ and that $t=2ap^{s+\varepsilon}-(2n+k+2)$, where $\varepsilon=0$ or 1 according as $[r/2]=0$ or $[r/2]\geqq 1$. If $L^n(p)$ is immersed in R^{2n+1+k} with a normal bundle ν, then there exists a natural homeomorphism*:

$$S^t(L^n(p))^{\nu} \approx L^{ap^{s+\varepsilon}-1}(p)/L^{ap^{s+\varepsilon}-n-2}(p).$$

Proof. As in the proof of Theorem 1, we see

$$\nu\oplus t = \{ap^{s+\varepsilon}-(n+1)\}\eta.$$

Then we have

$$\begin{aligned} S^t(L^n(p))^{\nu} &\approx (L^n(p))^{\nu\oplus t} = (L^n(p))^{\{ap^{s+\varepsilon}-(n+1)\}\eta} \\ &\approx L^{ap^{s+\varepsilon}-1}(p)/L^{ap^{s+\varepsilon}-n-2}(p). \end{aligned}$$

Let $\alpha=(E, p, X)$ be an oriented vector bundle of dimension k with the total space E, the base space X and the projection $p: E\to X$. Here, we assume that the base space X is a finite connected CW-complex. Denote by E_0 the subspace of E which consists of non-zero vectors. Then the following diagram is commutative (cf. [12]).

$$\begin{array}{ccc} H^q(E, E_0; Z_p) & \xrightarrow{j^*} & H^q(E; Z_p) \\ \approx\uparrow\phi & & \approx\uparrow p^* \\ H^{q-k}(X; Z_p) & \xrightarrow{\mu} & H^q(X; Z_p), \end{array}$$

where j^* is a map induced by the injection, ϕ is the Thom isomorphism and μ is defined by

$$\mu(y) = y \cdot \chi \qquad \text{for} \quad y \in H^{q-k}(X; Z_p),$$

where χ is the mod p Euler class of α.

If μ is an isomorphism, so is j^*. Therefore, if $\lambda: X \to X^\alpha$ is the inclusion map induced by the zero cross-section, then the induced map

$$\lambda^*: H^q(X^\alpha; Z_p) \to H^q(X; Z_p) \qquad \text{for} \quad k \leqq q \leqq \dim X$$

is also an isomorphism.

Theorem 3. *Let n and l be positive integers with $0 < l \leqq n$, and let $n = s(p-1) + r$ $(0 \leqq r < p-1)$. Assume that a is a positive integer such that $2ap^{s+\varepsilon} > 4n+3$ and that $t = 2ap^{s+\varepsilon} - 2(n+l+1)$, where $\varepsilon = 0$ or 1 according as $[r/2] = 0$ or $[r/2] \geqq 1$. If $L^n(p)$ is immersed in $R^{2n+1+2l}$ with a normal bundle ν whose Euler class is non-zero, then there is a map*

$$g: S^t(L^n(p)/L^{l-1}(p)) \to L^{ap^{s+\varepsilon}-l-1}(p)/L^{ap^{s+\varepsilon}-n-2}(p)$$

which induces isomorphisms of all cohomology groups with Z_p coefficients.

Proof. Since the mod p reduction induces an isomorphism:

$$H^{2l}(L^n(p); Z) \approx H^{2l}(L^n(p); Z_p),$$

the mod p Euler class $\bar{\chi}$ of ν is non-zero. The group $H^{2l}(L^n(p); Z_p)$ $(= Z_p)$ is generated by x^l, where x is a generator of $H^2(L^n(p); Z_p)$. Hence, $\bar{\chi} = mx^l$ for some m with $0 < m < p$, and so we have an isomorphism:

$$\mu = \bar{\chi}: H^{q-2l}(L^n(p); Z_p) \approx H^q(L^n(p); Z_p) \qquad \text{for} \quad 2l \leqq q \leqq 2n+1.$$

Therefore, if $\lambda: L^n(p) \to (L^n(p))^\nu$ is the natural inclusion, λ induces an isomorphism:

$$\lambda^* : H^q((L^n(p))^\vee ; Z_p) \approx H^q(L^n(p) ; Z_p) \qquad \text{for} \quad 2l \leqq q \leqq 2n+1 .$$

