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Local solutions for quasi-linear parabolic equations

By

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§ 0. Notations and introductions

Let Ω be a domain in $(t, x_1, \cdots, x_n)$-space, which is bounded by a lateral surface S and planes $t=0$, $t=T$. S is covered by $\{V_I\}_{(\text{finite})}$, and represented by $(t, x)=(t, F^I(t, \bar{x}'))$ in V_I. We denote

$$\Omega_\tau = \Omega \cap \{t=\tau\}, \quad \Omega_\tau^{\tau'} = \Omega \cap \{\tau<t<\tau'\},$$
$$S_\tau = S \cap \{t=\tau\}, \quad S_\tau^{\tau'} = S \cap \{\tau<t<\tau'\},$$

and we denote by $N(t, x)$ the inner normal direction at $(t, x)\in S_t$ in Ω_t.

Linear case. We consider the problem: Find the solution u satisfying the following conditions.

$$(*)\left\{\begin{array}{ll} Lu \equiv \dfrac{\partial}{\partial t}u - \sum\limits_{|\nu|\leqslant 2b} a_\nu(t, x)\left(\dfrac{\partial}{\partial x}\right)^\nu u = f(t, x) & \text{in } \Omega, \\ B_j u \equiv \sum\limits_{|\nu|\leqslant r_j} b_{j\nu}(t, x)\left(\dfrac{\partial}{\partial x}\right)^\nu u = f_j(t, x) & \text{on } S \quad (j=1, 2, \cdots, b), \\ u = u_0(x) & \text{on } \Omega_0, \end{array}\right.$$

where f, f_j, u_0 are given data $(0\leqslant r_j \leqslant 2b-1)$.

Assumptions (See [1], with respect to notations.)

i) $Re\, A_0(t, x; i\sigma) \leqslant -\delta|\sigma|^{2b}$ for $\sigma\in R^n$, $(t, x)\in\Omega$ $\left(A_0(t, x; i\sigma) = \sum\limits_{|\nu|=2b} a_\nu(t, x)(i\sigma)^\nu\right)$.

ii) $|R(t, x; p, \eta)| \geqslant \delta(|p|^\alpha + |\eta|)^{\sum_j (r_j - j+1)}$ $\left(\alpha = \dfrac{1}{2b}\right)$

for $Re\, p>0,\ \eta\in R^n,\ \eta\cdot N(t, x)=0,\ (t, x)\in S$

$$\left(R(t, x; p, \eta) = \det\left(\oint \frac{B_{0j}(t, x; i(\eta+zN(t, x)))z^{k-1}}{A_{0+}(t, x; p, \eta, z)}dz\right)_{jk}\right).$$

iii) $\sum_\nu |a_\nu|_{C^{k+\gamma}(\Omega)}+\sum_{j,\nu}|b_{j\nu}|_{C^{2b-r_j+k+\gamma}(S)}+\sum_{I,j}|F^I{}_j|_{C^{2b+k+\gamma}(V_I)}+\sum_I\left|\frac{1}{g_I}\right|_{C^0(V_I)}$
$= M_k<+\infty \qquad (k=0, 1, 2, \cdots,\ 0<\gamma<1).$

iv) $|f|_{C^{k+\gamma}(\Omega)}+\sum_j|f_j|_{C^{2b-r_j+k+\gamma}(S)}+|u_0|_{C^{2b+k+\gamma}(\Omega_0)}<+\infty$,

where we assume that $C^{2b+k+\gamma}$-class compatibility conditions on S_0 are satisfied for (*).

Hereafter we denote positive constants depending only on δ and M_k by the same letter C_k. Let $\tilde{L}=\frac{\partial}{\partial t}-\sum_\nu \tilde{a}_\nu\left(\frac{\partial}{\partial x}\right)^\nu$, $\tilde{f}$, $\tilde{u}_0$ be fixed extensions of L, f, u_0, satisfying

$$|\tilde{a}_\nu|_{C^{k+\gamma}((0,T)\times R^n)} \leqslant C_k|a_\nu|_{C^{k+\gamma}(\Omega)},$$
$$|\tilde{f}|_{C^{k+\gamma}((0,T)\times R^n)} \leqslant C_k|f|_{C^{k+\gamma}(\Omega)},$$
$$|\tilde{u}_0|_{C^{2b+k+\gamma}((0,T)\times R^n)} \leqslant C_k|u_0|_{C^{2b+k+\gamma}(\Omega_0)},$$
$$Re\sum_{|\nu|=2b}\tilde{a}_\nu(t, x)(i\sigma)^\nu \leqslant -C_k|\sigma|^{2b} \quad \text{for } \sigma\in R^n,\ (t, x)\in(0, T)\times R^n,$$

and we denote $\tilde{L}$, $\tilde{f}$, $\tilde{u}_0$ also by L, f, u_0.

Quasi-linear case. Let us consider the problem:

$$(P)\begin{cases} \frac{\partial}{\partial t}u-\sum_{|\nu|=2b}a_\nu(t, x; D^{2b-1}u)\left(\frac{\partial}{\partial x}\right)^\nu u = f(t, x; D^{2b-1}u) \quad \text{in } \Omega, \\ \sum_{|\nu|=r_j}b_{j\nu}(t, x; D^{r_j-1}u)\left(\frac{\partial}{\partial x}\right)^\nu u = f_j(t, x; D^{r_j-1}u) \\ \qquad\qquad \text{on } S\ (j=1, 2, \cdots, b), \\ u=u_0(x) \quad \text{on } \Omega_0, \end{cases}$$

where we denote

$$D^r u(t, x) = \left(u(t, x), \frac{\partial}{\partial x_1}u(t, x), \cdots, \left(\frac{\partial}{\partial x}\right)^\mu u(t, x), \cdots\right) \qquad |\mu|\leqslant r.$$

