J. Math. Kyoto Univ. (JMKYAZ)
48-2 (2008), 239–263

Malliavin calculus on extensions of abstract Wiener spaces

Dedicated to Wilfried Buchholz on the occasion of his 60$^{\text{th}}$ birthday

By

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Abstract

Malliavin calculus is developed in a uniform way for (possibly non separable) extensions of $L^p(W_{C_\mathbb{F}})$, where $W_{C_\mathbb{F}}$ is the Wiener measure on the space $C_\mathbb{F}$ of continuous functions from $[0,1]$ into any abstract Wiener Fréchet space $\mathbb{F}$ over a fixed separable Hilbert space $\mathbb{H}$. Since the continuous time line is available in $C_\mathbb{F}$, we can prove the Clark-Ocone formula for these extensions, we study time-anticipating Girsanov transformations and prove that Skorohod integral processes for finite chaos levels have continuous modifications.

We use a rich probability space with measure $\widehat{\Gamma}_\mathbb{H}$, which only depends on $\mathbb{H}$, such that for any $p \in [0,\infty[$, $L^p\left(W_{C_\mathbb{F}}\right)$ can be canonically embedded into $L^p\left(\widehat{\Gamma}_\mathbb{H}\right)$ for any abstract Wiener Fréchet space $\mathbb{F}$ over $\mathbb{H}$.

1. Introduction

This article studies in a uniform way Malliavin calculus for closed subspaces of the space $L^2(\widehat{\Gamma})$ of square integrable functions on a rich probability space $\left(\Omega, L_\Gamma(\mathcal{B}), \widehat{\Gamma}\right)$, which depends only on a fixed separable Hilbert space $\mathbb{H}$. These subspaces of $L^2(\widehat{\Gamma})$ have to fulfill certain closure conditions, characterized by so called "admissible sequences". They are given by sub σ-algebras of $L_\Gamma(\mathcal{B})$.

One of these subspaces, denoted by $L^2_\mathcal{W}$, is canonically (i.e., independent of the basis) isometric isomorphic to the L^2-space over any abstract Wiener Fréchet space $\mathbb{F}$, for which $\mathbb{H}$ is the Cameron Martin space. Another one, denoted by $L^2_{C_\mathcal{W}}$ is canonically isometric isomorphic to the L^2-space over the abstract Wiener space $C_\mathbb{F}$ of continuous functions, defined on $[0,1]$ with values in $\mathbb{F}$. The Cameron Martin space now is the Hilbert space $C_\mathbb{H}$ of absolutely continuous functions $f : [0,1] \to \mathbb{H}$, see Bogachev [5] or Kuelbs and Lepage [18]). We use an $\mathbb{F}$-valued Brownian motion to construct the mentioned isometries.

2000 *Mathematics Subject Classification*(*s*). Primary 55Q99, Secondary 55P60, 55P10.
AMS subject classification: Primary 60H07; Secondary 03H05.
Received October 26, 2006

Moreover, Malliavin calculus is developed for nonseparable extensions $L^2_{\mathcal{H}}$ of $L^2_{\mathcal{W}}$ and also of $L^2_{C_{\mathcal{W}}}$, which have not been studied yet in the standard literature as far as I know. It should be also indicated that the underlying probability space only depends on the fixed separable Hilbert $\mathbb{H}$ and not on the abstract Wiener space over $\mathbb{H}$. There are many non-isomorphic abstract Wiener spaces over $\mathbb{H}$. We prove a chaos decomposition simultaneously for all these L^2-spaces. The σ-algebra in $L^2_{\mathcal{H}}$ is not generated by Brownian motion, in contrast to the σ-algebra of $L^2_{C_{\mathcal{W}}}$.

The chaos decomposition is used to define simultaneously the Malliavin derivative, the Skorohod integral and Skorohod integral processes (in the cases of $L^2_{C_{\mathcal{W}}}$ and $L^2_{\mathcal{H}}$). We have the following applications:

(A) Skorohod integral processes for finite chaos levels have continuous modifications. This is an extension of a result of P.Imkeller [15] for the classical Wiener space $C_{\mathbb{R}}$. Whence we can replace the real numbers by any Fréchet space $\mathbb{F}$. Our prove is different from Imkeller's proof.