Since $(L^n(p))^\vee$ is $(2l-1)$-connected, there is a map f such that the following diagram is homotopy-commutative, where $q : L^n(p) \to L^n(p)/L^{l-1}(p)$ is the projection.

$$\begin{array}{ccc} L^n(p) & \xrightarrow{\lambda} & (L^n(p))^\vee \\ & {\scriptstyle q}\searrow \quad \nearrow{\scriptstyle f} & \\ & L^n(p)/L^{l-1}(p) & \end{array}$$

It is easily verified that the map f induces an isomorphism:

$$f^* : H^q(L^n(p))^\vee ; Z_p) \approx H^q(L^n(p)/L^{l-1}(p) ; Z_p) \qquad \text{for} \quad 0 \leqq q \leqq 2n+1 .$$

Let $S^t f$ be the t-fold suspension of the map f and let

$$\varphi : S^t(L^n(p))^\vee \simeq L^{ap^{s+\varepsilon}-1}(p)/L^{ap^{s+\varepsilon}-n-2}(p)$$

be a homeomorphism given in Theorem 2. Since the complex $S^t(L^n(p)/L^{l-1}(p))$ has dimension $2ap^{s+\varepsilon}-2l-1$, the image of a cellular approximation to the map $\varphi \circ S^t f$ is contained in the $(2ap^{s+\varepsilon}-2l-1)$-dimensional skeleton of $L^{ap^{s+\varepsilon}-1}(p)/L^{ap^{s+\varepsilon}-n-2}(p)$. Thus there exists a map g such that the following diagram is homotopy-commutative, where i is the inclusion.

$$\begin{array}{ccc} S^t(L^n(p)/L^{l-1}(p)) & \xrightarrow{S^t f} & S^t(L^n(p))^\vee \\ {\scriptstyle g}\downarrow & & \simeq\downarrow{\scriptstyle \varphi} \\ L^{ap^{s+\varepsilon}-l-1}(p)/L^{ap^{s+\varepsilon}-n-2}(p) & \xrightarrow{i} & L^{ap^{s+\varepsilon}-1}(p)/L^{ap^{s+\varepsilon}-n-2}(p) \end{array}$$

Then we can see that the map g induces isomorphisms of all cohomology groups with Z_p coefficients.

§ 4. Applications

In this section we apply Theorem 3 to get some non-immersion theorems for lens spaces. First, we recall the Pontrjagin class mod p of lens spaces $L^n(p)$ (cf. [9]).

Let $x \in H^2(L^n(p) ; Z_p)$ be a generator. The total Pontrjagin class mod p of $L^n(p)$ is given by the equation:

$$p(L^n(p)) = (1+x^2)^{n+1},$$

and the dual Pontrjagin class mod p is given by the equation:

$$\bar{p}(L^n(p)) = (1+x^2)^{-n-1} = \sum_{i=0}^{[n/2]} (-1)^i \binom{n+i}{i} x^{2i}.$$

Theorem 4. *Let p be a prime with $p \geqq 5$. Assume that either of the conditions (I) and (II) below is satisfied.*
(I) α and β are even integers such that $0 \leqq \alpha \leqq (2p-2)/3$ and $0 < \beta \leqq (2p-2)/3$, and l and k are integers such that $l > k \geqq 0$ and $l > 1$ if $\alpha > 0$, or $k > 1$ if $\alpha = 0$.
(II) α and β are odd integers such that $0 < \alpha \leqq (2p-1)/3$ and $0 < \beta \leqq (p-2)/3$, and l and k are integers such that $l > k \geqq 0$ and $l > 1$ if $\alpha > 1$, or $l > k \geqq 0$ and $l > 2$ if $\alpha = 1$.
Then $L^n(p) \nsubseteq R^{3n+1}$ for $n = \alpha p^l + \beta p^k$.