Assumptions

i) $Re\sum_{|\nu|=2b}a_\nu(t, x; U)(i\sigma)^\nu \leqslant -\delta(K)|\sigma|^{2b}$
for $\sigma\in R^n,\ (t, x)\in\Omega,\ |U|\leqslant K.$

ii) $|R(t, x; p, \eta; U)| \geqslant \delta(K)(|p|^{\alpha}+|\eta|)^{\sum_j(r_j-j+1)}$

for $Re\, p>0$, $\eta\in R^n$, $\eta\cdot N(t, x)=0$, $(t, x)\in S$, $|U|\leqslant K$.

iii) $$\sum_\nu |a_\nu|_{C^{k+\gamma}(\Omega,K)}+|f|_{C^{k+\gamma}(\Omega,K)}+\sum_j\Big(\sum_\nu|b_{j\nu}|_{C^{2b-r_j+k+\gamma}(S,K)}+|f_j|_{C^{2b-r_j+k+\gamma}(S,K)}\Big)+\sum_{I,j}|F^I{}_j|_{C^{2b+k+\gamma}(V_I)}+\sum_I\left|\frac{1}{g_I}\right|_{C^0(V_I)}=M_k(K)$$
$(k=0, 1, 2, \cdots)$,

where, for a function $g(t, x; U)$, we denote

$$|g|_{C^\beta(\Omega,K)}=\sum_{2b\nu_0+|\nu|+|\nu'|\leqslant\beta}\ \sup_{\substack{(t,x)\in\Omega\\ |U|<K}}\left|\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x}\right)^{\nu}\left(\frac{\partial}{\partial U}\right)^{\nu'}g(t, x; U)\right|.$$

$$+\sum_{\beta-2b<2b\nu_0+|\nu|+|\nu'|\leqslant\beta}\ \sup_{\substack{(t,x),\,(s,x)\in\Omega\\ |U|<K,}}$$
$$\times\frac{\left|\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x}\right)^{\nu}\left(\frac{\partial}{\partial U}\right)^{\nu'}g(t, x; U)-\left(\frac{\partial}{\partial s}\right)^{\nu_0}\left(\frac{\partial}{\partial x}\right)^{\nu}\left(\frac{\partial}{\partial U}\right)^{\nu'}g(s, x; U)\right|}{|t-s|^{\alpha(\beta-2b\nu_0-|\nu|-|\nu'|)}}$$

$$+\sum_{2b\nu_0+|\nu|+|\nu'|=[\beta]}\ \sup_{\substack{(t,x),\,(t,y)\in\Omega\\ |U|<K}}$$
$$\times\frac{\left|\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x}\right)^{\nu}\left(\frac{\partial}{\partial U}\right)^{\nu'}g(t, x; U)-\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial y}\right)^{\nu}\left(\frac{\partial}{\partial U}\right)^{\nu'}g(t, y; U)\right|}{|x-y|^{\beta-[\beta]}}$$

$$+\sum_{2b\nu_0+|\nu|+|\nu'|=[\beta]}\ \sup_{\substack{(t,x)\in\Omega\\ |U|,|V|<K}}$$
$$\times\frac{\left|\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x}\right)^{\nu}\left(\frac{\partial}{\partial U}\right)^{\nu'}g(t, x; U)-\left(\frac{\partial}{\partial t}\right)^{\nu_0}\left(\frac{\partial}{\partial x}\right)^{\nu}\left(\frac{\partial}{\partial V}\right)^{\nu'}g(t, x; V)\right|}{|U-V|}.$$

iv) $|u_0|_{C^{2b+k+\gamma}(\Omega_0)}=N_k<+\infty$, and $C^{2b+k+\gamma}$-class compatibility conditions on S_0 are satisfied for (P).

Recently the mixed problem with general boundary conditions has been treated by many mathematicians in linear case. On the other hand, Eidelman treated the Cauchy problem for quasi-linear equations in [3]. In the present paper, we consider the mixed problem in quasi-linear case. We shall show that, by using the results obtained in [1], we can obtain a local existence theorem.

In the following, we make some remarks in § 1 and § 2, and show the energy inequalities in linear case in § 3 (Proposition 2).

Finally in §4, we treat quasi-linear case. Our main result is

Theorem. *Under the assumptions stated above, there exists a unique (in C^{2b}) solution of (P) in $\Omega'(=\Omega\cap\{0<t<T'\})$, which belongs to $C^{2b+k+\gamma}(\Omega')$, where T' depends only on $|u_0|_{C^{2b}(\Omega_0)}$ with respect to the initial value $u_0(x)$.*

§1. Remarks on the fundamental solution

The fundamental solution $Z(t, x; \tau, \xi)$ of L has the following property.

Lemma 1.

$$\left|\int_0^t d\tau \int_{R^n} Z(t, x; \tau, \xi) f(\tau, \xi) d\xi\right|_{C^{2b+k+\gamma}} \leqslant C_k |f|_{C^{k+\gamma}},$$

$$\left|\int_\tau^t ds \int_{R^n} Z(t, x; s, y) f(s, y; \tau, \xi) dy\right|_{\hat{C}^{2b+k+\gamma}_{m-2b}} \leqslant C_k |f|_{\hat{C}^{k+\gamma}_m}.$$

Proof.

$Z(t, x; \tau, \xi) =$

$$Z_0(t-\tau, x-\xi; \tau, \xi)+\int_\tau^t ds \int_{R^n} Z_0(t-s, x-y; s, y)\varphi(s, y; \tau, \xi) dy,$$

$\varphi(t, x; \tau, \xi) =$

$$-(LZ_0)(t, x; \tau, \xi)+\int_\tau^t ds \int_{R^n} (-LZ_0)(t, x; s, y)\varphi(s, y; \tau, \xi) dy.$$