(B) The Clark Ocone formula holds for $L^2_{\mathcal{H}}$. Now the notion "non-time-anticipating" is needed and we use a filtration, which is larger than the Brownian filtration.

(C) Anticipating Girsanov transformations are studied in the manner of finite dimensional analysis similar to [26]. Let $b_{\mathbb{F}} : \Omega \times [0,1] \to \mathbb{F}$ be a Fréchet space valued Brownian motion and define $\sigma : \Omega \to C_{\mathbb{F}}$ by

$$\sigma(X)(r) := b_{\mathbb{F}}(X,r) + \int_0^r \varphi(X,s) d\widehat{\nu}(s),$$

where φ is an almost surely square integrable process with values in $\mathbb{H}$. We want to find conditions on φ, as mild as possible, under which σ follows the law of Brownian motion under a measure which is equivalent to $\widehat{\Gamma}$: it is not necessary to assume, like in [26], that φ is measurable with respect to the σ-algebra generated by the Brownian motion, thus, σ is not necessarily a shift on $C_{\mathbb{F}}$. Moreover, we do not need that φ fulfills certain smoothness conditions like, for example, in the work of Nualart [23]. The reason is the following. Since our probability space Ω is finite dimensional in the sense of a highly saturated model and since it is possible to represent φ by a smooth function (a consequence of the chaos decomposition), we only need conditions, which imply the transformation rule in elementary finite dimensional analysis. In particular the Carleman Fredholm determinant becomes a determinant in a finite dimensional setting.

2. Preliminaries

In order to construct the probability space Ω, we shall use a highly saturated model $\mathcal{V}$ of mathematics (see [2]).

In this model there exists an extension ${}^*\mathbb{N}$ of the positive integers, containing a number H such that each standard natural number divides H. Set $T := \{1, ..., H\}$. Fix a standard $n \in \mathbb{N}$. On the algebra of internal subsets $A \subseteq T^n$ we define an internal probability measure $\nu^n : A \mapsto \frac{|A|}{H^n}$, where $|A|$ denotes the internal number of elements of A. It is well known that the Lebesgue measure λ^n on $[0,1]^n$ is the image measure of the Loeb measure $\widehat{\nu^n}$ over ν^n under the standard part map $(t_1,...,t_n) \mapsto st\,(t_1,...,t_n) := \left(^\circ\frac{t_1}{H},...,^\circ\frac{t_n}{H}\right)$. If $(t_1,...,t_n) \in T^n$, we set $^\circ(t_1,...,t_n) := (^\circ t_1,...,^\circ t_n) := \left(^\circ\frac{t_1}{H},...,^\circ\frac{t_n}{H}\right)$. Denote the associated Loeb space by $(T^n, L_{\nu^n}(T^n), \widehat{\nu^n})$.

Fix a separable real Hilbert space $\mathbb{H}$. Recall from [26] that in the model $\mathcal{V}$ there exists a finite dimensional Euclidian space $\mathbb{L}$ (in the sense of this model) with the following two properties:

(1) $\mathbb{H}$ is a subspace of $\mathbb{L}$.

(2) Let ω be the internal dimension of $\mathbb{L}$. Fix an internal orthonormal basis $(\mathfrak{e}_i)_{i\leq\omega}$ of $\mathbb{L}$ and an orthonormal basis $(\mathfrak{b}_i)_{i\in\mathbb{N}}$ of $\mathbb{H}$. Then for all $n\in\mathbb{N}$ and all f,g in the n-**fold tensor product** $\mathbb{H}^{\otimes n}$ of $\mathbb{H}$, the **Hilbert Schmidt scalar product** $\langle f,g\rangle_{\mathbb{H}^n} := \sum_{i_1,...,i_n\in\mathbb{N}} (f\cdot g)\,(\mathfrak{b}_{i_1},...,\mathfrak{b}_{i_n})$ of f and g is infinitely close to $\langle {}^*f, {}^*g\rangle_{\mathbb{L}^n} := \sum_{i_1,...,i_n\leq\omega} ({}^*f\cdot{}^*g)\,(\mathfrak{e}_{i_1},...,\mathfrak{e}_{i_n})$. The corresponding norms are denoted by $\|\cdot\|_{\mathbb{H}^n}, \|\cdot\|_{\mathbb{L}^n}$, respectively. If $n=1$, we drop the index 1.