Proof. First, we consider the case when the condition (I) is satisfied. Suppose that $L^n(p) \subseteq R^{3n-1}$ for $n = \alpha p^l + \beta p^k$. Let ν be an oriented normal vector bundle of dimension n. The highest dimensional non-zero Pontrjagin class mod p of ν is

$$\bar{p}_{n/2} = (-1)^{n/2} \binom{n+n/2}{n/2} x^n \quad (\in H^{2n}(L^n(p); Z_p))$$

where x is a generator of $H^2(L^n(p); Z_p)$, since

$$\binom{n+n/2}{n/2} = \binom{\frac{3\alpha}{2}p^l + \frac{3\beta}{2}p^k}{\frac{\alpha}{2}p^l + \frac{\beta}{2}p^k} \equiv \binom{\frac{3\alpha}{2}}{\frac{\alpha}{2}} \cdot \binom{\frac{3\beta}{2}}{\frac{\beta}{2}} \not\equiv 0 \pmod{p}^{1)}.$$

Let $\bar{\chi}$ $(\in H^n(L^n(p); Z_p))$ be the Euler class mod p of ν. It is well known (e.g., [8], Theorem 31) that

$$\bar{\chi}^2 = \bar{p}_{n/2}.$$

1) If $a = \sum_i a_i p^i$ and $b = \sum_i b_i p^i$ are p-adic expansions, then

$$\binom{a}{b} \equiv \prod_i \binom{a_i}{b_i} \pmod{p}.$$

Thus we have $\bar{\chi} \neq 0$. Let s and r be integers given by the equation:

$$\alpha p^l + \beta p^k = s(p-1) + r \qquad (0 \leqq r < p-1),$$

and let $v(=s+\varepsilon)$ be s or $s+1$ according as $[r/2]=0$ or $[r/2] \geqq 1$ respectively. Denote by a an integer with $2ap^v > 4n+3$.

Now, by Theorem 3, there exists a map

$$g : S^t(L^n(p)/L^{(n-2)/2}(p)) \to L^{ap^v-(n+2)/2}(p)/L^{ap^v-n-2}(p)$$

which induces isomorphisms of all cohomology groups with Z_p coefficients, where t is a positive integer given by

$$t = 2ap^v - 3n - 2.$$

Let

$$E^t : H^{q-t}(L^n(p)/L^{(n-2)/2}(p); Z_p) \approx H^q(S^t(L^n(p)/L^{(n-2)/2}(p)); Z_p)$$

be the t-fold suspension isomorphism and let $(E^t)^{-1} \circ g^* = G$. Since E and g^* commute with Steenrod reduced power operations respectively, so is G.

Define a positive integer q by the equation:

$$q = 2ap^v - n - 2p^2$$

and consider the following commutative diagram, where the two $\mathcal{P}^1$ are first Steenrod reduced power operations mod p.

$$\begin{array}{ccc}
H^q(L^{ap^v-(n+1)/2}(p)/L^{ap^v-n-2}(p); Z_p) & \xrightarrow[\approx]{G} & H^{q-t}(L^n(p)/L^{(n-3)/2}(p); Z_p) \\
\downarrow \mathcal{P}^1 & & \downarrow \mathcal{P}^1 \\
H^{q+2(p-1)}(L^{ap^v-(n+1)/2}(p)/L^{ap^v-n-2}(p); Z_p) & \xrightarrow[\approx]{G} & H^{q-t+2(p-1)}(L^n(p)/L^{(n-3)/2}(p); Z_p)
\end{array}$$

It is easily seen that each group in the diagram is non-zero. Note that

$$q - \{2(ap^v - n - 2) + 1\} = \alpha p^l + \beta p^k - 2p^2 + 3 \geqq 3$$

by the assumption. Therefore, the two operations $\mathcal{P}^1$ on the left and right in the diagram are equivalent to the following two operations respectively:

$$\mathcal{P}^1: H^q(L^{ap^v-(n+2)/2}(p); Z_p) \to H^{q+2(p-1)}(L^{ap^v-(n+2)/2}(p); Z_p),$$
$$\mathcal{P}^1: H^{q-t}(L^n(p); Z_p) \to H^{q-t+2(p-1)}(L^n(p); Z_p).$$

Let $k>0$. For a generator $x^{q/2}$ of $H^q(L^{ap^v-(n+2)/2}(p); Z_p)$ we have $\mathcal{P}^1 x^{q/2}=0$, since $q/2\equiv 0 \pmod p$[2]. On the other hand, for a generator $x^{(q-t)/2}$ of $H^{q-t}(L^n(p); Z_p)$ we have $\mathcal{P}^1 x^{(q-t)/2} \neq 0$, since $(q-t)/2 \not\equiv 0 \pmod p$.