In order to show

$$\left|\int_0^t d\tau \int (LZ_0)(t, x; \tau, \xi) f(\tau, \xi) d\xi\right|_{C^{k+\gamma}} \leqslant C_k |f|_{C^{k+\gamma}},$$

$$\left|\int_\tau^t ds \int (LZ_0)(t, x; s, y) f(s, y; \tau, \xi) dy\right|_{\hat{C}^{k+\gamma}_{m-\gamma}} \leqslant C_k |f|_{\hat{C}^{k+\gamma}_m},$$

let us see the principal part

$$g(t, x) = \int_0^t d\tau \int \sum_{|\nu|=2b} (a_\nu(t, x)-a_\nu(\tau, \xi))\left(\frac{\partial}{\partial x}\right)^\nu Z_0(t-\tau, x-\xi; \tau, \xi) f(\tau, \xi) d\xi.$$

$g(t, x+\Delta)-g(t, x) =$

$$\int_{t-|\Delta|^{2b}}^t d\tau \int \sum_\nu (a_\nu(t, x+\Delta)-a_\nu(\tau, \xi))\left(\frac{\partial}{\partial x}\right)^\gamma Z_0(t-\tau, x+\Delta-\xi; \tau, \xi)$$

$$\times f(\tau, \xi)d\xi - \int_{t-|\Delta|^{2b}}^{t} d\tau \int \sum_{\nu} (a_\nu(t, x) - a_\nu(\tau, \xi)) \left(\frac{\partial}{\partial x}\right)^{\nu} Z_0(t-\tau, x-\xi ; \tau, \xi)$$

$$\times f(\tau, \xi)d\xi + \sum_{\nu} (a_\nu(t, x+\Delta) - a_\nu(t, x)) \int_0^{t-|\Delta|^{2b}} d\tau$$

$$\times \int \left(\frac{\partial}{\partial x}\right)^{\nu} Z_0(t-\tau, x+\Delta-\xi ; \tau, \xi) f(\tau, \xi) d\xi$$

$$+ \int_0^{t-|\Delta|^{2b}} d\tau \int \sum_{\nu} (a_\nu(t, x) - a_\nu(\tau, \xi)) \left\{ \left(\frac{\partial}{\partial x}\right)^{\nu} Z_0(t-\tau, x+\Delta-\xi ; \tau, \xi) \right.$$

$$\left. - \left(\frac{\partial}{\partial x}\right)^{\nu} Z_0(t-\tau, x-\xi ; \tau, \xi) \right\} f(\tau, \xi) d\xi .$$

The coefficients of the third term are equal to

$$\int_0^{t-|\Delta|^{2b}} d\tau \int \left\{ \left(\frac{\partial}{\partial x}\right)^{\nu} Z_0(t-\tau, x+\Delta-\xi ; \tau, \xi) - \left(\frac{\partial}{\partial x}\right)^{\nu} Z_0(t-\tau, x+\Delta-\xi ; t, y)_{y=x} \right\}$$

$$\times f(\tau, \xi) d\xi + \int_0^{t-|\Delta|^{2b}} d\tau \int \left(\frac{\partial}{\partial x}\right)^{\nu} Z_0(t-\tau, x+\Delta-\xi ; t, y)_{y=x} \{f(\tau, \xi) - f(t, x)\} d\xi ,$$

because

$$\int \left(\frac{\partial}{\partial x}\right)^{\nu} Z_0(t-\tau, x+\Delta-\xi ; t, y)_{y=x} f(t, x) d\xi = 0 .$$

Thus we have $|g(t, x+\Delta) - g(t, x)| \leqslant C_0 |\Delta|^{\gamma} |f|_{C^\gamma}$. In the same way, we have $|g|_{C^\gamma} \leqslant C_0 |f|_{C^\gamma}$. Proof of the rest is shown easily.

Now let us denote

$$v(t, x) = u_0(x) + \int_0^t d\tau \int_{R^n} Z(t, x ; \tau, \xi) \{f(\tau, \xi) + Au_0(\tau, \xi)\} d\xi ,$$

which belongs to $C^{2b+k+\gamma}((0, T) \times R^n)$ and is a unique solution of

$$(**) \quad \begin{cases} Lv = f(t, x) & \text{in } (0, T) \times R^n , \\ v\big|_{t=0} = u_0(x) . \end{cases}$$

And we have

Proposition 1. *Let* $C_k = C(\delta, M_k)$.

i) $$|v - u_0|_{C^{2b-1+\varepsilon}((0,t) \times R^n)} \leqslant C_0 \int_0^t (t-\tau)^{-\alpha(2b-1+\varepsilon)} \{ |f|_{C^0((0,\tau) \times R^n)} + |u_0|_{C^{2b}(R^n)} \} d\tau \quad \text{for } 0 < t \leqslant T \ (0 \leqslant {}^{\nu}\varepsilon \leqslant \gamma) ,$$

ii) $|v|_{C^{2b+k+\gamma}((0,T)\times R^n)}\leqslant C_k(|f|_{C^{k+\gamma}((0,T)\times R^n)}+|u_0|_{C^{2b+k+\gamma}(R^n)})$.

Let us denote $u=v+w$, then we have

$$(***)\quad \begin{cases} Lw=0 & \text{in } \Omega\,, \\ B_jw=g_j & \text{on } S\,, \\ w=0 & \text{on } \Omega_0\,, \end{cases}$$

where $g_j=f_j-B_jv|_S$. Now we denote

$$C_0^\beta=\left\{f\in C^\beta;\ \left(\frac{\partial}{\partial t}\right)^m f(t,x)\big|_{t=0}=0\,,\ 0\leqslant m<[\alpha\beta]\right\},$$

then g_j belongs to $C_0^{2b-r_j+k+\gamma}(S)$, and

$$|g_j|_{C^{2b-r_j+k+\gamma}(S)}\leqslant C_k\{|f|_{C^{k+\gamma}(\Omega)}+|f_j|_{C^{2b-r_j+k+\gamma}(S)}+|u_0|_{C^{2b+k+\gamma}(\Omega_0)}\}\,.$$

In fact, for any $u\in C^{2b+k+\gamma}(\Omega)$, we denote

$$\left(\frac{\partial}{\partial t}\right)^i u\big|_{t=0}=u_i\,,$$

and then

$$\left(\frac{\partial}{\partial t}\right)^i Au\big|_{t=0}=A^{(i)}(u_0,u_1,\cdots,u_i)\,,$$
$$\left(\frac{\partial}{\partial t}\right)^i B_ju\big|_{t=0}=B_j^{(i)}(u_0,u_1,\cdots,u_i)\,.$$