Let $F,G\in\mathbb{L}^{\otimes n}\cup{}^*\mathbb{H}^{\otimes n}$. Then F is called **infinitely close to** G, written $F\approx_{\mathbb{L}^n} G$, if $\|F-G\|_{\mathbb{L}^n}\approx 0$; F is called **nearstandard** if there exists a $g\in\mathbb{H}^{\otimes n}$ such that $F\approx_{\mathbb{L}^n} {}^*g$. Then g is called the **standard part of** F and is denoted by $^\circ F$.

Let Γ be the internal $H\cdot\omega$-fold product of the internal centered Gaußian distribution of infinitesimal variance $\frac{1}{H}$ on the internal Borel algebra $\mathcal{B} = \mathcal{B}\left(\mathbb{L}^H\right)$ of $\mathbb{L}^H$. Our basic probability space is the Loeb space $\left(\Omega,\mathcal{D},\widehat{\Gamma}\right)$ over $\left(\mathbb{L}^H,\mathcal{B},\Gamma\right)$, where $\Omega := \mathbb{L}^H$, $\mathcal{D} := L_\Gamma(\mathcal{B})$ is the completed Loeb σ-algebra over $\mathcal{B}$ and $\widehat{\Gamma}$ is the Loeb measure over Γ.

The following result (Theorem 2.1) says that there exists a standard part map in the form of a Brownian motion $b_\mathbb{F}$, which is a surjective mapping from $\Omega=\mathbb{L}^H$ onto the space $C_\mathbb{F}$ of continuous functions from $[0,1]$ into $\mathbb{F}$. Moreover, $b_\mathbb{F}(\cdot,1)$ is a standard part map from $\mathbb{L}^H$ onto $\mathbb{F}$. It is left to the reader to prove Theorem 2.1, which is a straightforward extension of the corresponding result for Banach spaces in [25] and [26]. Banach and Fréchet space valued Brownian motions are well established in the literature (see [12], [11], [19]). In these articles the probability space depends on the Fréchet space $\mathbb{F}$, in our approach the probability space only depends on the underlying Hilbert space $\mathbb{H}$; there exists uncountable many abstract Wiener Fréchet space over $\mathbb{H}$. Moreover, the Brownian motion is pathwisely constructed as a continuous process.

Theorem 2.1 is an extension of Cutland's [8] construction of finite dimensional Brownian motion to the infinite dimensional setting. The process $b_\mathbb{F}$ is the standard part $^\circ B$ (with respect to the locally convex topology on $\mathbb{F}$) of the **internal Brownian motion**

$$B:\Omega\times T\to\mathbb{L}, (X,n)\mapsto\sum_{s\leq n, s\in T} X_s.$$

Note that this internal process B only depends on the underlying Hilbert space $\mathbb{H}$.

Theorem 2.1. *Fix an abstract Wiener Fréchet space $\mathbb{F}$ over $\mathbb{H}$. The mapping $b_{\mathbb{F}} : \Omega \times [0,1] \to \mathbb{F}$, defined by*

$$b_{\mathbb{F}}\left(\cdot, {}^{\circ} n\right) := b_{\mathbb{F}}\left(\cdot, {}^{\circ} \frac{n}{H}\right) := {}^{\circ} B(\cdot, n),$$

is well defined $\widehat{\Gamma}$-a.s. and a Brownian motion. If $b_{\mathbb{F}}(X, \cdot)$ is not well defined or not continuous, we may set $b_{\mathbb{F}}(X, \cdot) := 0$.

Moreover, $X \mapsto b_{\mathbb{F}}(X, \cdot)$ is a surjective mapping from Ω onto the space $C_{\mathbb{F}}$ of continuous functions from $[0,1]$ into $\mathbb{F}$.

Let $\mathcal{W}_{C_{\mathbb{F}}}$ and $\mathcal{W}_{\mathbb{F}}$ be the σ-algebras on Ω generated by $b_{\mathbb{F}}$, $b_{\mathbb{F}}(\cdot, 1)$, respectively, augmented by the $\widehat{\Gamma}$-nullsets.