Therefore, in the diagram, $\mathcal{P}^1$ on the left is trivial, while $\mathcal{P}^1$ on the right is non-trivial. This is a contradiction.

If $k=0$, consider the following commutative diagram, where the two $\mathcal{P}^p$ are p-th reduced power operations mod p.

$$\begin{array}{ccc} H^q(L^{ap^v-(n+2)/2}(p)/L^{ap^v-n-2}(p); Z_p) & \xrightarrow[\approx]{G} & H^{q-t}(L^n(p)/L^{(n-2)/2}(p); Z_p) \\ \downarrow \mathcal{P}^p & & \downarrow \mathcal{P}^p \\ H^{q+2(p-1)p}(L^{ap^v-(n+2)/2}(p)/L^{ap^v-n-2}(p); Z_p) & \xrightarrow[\approx]{G} & H^{q-t+2(p-1)p}(L^n(p)/L^{(n-2)/2}(p); Z_p) \end{array}$$

As is easily seen, each group in the diagram is non-zero. Note that

$$q-\{2(ap^v-n-2)+1\} = \alpha p^l+\beta-2p^2+3>3$$

by the assumption. Thus we carry the proof as in the above case.

For a generator $x^{q/2}$ of $H^q(L^{ap^v-(n+2)/2}(p); Z_p)$ we have

$$\mathcal{P}^p x^{q/2} = \binom{q/2}{p} x^{q/2+(p-1)p} = -x^{q/2+(p-1)p} \neq 0,$$

since $q/2=(a-1)p^v+(p-1)p^{v-1}+\cdots+(p-1)p^{l+1}+(p-1-\alpha/2)p^l+(p-1)p^{l-1}+\cdots+(p-1)p^3+(p-2)p^2+(p-1)p+(p-\beta/2)$. On the other hand, for a generator $x^{(q-t)/2}$ of $H^{q-t}(L^n(p); Z_p)$ we have

2) $\mathcal{P}^r x^i = \binom{i}{r} x^{i+(p-1)r}$.

$$\mathcal{P}^p x^{(q-t)/2} = \binom{(q-t)/2}{p} x^{(q-t)/2+(p-1)p} = 0,$$

since $(q-t)/2=(\alpha-1)p^l+(p-1)p^{l-1}+\cdots+(p-1)p^2+(\beta+1)$.

Therefore, in the diagram, $\mathcal{P}^p$ on the left is non-trivial, while $\mathcal{P}^p$ on the right is trivial. This is a contradiction.

Next, we consider the case when the condition (II) is satisfied. Suppose that $L^n(p)\subseteq R^{3n+1}$ for $n=\alpha p^l+\beta p^k$. Let ν be an oriented normal bundle of dimension n. The highest dimensional non-zero Pontrjagin class mod p of ν is $\bar{p}_{n/2}$ $(\in H^{2n}(L^n(p)\,;Z_p))$, because

$$\binom{n+n/2}{n/2} = \binom{\frac{3\alpha-1}{2}p^l+\frac{p-1}{2}p^{l-1}+\cdots+\frac{p-1}{2}p^{k+1}+\frac{p+3\beta}{2}p^k}{\frac{\alpha-1}{2}p^l+\frac{p-1}{2}p^{l-1}+\cdots+\frac{p-1}{2}p^{k+1}+\frac{p+\beta}{2}p^k}$$

$$\equiv \binom{\frac{3\alpha-1}{2}}{\frac{\alpha-1}{2}} \cdot \binom{\frac{p+3\beta}{2}}{\frac{p+\beta}{2}} \not\equiv 0 \pmod{p}.$$

The rest of the proof is similar to the above case (I), so we omit the details.

Thus, the proof of the theorem is completed.

If the number of the non-zero terms of the p-adic expansions of n is greater than 2, there are many types of theorems corresponding to theorem 4. For example, we obtain the following result.