If u satisfies $\frac{\partial}{\partial t}u=Au+f$ in Ω and $u=u_0$ on Ω_0, $u_1,u_2,\cdots,u_m$ ($m=[\alpha(2b+k)]$) are determined by

$$u_1=A^{(0)}(u_0)+f\big|_{t=0},$$
$$u_2=A^{(1)}(u_0,u_1)+\left(\frac{\partial}{\partial t}\right)f\big|_{t=0},$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$u_m=A^{(m-1)}(u_0,u_1,\cdots,u_{m-1})+\left(\frac{\partial}{\partial t}\right)^{m-1}f\big|_{t=0}.$$

Then the compatibility conditions for (*) are described in the following way:

$$B_i^{(i)}(u_0,u_1,\cdots,u_i)=\left(\frac{\partial}{\partial t}\right)^i f_j$$
$$\text{on}\quad S_0\ \ (0\leqslant i\leqslant[\alpha(2b-r_j+k)]\,,\ j=1,2,\cdots,b)\,.$$

Now, since $Lv=f$ in Ω, $v=u_0$ on Ω_0, we have

$$\left(\frac{\partial}{\partial t}\right)^i v\Big|_{t=0}=u_i\,,$$

and then

$$\left(\frac{\partial}{\partial t}\right)^i B_j v\Big|_{t=0}=B_j^{(i)}(u_0, u_1, \cdots, u_i)=\left(\frac{\partial}{\partial t}\right)^i f_j$$
$$\text{on}\quad S_0\ (0\leqslant i\leqslant[\alpha(2b-r_j+k)])\,,$$

which implies $g_j\in C_0^{2b-r_j+k+\gamma}(S)$.

§ 2. Remarks on fractional powers

Let us denote $\mathfrak{L}=\left(\frac{\partial}{\partial t}\right)+(-\Delta)^b$, where Δ is Laplace-Beltrami's operator on S_t, and its fractional powers by $\mathfrak{L}^\sigma$(σ: real). Then we have

Lemma 2. *$\mathfrak{L}^\sigma$ is a one-to-one bicontinuous operator from $C_0^\beta(S)$ (resp. $\hat{C}_m^\beta(S, S)$) to $C_0^{\beta-2b\sigma}(S)$ (resp. $\hat{C}_{m+2b\sigma}^{\beta-2b\sigma}(S, S)$), and its operator norm is bounded by an absolute constant depending only on C_k, σ, β, m, where β and $\beta-2b\sigma$ are numbers in $(0, 2b+k+\gamma]$ and not equal to integers, m and $m+2b\sigma$ are less than $n-1+2b$.*

Proof. Since $\mathfrak{L}$ has the same properties as stated in Lemma 1 on L, and $\mathfrak{L}^\sigma f=\mathfrak{L}^{-l}\mathfrak{L}^{\sigma+l-l'}\mathfrak{L}^{l'}f$, $f\in C_0^\beta$ (l, l': positive integers, $l'\leqslant\beta$, $-1<\sigma+l-l'<1$), we need prove the case where $-1<\sigma<1$, $0<\beta<2b$. On the other hand, we have proved in [1] the case where $0\leqslant\beta\leqslant 2b+k-1+\gamma$. Here we only remark the rest case where $k=0$ and $2b-1+\gamma<\beta<2b$. We shall show the following:

Let $\sigma_0=\frac{2}{3}\alpha\gamma$, then $\mathfrak{L}^{\sigma_0}$ is a continuous operator from $C_0^\beta(S)$ to $C_0^{\beta-2b\sigma_0}(S)$ $(2b-1-\gamma<\beta<2b)$.

In fact, for $f\in C_0^\beta$, we construct a mollifier f_ε:

$$f(t, x)=\sum_I f^I(t, \bar{x}(t, x))\,,\quad f_\varepsilon(t, x)=\sum f_\varepsilon^I(t, \bar{x}(t, x))\,,$$

$$f_\varepsilon^I(t, x)=\iint_{R^n}\rho_\varepsilon(t-\tau, x-\xi)f^I(\tau, \xi)d\tau d\xi\,,\ \ \rho_\varepsilon(t, x)=\varphi_{\varepsilon^{2b}}(t)\varphi_\varepsilon(x_1)\cdots\varphi_\varepsilon(x_{n-1}),$$

$$\varphi_\varepsilon(t)=\frac{1}{\varepsilon}\varphi\left(\frac{t}{\varepsilon}\right),\quad 0\leqslant\varphi(t)\leqslant 1(|t|<1)\,,\quad \varphi(t)=0\ (|t|\geqslant 1)\,,$$

$$\varphi(t)=\varphi(-t)\,,\quad \int_{-\infty}^{\infty}\varphi(t)dt=1\,.$$

Then we have $f_\varepsilon\in C_0^{2b+\gamma}$ (, where the suffix o is understood as "comppact support" in t), and we can verify

$$|f_\varepsilon|_{C^s}\leqslant C\varepsilon^{-(s-\beta)}|f|_{C^\beta}\qquad (s\geqslant\beta)\,,$$
$$|f-f_\varepsilon|_{C^{s'}}\leqslant C\varepsilon^{\beta-s'}|f|_{C^\beta}\qquad (\beta-1<s'<\beta)\,.$$

Let us fix $s=2b+\gamma/3$, $s'=2b-1+\gamma$. We know

$$|\mathfrak{L}^{\sigma_0}f_\varepsilon|_{C^{s-2b\sigma_0}}\leqslant C|f_\varepsilon|_{C^s}\,,$$
$$|\mathfrak{L}^{\sigma_0}f|_{C^{s'-2b\sigma_0}}\leqslant C|f|_{C^{s'}}\,.$$