Proposition 2.2. *The σ-algebras $\mathcal{W}_{C_{\mathbb{F}}}$ and $\mathcal{W}_{\mathbb{F}}$ do not depend on the abstract Wiener Fréchet space $\mathbb{F}$ over a fixed Hilbert $\mathbb{H}$.*

Proof. Since the topological dual $\mathbb{F}'$ of $\mathbb{F}$ is dense in $\mathbb{H}' = \mathbb{H}$, $\mathcal{W}_{C_{\mathbb{F}}}$ is generated by the random variables ${}^{\circ}\langle a, B_n \rangle$ with $n \in T$ and $a \in \mathbb{H}$ and the $\widehat{\Gamma}$-nullsets. Since the definition of B only depends on the Hilbert space $\mathbb{H}$, $\mathcal{W}_{C_{\mathbb{F}}}$ only depends on $\mathbb{H}$. In the same manner, $\mathcal{W}_{\mathbb{F}}$ is generated by ${}^{\circ}\langle a, B_H \rangle$ with $a \in \mathbb{H}$ and the $\widehat{\Gamma}$-nullsets, thus, $\mathcal{W}_{\mathbb{F}}$ only depends on $\mathbb{H}$. □

Therefore, we may set $\mathcal{W}_C := \mathcal{W}_{C_{\mathbb{F}}}$ and $\mathcal{W} := \mathcal{W}_{\mathbb{F}}$. The image measures $W_{C_{\mathbb{F}}}$ on $C_{\mathbb{F}}$ of $\widehat{\Gamma}$ under $b_{\mathbb{F}}$ and $W_{\mathbb{F}}$ on $\mathbb{F}$ of $\widehat{\Gamma}$ under $b_{\mathbb{F}}(\cdot, 1)$ are called the **Wiener measure** on $C_{\mathbb{F}}$, $\mathbb{F}$, respectively. The next simple result tells us that all L^p-spaces on abstract Wiener Fréchet spaces over a fixed Hilbert space $\mathbb{H}$ can be identified.

Corollary 2.3. *Fix abstract Wiener Fréchet spaces $\mathbb{F}$ and $\mathbb{G}$ over a separable Hilbert space $\mathbb{H}$ and $p \in [0, \infty[$.*

(1) *The spaces $L^p_{\mathcal{W}_C}(\widehat{\Gamma}), L^p(W_{C_{\mathbb{F}}})$ and $L^p(W_{C_{\mathbb{G}}})$ are canonically (i.e., basis independent) homeomorphic isomorphic (isometric isomorphic in case $p \geq 1$).*

(2) *The same holds for the spaces $L^p_{\mathcal{W}}(\widehat{\Gamma}), L^p(W_{C_{\mathbb{F}}})$ and $L^p(W_{C_{\mathbb{G}}})$.*

Proof. (1) Since $b_{\mathbb{F}}$ is a surjective mapping from Ω onto $C_{\mathbb{F}}$ and $\mathcal{W}_C$ is generated by $b_{\mathbb{F}}$, the mapping $\iota : L^p(W_{C_{\mathbb{F}}}) \to L^p_{\mathcal{W}_C}(\widehat{\Gamma})$ with $\iota(\varphi)(X) := \varphi\left(b_{\mathbb{F}}(X, \cdot)\right)$ defines a homeomorphic isomorphism (isometric isomorphism in case $p \geq 1$) from $L^p(W_{C_{\mathbb{F}}})$ onto $L^p_{\mathcal{W}_C}(\widehat{\Gamma})$.

(2) The proof of assertion (2) is similar. □

Corollary 2.3 indicates how large Loeb spaces are. Here we see that any L^p space on any abstract Wiener Fréchet space over a fixed Hilbert space can be canonically embedded into the L^p-space on the Loeb measure $\widehat{\Gamma}$.

3. Chaos Decomposition for extensions of abstract Wiener space

Malliavin calculus can be founded on chaos decomposition of the Wiener space. In order to establish chaos decomposition results for spaces different from $L^p(W_{C_\mathbb{F}})$ (in particular, for extensions of $L^p(W_{C_\mathbb{F}})$), we use the notion of admissible sequence introduced in [28] for the classical Wiener space. To this end we use internal iterated Itô integrals and their standard parts.