Theorem 4′. *Let p be a prime with $p\geqq 5$. Assume that either of the conditions (I') and (II') below is satisfied.*

(I') m is an integer with $m>2$; α_i $(i=1,2,\cdots,m)$ are even integers such that $0<\alpha_i\leqq(2p-2)/3$; and k_i $(i=1,2,\cdots,m)$ are integers such that $k_m>k_{m-1}>\cdots>k_2>k_1\geqq 0$.

(II') m is an even integer with $m>2$; α_i $(i=1,2,\cdots,m)$ are odd integers such that $0<\alpha_i\leqq(2p-1)/3$ if i is even and $0<\alpha_i\leqq(p-2)/3$ if i is odd; and k_i $(i=1,2,\cdots,m)$ are integers such that $k_m>k_{m-1}>\cdots>k_2>k_1\geqq 0$.

Then $L^n(p) \not\subseteq R^{3n+1}$ *for* $n = \sum_{i=1}^{m} \alpha_i p^{k_i}$.

The proof is quite similar to that of Theorem 4, so we shall omit the proof.

Theorem 5. *Let p be an odd prime. Assume that either of the conditions (III) and (IV) below is satisfied.*
(III) α is an even integer with $0 \leqq \alpha \leqq (2p-2)/3$, β is an odd integer with $0 < \beta \leqq (2p-1)/3$, and l and k are integers such that $l > k \geqq 0$ and $l > 1$ if $\alpha > 0$, or $k > 1$ if $\alpha = 0$.
(IV) α is an odd integer with $0 < \alpha \leqq (2p-1)/3$, β is an even integer with $0 \leqq \beta \leqq (p-1)/3$, and l and k are integers such that $l > k > 0$; $l > 1$ if $k = 0$ and $\alpha > 1$; or $l > 2$ if $k = 0$ and $\alpha = 1$.
Then $L^n(p) \not\subseteq R^{3n}$ for $n = \alpha p^l + \beta p^k$.

We are indebted to Professor Y. Saito for the proof of the theorem.

Proof. First, we consider the case when the condition (III) is satisfied. Suppose that $L^n(p) \subseteq R^{3n}$ for $n = \alpha p^l + \beta p^k$. Let ν be an oriented normal bundle of dimension $n-1$. The highest dimensional non-zero Pontrjagin class mod p of ν is

$$\bar{p}_{(n-1)/2} = (-1)^{(n-1)/2} \binom{n+(n-1)/2}{(n-1)/2} x^{n-1} \quad (\in H^{2n-2}(L^n(p); Z_p)),$$

where x is a generator of $H^2(L^n(p); Z_p)$, since

$$\binom{n+(n-1)/2}{(n-1)/2} = \begin{pmatrix} \frac{3\alpha}{2} p^l + \frac{3\beta-1}{2} p^k + \frac{p-1}{2} p^{k-1} + \cdots + \frac{p-1}{2} p + \frac{p-1}{2} \\ \frac{\alpha}{2} p^l + \frac{\beta-1}{2} p^k + \frac{p-1}{2} p^{k-1} + \cdots + \frac{p-1}{2} p + \frac{p-1}{2} \end{pmatrix}$$

$$\equiv \begin{pmatrix} \frac{3\alpha}{2} \\ \frac{\alpha}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{3\beta-1}{2} \\ \frac{\beta-1}{2} \end{pmatrix} \not\equiv 0 \pmod{p}.$$

Let $\bar{\chi} (\in H^{n-1}(L^n(p); Z_p))$ be the Euler class mod p of ν. Since $\bar{\chi}^2 = \bar{p}_{(n-1)/2}$, we have $\bar{\chi} \neq 0$. Let s and r be integers given by the equation:

$$\alpha p^l + \beta p^k = s(p-1) + r \qquad (0 \leqq r < p-1),$$

and let $v(=s+\varepsilon)$ be s or $s+1$ according as $[r/2]=0$ or $[r/2]\geqq 1$ respectively. Denote by a an integer with $2ap^v>4n+3$.