Now we have easily, for $|\nu|\leqslant 2b-1$,

$$\left|\left(\frac{\partial}{\partial x}\right)^{\nu}(\mathfrak{L}^{\sigma_0}f)(t,x)\right|\leqslant C|f|_{C^s}\,.$$

Next, for $|\nu|=2b-1$,

$$\begin{aligned}&\left(\frac{\partial}{\partial x}\right)^{\nu}(\mathfrak{L}^{\sigma_0}f)(t,x+\Delta)-\left(\frac{\partial}{\partial x}\right)^{\nu}(\mathfrak{L}^{\sigma_0}f)(t,x)\\ &\quad=\left\{\left(\frac{\partial}{\partial x}\right)^{\nu}(\mathfrak{L}^{\sigma_0}(f-f_\varepsilon))(t,x+\Delta)-\left(\frac{\partial}{\partial x}\right)^{\nu}(\mathfrak{L}^{\sigma_0}(f-f_\varepsilon))(t,x)\right\}\\ &\quad\quad+\left\{\left(\frac{\partial}{\partial x}\right)^{\nu}(\mathfrak{L}^{\sigma_0}f_\varepsilon)(t,x+\Delta)-\left(\frac{\partial}{\partial x}\right)^{\nu}(\mathfrak{L}^{\sigma_0}f_\varepsilon)(t,x)\right\}=I_1+I_2\,,\end{aligned}$$

$$\begin{aligned}|I_1|&\leqslant C|\Delta|^{s'-2b\sigma_0}|\mathfrak{L}^{\sigma_0}(f-f_\varepsilon)|_{C^{s'-2b\sigma_0}}\leqslant C|\Delta|^{s'-2b\sigma_0}|f-f_\varepsilon|_{C^{s'}}\\ &\leqslant C|\Delta|^{s'-2b\sigma_0}\varepsilon^{s'-\beta}|f|_{C^\beta}\,,\\ |I_2|&\leqslant C|\Delta|^{s-2b\sigma_0}|\mathfrak{L}^{\sigma_0}f_\varepsilon|_{C^{s-2b\sigma_0}}\leqslant C|\Delta|^{s-2b\sigma_0}|f_\varepsilon|_{C^s}\\ &\leqslant C|\Delta|^{s-2b\sigma_0}\varepsilon^{-(s-\beta)}|f|_{C^\beta}\,.\end{aligned}$$

Since ε is any positive number, we take $\varepsilon=|\Delta|$, then

$$|I_1|+|I_2|\leqslant C|\Delta|^{\beta-2b\sigma}|f|_{C^\beta}\,.$$

Hölder continuity in t is shown analogously.

Here we have Lemma 2, because

$$\mathfrak{L}^{\sigma}f=\mathfrak{L}^{\sigma-l\sigma_0}(\mathfrak{L}^{\sigma_0})^l f\,,$$

where l is the minimal integer such that $\beta-\alpha l\sigma_0\leqslant 2b-1+\gamma$.

§ 3. Energy inequalities

Let us denote

$$G_j^I(t, \bar{x}\,;\tau, \bar{\xi}') = \bar{F}\left[(\mathfrak{L}_0{}^I)^{-\beta_j}\frac{R_j^I}{R^I}\right],$$

where $\beta_j=\alpha(2b-r_j+k+\varepsilon)$, $0<\varepsilon<\gamma$, and denote

$$\begin{aligned} &G_j(t, x\,;\tau, \xi) \\ &\quad = \sum_I \alpha_I(t, x)G_j{}^I(t-\tau, \bar{x}(t, x)-\bar{\xi}'(\tau, \xi)\,;\tau, \bar{\xi}'(\tau, \xi))\alpha_I(\tau, \xi)\frac{1}{\sqrt{g_I(\tau, \xi)}} \end{aligned}$$

which belongs to $\hat{C}^{2b+k+\gamma}_{n-1+2b-(2b+k+\varepsilon)}(\Omega, S)$. Then we have an extension $\widetilde{LG_j}$ of LG_j, such that $\widetilde{LG_j}$ belongs to $\hat{C}^{k+\gamma}_{n-1+2b-(k+\varepsilon+\gamma)}((0, T)\times R^n, S)$. Now we denote

$$E_j(t, x\,;\tau, \xi)=G_j(t, x\,;\tau, \xi)-\int_\tau^t ds\int_{R^n} Z(t, x\,;s, y)\widetilde{LG_j}(s, y\,;\tau, \xi)dy\,,$$

then we have

$$\begin{aligned} B_i\int_0^t d\tau\int_{S_\tau} E_j(t, x\,;\tau, \xi)\varphi(\tau, \xi)dS &= \delta_{ij}\mathfrak{L}^{-\beta_j}\varphi(t, x) \\ &\quad +\int_0^t d\tau\int_{S_\tau} E_{ij}(t, x\,;\tau, \xi)\varphi(\tau, \xi)dS \end{aligned}$$

where E_{ij} belongs to $\hat{C}^{2b-r_j+k+\gamma}_{n-1+2b-(2b-r_j+k+\varepsilon)-\gamma}(S, S)=\hat{C}^{2b\beta_i+\gamma-\varepsilon}_{n-1+2b-2b\beta_i-\gamma}(S, S)$. When $E_{ij}\in\hat{C}^{2b\beta_i+\gamma'}_{n-1+2b-2b\beta_i-\gamma}$, $g_i\in C_0^{2b\beta_i+\gamma'}$, $\varphi_i\in C_0^{\gamma'}$ $(\gamma'=\gamma-\varepsilon>0)$ the following equations are equivalent:

$$g_i(t, x) = \mathfrak{L}^{-\beta_i}\varphi_i(t, x)+\sum_j\int_0^t d\tau\int_{S_\tau} E_{ij}(t, x\,;\tau, \xi)\varphi_j(\tau, \xi)dS$$

$$\mathfrak{L}^{\beta_i}g_i(t, x) = \varphi_i(t, x)+\sum_j\int_0^t d\tau\int_{S_\tau} \mathfrak{L}^{\beta_i}E_{ij}(t, x\,;\tau, \xi)\varphi_j(\tau, \xi)dS\,.$$