Now, by Theorem 3, there is a map

$$g: S^t(L^n(p)/L^{(n-3)/2}(p)) \to L^{ap^v-(n+1)/2}(p)/L^{ap^v-n-2}(p)$$

which induces isomorphisms of all cohomology groups with Z_p coefficients, where t is a positive integer given by

$$t = 2ap^v-3n-1\,.$$

Let

$$E^t: H^{q-t}(L^n(p)/L^{(n-3)/2}(p)\,;Z_p) \approx H^q(S^t(L^n(p)/L^{(n-3)/2}(p))\,;Z_p)$$

be the t-fold suspension isomorphism and let $(E^t)^{-1}\circ g^*=G$. Since E and g^* commute with the Steenrod reduced power operation respectively, so is G.

If $k>0$, we define a positive integer q by the equation:

$$q = 2ap^v-n-p^2\,,$$

and consider the following commutative diagram.

$$\begin{array}{ccc}
H^q(L^{ap^v-(n+2)/2}(p)/L^{ap^v-n-2}(p)\,;Z_p) & \xrightarrow[\approx]{G} & H^{q-t}(L^n(p)/L^{(n-2)/2}(p)\,;Z_p) \\
\downarrow \mathcal{P}^1 & & \downarrow \mathcal{P}^1 \\
H^{q+2(p-1)}(L^{ap^v-(n+2)/2}(p)/L^{ap^v-n-2}(p)\,;Z_p) & \xrightarrow[\approx]{G} & H^{q-t+2(p-1)}(L^n(p)/L^{(n-2)/2}(p)\,;Z_p)
\end{array}$$

As in the proof of Theorem 4, we can show that $\mathcal{P}^1$ on the left is trivial and that $\mathcal{P}^1$ on the right is non-trivial. This is a contradiction.

If $k=0$, we can give the proof by defining $q=2ap^v-n-2p^2-1$ and using reduced power operations $\mathcal{P}^p$, similarly in the proof of Theorem 4. We omit the details here.

Secondly, we consider the case when the condition (IV) is satisfied. Suppose that $L^n(p)\subseteq R^{3n}$ for $n=\alpha p^l+\beta p^k$. Let ν be an oriented normal bundle of dimension $n-1$. The highest dimensional non-zero Pontrjagin class mod p of ν is $\bar{p}_{(n-1)/2}$ ($\in H^{2n-2}(L^n(p); Z_p)$), because

$$\binom{n+(n-1)/2}{(n-1)/2} =$$

$$\begin{pmatrix} \frac{3\alpha-1}{2}p^l+\frac{p-1}{2}p^{l-1}+\cdots+\frac{p-1}{2}p^{k+1}+\frac{3\beta+p-1}{2}p^k+\frac{p-1}{2}p^{k-1}+\cdots+\frac{p-1}{2} \\ \frac{\alpha-1}{2}p^l+\frac{p-1}{2}p^{l-1}+\cdots+\frac{p-1}{2}p^{k+1}+\frac{\beta+p-1}{2}p^k+\frac{p-1}{2}p^{k-1}+\cdots+\frac{p-1}{2} \end{pmatrix}$$

$$\equiv \begin{pmatrix} \frac{3\alpha-1}{2} \\ \frac{\alpha-1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{3\beta+p-1}{2} \\ \frac{\beta+p-1}{2} \end{pmatrix} \not\equiv 0 \quad (\mathrm{mod}\ p).$$

The rest of the proof is similar to the above case (*III*), so we omit the details.

Thus the proof of the theorem is completed.

As an example of a corresponding result to Theorem 5 in case when the number of non-zero terms of the p-adic expansion of n is greater than 2, we have the following

Theorem 5′. *Let p be an odd prime. Assume that the condition (III′) below is satisfied.*
(III′) m is an integer with $m>2$; r is an integer with $1\leqq r\leqq m$; α_i $(i=1, 2, \cdots, r-1)$ are even integers with $0<\alpha_i\leqq(p-1)/3$, α_r is an odd integer with $0<\alpha_r\leqq(2p-1)/3$ and α_i $(i=r+1, \cdots, m)$ are even integers with $0<\alpha_i\leqq(2p-2)/3$; and k_i $(i=1, 2, \cdots, m)$ are integers such that $k_m>k_{m-1}>\cdots>k_2>k_1\geqq 0$.

Then $L^n(p)\not\subseteq R^{3n}$ *for* $n=\sum_{i=1}^{m} a_i p^{k_i}$.

We can prove this theorem by the similar way to Theorem 5, so we omit the proof.

In the end, we shall discuss the exceptional case $p=3$.

Theorem 6. *Let k be a positive integer. If $n=3^k$, $L^n(3)\not\subseteq R^{3n}$.*

Proof. If $k>1$, the assertion is true by Theorem 5. We consider the case $k=1$. Suppose that $L^3(3)\subseteq R^9$. Let ν be an oriented normal bundle of dimension 2. The highest dimensional non-zero Pontrjagin class mod 3 of ν is

$$\bar{p}_1 = -x^2 \ (\in H^4(L^3(3)\,;Z_3))\,,$$

where x is a generator of $H^2(L^3(3)\,;Z_3)$.

Let $\bar{\chi}(\in H^2(L^3(3)\,;Z_3))$ be the Euler class mod 3 of ν. Then we have $\bar{\chi}^2=\bar{p}_1=-x^2$. But, this is impossible. Therefore, $L^3(3)\not\subseteq R^9$.

Theorem 7. *Let l and k be integers with $l-1>k\geqq 0$. If $n=3^l+3^k$, $L^n(3)\not\subseteq R^{3n-1}$.*

Proof. Suppose that $L^n(3)\subseteq R^{3n-1}$. Let ν be an oriented normal bundle of dimension $n-2$. The highest dimensional non-zero Pontrjagin class mod 3 of ν is

$$\bar{p}_{(n-2)/2} = (-1)^{n/2}x^{n-2} \ (\in H^{2n-4}(L^n(3)\,;Z_3))$$

because

$$\binom{n+n/2}{n/2} \equiv 0 \pmod 3 \quad \text{and} \quad \binom{n+(n-2)/2}{(n-2)/2} \equiv -1 \pmod 3\,.$$

Let $\bar{\chi}\,(\in H^{n-2}(L^n(3)\,;Z_3))$ be the Euler class mod 3 of ν. Since $\bar{\chi}^2=\bar{p}_{(n-2)/2}$, we get $\bar{\chi}\neq 0$. Note that $\varepsilon=0$ (i.e., $v=s$) and that $2\cdot 3^s>4n+3$ for $n\geqq 10$, where $s=n/2$.

Therefore, by Theorem 3, there is a map

$$g\colon S^t(L^n(3)/L^{(n-4)/2}(3)) \to L^{3^s-n/2}(3)/L^{3^s-n-2}(3)$$

which induces isomorphisms of all cohomology groups with Z_3 coefficients, where t is a positive integer given by

$$t = 2\cdot 3^s-3n\,.$$

Let

$$E^t\colon H^{q-t}(L^n(3)/L^{(n-4)/2}(3)\,;Z_3) \approx H^q(S^t(L^n(3)/L^{(n-4)/2}(3))\,;Z_3)$$

be the t-fold suspension isomorphism and let $(E^t)^{-1}\circ g^*=G$. Since E and g^* commute with the reduced power operations respectively, so is G.

Define a positive integer q by the equation:

$$q = 2\cdot 3^s-n-2\cdot 3^{k+2}$$

and consider the following commutative diagram, where the two $\mathcal{P}^{3^{k+1}}$ are 3^{k+1}-th Steenrod reduced power operations mod 3.

$$\begin{array}{ccc} H^q(L^{3^s-n/2}(3)/L^{3^s-n-3}(3)\,;\,Z_3) & \underset{\approx}{\overset{G}{\longrightarrow}} & H^{q-t}(L^n(3)/L^{(n-4)/2}(3)\,;\,Z_3) \\ \downarrow \mathcal{P}^{3^{k+1}} & & \downarrow \mathcal{P}^{3^{k+1}} \\ H^{q+4\cdot 3^{k+1}}(L^{3^s-n/2}(3)/L^{3^s-n-2}(3)\,;\,Z_3) & \underset{\approx}{\overset{G}{\longrightarrow}} & H^{q-t+4\cdot 3^{k+1}}(L^n(3)/L^{(n-4)/2}(3)\,;\,Z_3) \end{array}$$