Now we denote

$$K_{ij}(t, x\,;\tau, \xi) = (-\mathfrak{L}^{\beta_i}E_{ij})(t, x\,;\tau, \xi)\,,$$

$$\begin{aligned} \Phi_{ij}(t, x\,;\tau, \xi) &= K_{ij}(t, x\,;\tau, \xi)+\sum_k\int_\tau^t ds\int_{S_s} K_{ik}(t, x\,;s, y) \\ &\qquad \times\Phi_{kj}(s, y\,;\tau, \xi)dS(\in\hat{C}^{\gamma'}_{n-1+2b-\gamma}) \end{aligned}$$

$$\mathscr{E}_j(t, x;\tau, \xi) = G_j(t, x;\tau, \xi)+\sum_i\int_\tau^t ds\int_{S_s} G_i(t, x;s, y)\Phi_{ij}(s, y;\tau, \xi)dS\,.$$

Here we have the solution of (***)

$$w(t,x)=\sum_j\int_0^t d\tau\int_{S_\tau}\mathcal{E}_j(t,x;\tau,\xi)\mathfrak{L}^{\beta_j}g_j(\tau,\xi)dS\,.$$

Then we have easily,

$$|w|_{C^{2b+k+\gamma}(\Omega)}\leqslant C_k\sum_j|\mathfrak{L}^{\beta_j}g_j|_{C^{\gamma'}(S)}\leqslant C_k\sum_j|g_j|_{C^{2b-r_j+k+\gamma}(S)}\,,$$

We remark that the above constructions are also correct for $-1-\gamma<\varepsilon<0$, $k=0$ (then $\Phi\in\hat{C}^{\gamma}_{n-1+2b-\gamma}(S,S)$). w is uniquely determined for any ε, because of the uniqueness theorem for (***) ([2]). Let $\varepsilon=-1+\gamma+\varepsilon'$ $(0<\varepsilon'<1-\gamma)$, then

$$|w|_{C^{2b-1+\gamma}(\Omega_0^t)}\leqslant C_0\int_0^t(t-\tau)^{-\alpha(2b-\varepsilon')}\sum_j|g_j|_{C^{2b-r_j}(S_0^\tau)}d\tau\,,$$

Let $\varepsilon=-1+\varepsilon''$ $(0<\varepsilon''<\gamma)$, then

$$|w|_{C^{2b-1}(\Omega_0^t)}\leqslant C_0\int_0^t(t-\tau)^{-\alpha(2b-\varepsilon'')}\sum_j|g_j|_{C^{2b-r_j-1+\gamma}(S_0^\tau)}d\tau\,.$$

Here, together with Prop. 1, we have

Proposition 2.

i) $$|u-u_0|_{C^{2b-1+\gamma}(\Omega_0^t)}\leqslant C_0\int_0^t(t-\tau)^{-\alpha(2b-\varepsilon')}\{|f|_{C^0(\Omega_0^\tau)}+\sum_j|f_j|_{C^{2b-r_j}(S_0^\tau)}+|u_0|_{C^{2b}(\Omega_0)}\}d\tau\quad \textit{for } 0<t\leqslant T\ (0<\varepsilon'<1-\gamma),$$

ii) $$|u-u_0|_{C^{2b-1}(\Omega_0^t)}\leqslant C_0\int_0^t(t-\tau)^{-\alpha(2b-\varepsilon'')}\{|f|_{C^0(\Omega_0^\tau)}+\sum|f_j|_{C^{2b-r_j-1+\gamma}(S_0^\tau)}+|u_0|_{C^{2b}(\Omega_0)}\}d\tau\quad \textit{for } 0<t\leqslant T\ (0<\varepsilon''<\gamma),$$

iii) $$|u|_{C^{2b+k+\gamma}(\Omega)}\leqslant C_k(|f|_{C^{k+\gamma}(\Omega)}+\sum|f_j|_{C^{2b-r_j+k+\gamma}(S)}+|u_0|_{C^{2b+k+\gamma}(\Omega_0)})\,,$$

where $C_0=C(\delta,M_0)$, $C_k=C(\delta,M_k)$.

Remark. *If* T *varies in* $0<T\leq T_0$, $C(\delta,M)$ *does not depend on* T, *it depends only on* T_0.

§ 4. Quasi-linear equations

Lemma 3. *Let*

$$g'(t,x)=g(t,x;U(t,x))\,,\quad g''(t,x)=g(t,x;V(t,x))\,,$$

where

$$|U|, \quad |V| < K, \quad |U|_{C^{h+\gamma}(\Omega)}, \quad |V|_{C^{h+\gamma}(\Omega)} < K',$$
$$|g|_{C^{h+\gamma}(\Omega, K)} < M \qquad (h=0, 1, 2, \cdots).$$

Then

i) $|g'|_{C^{h+\gamma}(\Omega)} < C$,

ii) $|g'-g''|_{C^0(\Omega)} \leqslant C|U-V|_{C^0(\Omega)}$ *for* $h=0$,

$|g'-g''|_{C^{h-1+\gamma}(\Omega)} \leqslant C|U-V|_{C^{h-1+\gamma}(\Omega)}$ *for* $h \geqslant 1$,

where $C=C(M, K')$ (*independent of* T).

Proof. Let $h=0$.

i) $|g'(t, x)| = |g(t, x; U(t, x))| \leqslant |g|_{C^0}$,

$$|g'(t+\Delta_0, x+\Delta)-g'(t, x)| = |g(t+\Delta_0, x+\Delta; U(t+\Delta_0, x+\Delta)) - g(t, x; U(t, x))| \leqslant |g(t+\Delta_0, x+\Delta; U(t+\Delta_0, x+\Delta)) - g(t, x; U(t+\Delta_0, x+\Delta))| + |g(t, x; U(t+\Delta_0, x+\Delta)) - g(t, x; U(t, x))| \leqslant (|g|_{C^\gamma} + |g|_{C^0}|U|_{C^\gamma})(|\Delta_0|^{\alpha\gamma} + |\Delta|^\gamma),$$

ii) $$|(g'-g'')(t, x)| = |g(t, x; U(t, x)) - g(t, x; V(t, x))| \leqslant |g|_{C^0}|U(t, x)-V(t, x)|,$$

Let $h=1$.

i) $$\frac{\partial}{\partial x_i} g'(t, x) = g_{x_i}(t, x; U(t, x)) + \sum_\mu g_{U_\mu}(t, x; U(t, x)) \frac{\partial}{\partial x_i} U_\mu(t, x).$$

$$\left|\frac{\partial}{\partial x_i} g'(t, x)\right| \leqslant |g|_{C^1} + |g|_{C^1}|U|_{C^1},$$

$$\left|\frac{\partial}{\partial x_i} g'(t+\Delta_0, x+\Delta) - \frac{\partial}{\partial x_i} g'(t, x)\right| \leqslant \{|g|_{C^{1+\gamma}} + |g|_{C^1}|U|_{C^\gamma} + (|g|_{C^{1+\gamma}} + |g|_{C^1}|U|_{C^\gamma})|U|_{C^1} + |g|_{C^1}|U|_{C^{1+\gamma}}\}(|\Delta_0|^{\alpha\gamma} + |\Delta|^\gamma),$$

$$|g'(t+\Delta_0, x)-g'(t, x)| \leqslant (|g|_{C^{1+\gamma}} + |g|_{C^0}|U|_{C^{1+\gamma}})|\Delta_0|^{\alpha(1+\gamma)}.$$

ii) $$(g'-g'')(t, x) = \sum_\mu \int_0^1 g_{U_\mu}(t, x; (1-\theta)V(t, x) + \theta U(t, x))d\theta(U_\mu(t, x) - V_\mu(t, x)) = \sum_\mu \varphi_\mu(t, x)(U_\mu(t, x) - V_\mu(t, x)),$$

$$|\varphi_\mu(t, x)| \leqslant |g|_{C^1},$$

$$|\varphi_\mu(t+\Delta_0, x+\Delta) - \varphi_\mu(t, x)| \leqslant \{|g|_{C^{1+\gamma}} + |g|_{C^1}(|U|_{C^\gamma} + |V|_{C^\gamma})\} \times (|\Delta_0|^{\alpha\gamma} + |\Delta|^\gamma).$$

Cases when $h\geqslant 2$ are shown in the same way.

Hereafter we fix $K=2|u_0|_{C^{2b}(R^n)}$, and denote $\delta=\delta(K)$, $M_k=M_k(K)$.

When u satisfies the conditions:

$$(H_k)\qquad |u|_{C^{2b-1+\gamma}(\Omega)}\leqslant K,\qquad |u|_{C^{2b+k+\gamma}(\Omega)}\leqslant K_k,$$

we say that u satisfies (H_k) in Ω. We denote

$$a'_\nu(t,x)=a_\nu(t,x;D^{2b-1}u),\qquad a''_\nu(t,x)=a_\nu(t,x;D^{2b-1}v),$$
$$b'_{j\nu}(t,x)=b_{j\nu}(t,x;D^{r_j-1}u),\qquad b''_{j\nu}(t,x)=b_{j\nu}(t,x;D^{r_j-1}v),\qquad \text{etc..}$$

Then we have

Corollary. *Let u, v satisfy (H_{k-1}), then we have*

i) $$(\sum_\nu|a'_\nu|_{C^{k+\gamma}}+|f'|_{C^{k+\gamma}})+\sum_j(\sum_\nu|b'_{j\nu}|_{C^{2b-r_j+k+\gamma}}+|f'_j|_{C^{2b-r_j+k+\gamma}})\leqslant M'(M_k,K_{k-1}),$$

ii) $$(\sum_\nu|a'_\nu-a''_\nu|_{C^0}+|f'-f''|_{C^0})+\sum_j(\sum_\nu|b'_{j\nu}-b''_{j\nu}|_{C^{2b-r_j-1+\gamma}}+|f'_j-f''_j|_{C^{2b-r_j-1+\gamma}})\leqslant M''(M_0,K)|u-v|_{C^{2b-1}}.$$

In fact, since

$$|D^{2b-1}u|<K,\qquad |D^{2b-1}u|_{C^{k+\gamma}}<K_{k-1},$$
$$|D^{r_j-1}u|<K,\qquad |D^{r_j-1}u|_{C^{2b-r_j+k+\gamma}}<K_{k-1},$$

we can use Lemma 3, only remarking

$$|D^{2b-1}u-D^{2b-1}v|_{C^0}\leqslant|u-v|_{C^{2b-1}},$$
$$|D^{r_j-1}u-D^{r_j-1}v|_{C^{2b-r_j-1+\gamma}}\leqslant|u-v|_{C^{2b-1}}.$$

Now we shall show (P) is solved by the method of successive approximations. At first we consider

$$(P')\quad\begin{cases}\dfrac{\partial}{\partial t}u-\sum\limits_{|\nu|=2b}a_\nu(t,x;D^{2b-1}v)\left(\dfrac{\partial}{\partial x}\right)^\nu u=f(t,x;D^{2b-1}v) & \text{in }\Omega,\\ \sum\limits_{|\nu|=r_j}b_{j\nu}(t,x;D^{r_j-1}v)\left(\dfrac{\partial}{\partial x}\right)^\nu u=f_j(t,x;D^{r_j-1}v) & \\ & \text{on }S\ (j=1,2,\cdots,b),\\ u=u_0(x)\qquad \text{on }\Omega_0, & \end{cases}$$

where v is given and satisfies

$$|v|_{C^{2b-1+\gamma}(\Omega)} \leqslant K,$$

and $C^{2b+\gamma}$-class compatibility conditions are satisfied. Then we have

$$|u-u_0|_{C^{2b-1+\gamma}(\Omega_0^t)} \leqslant C(\delta, M_0, K)t^{\alpha\varepsilon'} \qquad \text{for} \quad 0<t\leqslant T,$$
$$|u|_{C^{2b+\gamma}(\Omega)} \leqslant C(\delta, M_0, N_0, K).$$