If $l>k+2$,

$$q-\{2(3^s-n-2)+1\} = 3^l+3^k-2\cdot 3^{k+2}+3\geqq 13\,.$$

Hence, we see that the two operations $\mathcal{P}^{3^{k+1}}$ on the left and on the right in the diagram are equivalent to the following two operations respectively:

$$\mathcal{P}^{3^{k+1}}:\ H^q(L^{3^s-n/2}(3)\,;\,Z_3) \to H^{q+4\cdot 3^{k+1}}(L^{3^s-n/2}(3)\,;\,Z_3)\,,$$
$$\mathcal{P}^{3^{k+1}}:\ H^{q-t}(L^n(3)\,;\,Z_3) \to H^{q-t+4\cdot 3^{k+1}}(L^n(3)\,;\,Z_3)\,.$$

For a generator $x^{q/2}$ of $H^q(L^{3^s-n/2}(3)\,;\,Z_3)$ we have

$$\mathcal{P}^{3^{k+1}}x^{q/2} = \binom{q/2}{3^{k+1}}x^{q/2+2\cdot 3^{k+1}} = x^{q/2+2\cdot 3^{k+1}} \neq 0\,,$$

since $q/2=2\cdot 3^{s-1}+\cdots+2\cdot 3^l+3^{l-1}+\cdots+3^{k+3}+3^{k+1}+3^k$. On the other hand, for a generator $x^{(q-t)/2}$ of $H^{q-t}(L^n(3)\,;\,Z_3)$ we have

$$\mathcal{P}^{3^{k+1}}x^{(q-t)/2} = \binom{(q-t)/2}{3^{k+1}}x^{(q-t)/2+2\cdot 3^{k+1}} = 0\,,$$

since $(q-t)/2=2\cdot 3^{l-1}+\cdots+2\cdot 3^{k+2}+3^k$.

Therefore, $\mathcal{P}^{3^{k+1}}$ on the left in the above diagram is non-trivial, while $\mathcal{P}^{3^{k+1}}$ on the right is trivial. This is a contradiction.

If $l=k+2$, $n/2=3^{k+1}+2\cdot 3^k\equiv 1$ (mod 2). Then we have $\bar{p}_{(n-2)/2} = -x^{n-2}$. But, this is inconsistent with the fact that $\bar{\chi}^2=\bar{p}_{(n-2)/2}$.

Thus the proof of the theorem is completed.

§ 5. Remarks

T. Kambe has proved in [6] the following non-immersion theorem for lens spaces.

Let p be an odd prime. Then $L^n(p)\not\subseteq R^{2n+2L(n,p)}$, where $L(n,p)$

is the integer defined by

$$L(n, p) = \max\left\{i \leqq [n/2] \,\middle|\, \binom{n+i}{i} \not\equiv 0 \mod p^{1+[(n-2i)/(p-1)]}\right\}$$

From this theorem, the following results are obtained.

1) *Let p be a prime with $p \geqq 5$. α and β are integers defined in (I) or (II), and l and k are integers such that $l > k \geqq 0$.*

Then $L^n(p) \not\subseteq R^{3n}$ for $n = \alpha p^l + \beta p^k$.

2) *Let p be an odd prime. α and β are integers defined in (III) or (IV), and l and k are integers such that $l > k \geqq 0$.*

Then $L^n(p) \not\subseteq R^{3n-1}$ for $n = \alpha p^l + \beta p^k$.

3) *Let l and k be integers with $l \geqq k \geqq 0$.*

Then $L^n(3) \not\subseteq R^{3n-2}$ for $n = 3^l + 3^k$.

These results 1), 2) and 3) are also obtained from the well known theorem:

Let M^n be an n-dimensional manifold. If $M^n \subseteq R^{n+k}$, then $\bar{p}_i(M^n) = 0$ except 2-torsions for $i > [k/2]$, where $\bar{p}_i(M^n)$ ($\in H^{4i}(M^n; Z)$) is the i-th normal Pontrjagin class.

As for the immersion theorem for lens spaces, recently F. Uchida has proved in [13] the following result:

Let p be an odd prime,

1) *If n is odd, $L^n(p) \subseteq R^{3n+3}$.*

2) *If n is even, $L^n(p) \subseteq R^{3n+4}$.*

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