In fact, by virtue of Prop. 2 and Cor. of Lem. 3,

$$|u-u_0|_{C^{2b-1+\gamma}(\Omega_0^t)} \leqslant C(\delta, M'(M_0, K))\int_0^t (t-\tau)^{-\alpha(2b-\varepsilon')}d\tau\left\{M'(M_0, K)+\frac{1}{2}K\right\},$$
$$|u|_{C^{2b+\gamma}(\Omega)} \leqslant C(\delta, M'(M_0, K))\{M'(M_0, K)+N_0\}.$$

Now we restrict the interval $(0, T)$ to $(0, T')$, where

$$C(\delta, M_0, K)T'^{\alpha\varepsilon'} = \frac{1}{2}K,$$

and we denote $\Omega' = \Omega_0^{T'}$, $S' = S_0^{T'}$. Then we have

$$|u|_{C^{2b-1+\gamma}(\Omega')} \leqslant K, \qquad |u|_{C^{2b+\gamma}(\Omega')} \leqslant K_0.$$

Now let $u_0 = u_0(x)$ and, for $m=1, 2, 3, \cdots$,

$$(P_m) \begin{cases} \dfrac{\partial}{\partial t} u_m - \sum\limits_{|\nu|=2b} a_\nu(t, x\,; D^{2b-1}u_{m-1})\left(\dfrac{\partial}{\partial x}\right)^\nu u_m = f(t, x\,; D^{2b-1}u_{m-1}) & \text{in } \Omega', \\ \sum\limits_{|\nu|=r_j} b_{j\nu}(t, x\,; D^{r_j-1}u_{m-1})\left(\dfrac{\partial}{\partial x}\right)^\nu u_m = f_j(t, x\,; D^{r_j-1}u_{m-1}) & \text{on } S' \\ & (j=1, 2, \cdots, b), \\ u_m = u_0(x) \qquad \text{on } \Omega_0. \end{cases}$$

Then $C^{2b+\gamma}$-class compatibility conditions are satisfied for every (P_m), and

$$|u_m|_{C^{2b-1+\gamma}(\Omega')} \leqslant K, \qquad |u_m|_{C^{2b+\gamma}(\Omega')} \leqslant K_0.$$

Therefore a subsequence of $\{u_m\}$ converges to u in $C^{2b+\gamma'}(\Omega')$ $(0<\gamma'<\gamma)$, and u belongs to $C^{2b+\gamma}(\Omega')$. On the other hand, let $u_{m+1}-u_m = v_m$, then

$$\begin{cases} \dfrac{\partial}{\partial t} v_m - \sum\limits_{|\nu|=2b} a_\nu(t, x\,; D^{2b-1}u_m)\left(\dfrac{\partial}{\partial x}\right)^\nu v_m = F^m(t, x) \qquad \text{in } \Omega', \\ \sum\limits_{|\nu|=r_j} b_{j\nu}(t, x\,; D^{r_j-1}u_m)\left(\dfrac{\partial}{\partial x}\right)^\nu v_m = F_j^m(t, x) \qquad \text{on } S' \ (j=1, 2, \cdots, b), \\ v_m = 0 \qquad \text{on } \Omega_0, \end{cases}$$

where

$$F^m(t, x) = \sum_{|\nu|=2b} \{a_\nu(t, x ; D^{2b-1}u_m) - a_\nu(t, x ; D^{2b-1}u_{m-1})\} \left(\frac{\partial}{\partial x}\right)^\nu u_m$$
$$+ \{f(t, x ; D^{2b-1}u_m) - f(t, x ; D^{2b-1}u_{m-1})\} ,$$
$$F_j^m(t, x) = - \sum_{|\nu|=r_j} \{b_{j\nu}(t, x ; D^{r_j-1}u_m) - b_{j\nu}(t, x ; D^{r_j-1}u_{m-1})\} \left(\frac{\partial}{\partial x}\right)^\nu u_m$$
$$+ \{f_j(t, x ; D)^{r_j-1}u_m) - f_j(t, x ; D^{r_j-1}u_{m-1})\} .$$

Since, by virtue of Cor. of Lem. 3,

$$|F^m|_{C^0(\Omega')} \leqslant C|v_{m-1}|_{C^{2b-1}(\Omega')} ,$$
$$|F_j^m|_{C^{2b-r_j-1+\gamma}(S')} \leqslant C|v_{m-1}|_{C^{2b-1}(\Omega')} ,$$

we have, by virtue of Prop. 2,

$$|v_m|_{C^{2b-1}(\Omega_0^t)} \leqslant C\int_0^t (t-\tau)^{-\sigma(2b-\varepsilon'')}|v_{m-1}|_{C^{2b-1}(\Omega_0^\tau)}d\tau \qquad (0<t\leqslant T') ,$$

then we have

$$|v_m|_{C^{2b-1}(\Omega')} \leqslant \frac{C^m}{\Gamma(1+\alpha\varepsilon''m)} ,$$

Therefore $\{u_m\}$ converges to u in $C^{2b-1}(\Omega')$ and u satisfies (P) in Ω'.

Let u and v be solutions of (P), belonging to $C^{2b}(\Omega')$, then we have in the same way

$$|u-v|_{C^{2b-1}(\Omega')} \leqslant \frac{C^m}{\Gamma(1+\alpha\varepsilon''m)} \xrightarrow[m\to\infty]{} 0 ,$$

therefore $u=v$ in Ω'.

Finally with respect to the regularity, it can be seen that u belongs to $C^{2b+k+\gamma}(\Omega')$. In fact, since u satisfies (P) and belongs $C^{2b+\gamma}(\Omega')$, we have, by virtue of Prop. 2 and Cor. of Lem. 3, that u belongs to $C^{2b+1+\gamma}(\Omega')$, and so on. Thus we have Theorem stated in § 0.

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