Let us first recall the definition of S-integrability. Fix an internal probability space $(\Lambda, \mathcal{C}, P)$. A $\mathcal{C}$-measurable function $G : \Lambda \to {}^*\mathbb{R}$, is called S_P**-integrable** if $\int_{\{|F| \geq K\}} |F| \, dP \approx 0$ for all unlimited $K \in^* \mathbb{N}$, in which case we will write $F \in SL^1(P)$.

Fix an internal function $F : T^n \to \mathbb{L}^{\otimes_n}$. The mapping $I_n(F) : \Omega \to {}^*\mathbb{R}$, simply defined by

$$I_n(F)(X) := \sum_{t_1 < ... < t_n} F(t_1, ..., t_n)(X_{t_1}, ..., X_{t_n})$$

is called the **(n-fold) internal iterated Itô integral of** F. If $n = 1$, we set $I(F) := I_1(F)$.

If $F \in SL^2(\nu^n, \mathbb{L}^{\otimes_n})$, i.e., $\|F\|^2_{\mathbb{L}^n}$ is $SL^1(\nu^n)$, then $\mathbb{E}\,|I_n(F)|^p$ is limited for all standard $p \in \mathbb{R}^+$ (see [26]). In this case ${}^\circ I_n(F)$ exists $\widehat{\Gamma}$-a.s. and $|{}^\circ I_n(F)|^p$ is $\widehat{\Gamma}$-integrable. If, in addition, F is nearstandard $\widehat{\nu^n}$-a.s, we may define

$$I_n({}^\circ F) := {}^\circ I_n(F).$$

It is easy to see that $I_n({}^\circ F)$ is well defined. It is called the **(n-fold) iterated Ito integral of** ${}^\circ F$. Note that, if $F \in SL^2(\nu^n, \mathbb{L}^{\otimes_n})$ and ${}^\circ F$ exists $\widehat{\nu^n}$-a.s., then ${}^\circ F$ belongs to the space $L^2\left(\widehat{\nu^n}, \mathbb{H}^{\otimes_n}\right)$ of Bochner $\widehat{\nu^n}$ square integrable functions from T^n into the n-fold tensor product $\mathbb{H}^{\otimes_n}$ of $\mathbb{H}$. Moreover, note that for all $f \in L^2\left(\widehat{\nu^n}, \mathbb{H}^{\otimes_n}\right)$, $g \in L^2\left(\widehat{\nu^m}, \mathbb{H}^{\otimes_m}\right)$

$$\mathbb{E}\left(I_n(f) \cdot I_m(g)\right) = \begin{cases} \int_{T^n_<} \langle f, g \rangle_{\mathbb{H}^n} \, d\widehat{\nu^n}, & \text{if } n = m \\ 0, & \text{if } n \neq m. \end{cases}$$

Now assume, in addition, that there exists a function $f : [0,1]^n \to \mathbb{H}^{\otimes_n}$ such that $f \circ st = {}^\circ F$ $\widehat{\nu^n}$-a.s. (recall that any measurable function $f : [0,1]^n \to \mathbb{H}^{\otimes_n}$ is of this shape). Then F is called a **lifting** of f and of $f \circ st$ as well. It follows that $f \in L^2\left(\lambda^n, \mathbb{H}^{\otimes_n}\right)$ and $I_n(f) := I_n({}^\circ F)$ is $\widehat{\Gamma}$-a.s. well defined. Using the standard part map $b_\mathbb{F}$, the integral $I_n(f)$ can be converted to a standard mapping $I^\mathbb{F}_n(f)$ with domain $C_\mathbb{F}$: By Theorem 2.1, each $g \in C_\mathbb{F}$ has the form $g := b_\mathbb{F}(X, \cdot)$. Then we set

$$I^\mathbb{F}_n(f)(b_\mathbb{F}(X, \cdot)) := I_n(f)(X).$$

Since $I_n(f)$ is $\mathcal{W}_C$-measurable, $I^\mathbb{F}_n(f)$ is well defined and is the usual **standard iterated Itô integral on** $L^2(W_{C_\mathbb{F}})$ (see [26]).

To establish chaos decompositions, fix a vector space $\mathfrak{H} \subseteq SL^2(\nu, \mathbb{L})$ over $\mathbb{R}$ such that the standard part ${}^\circ F : T \to \mathbb{H}$ of F exists $\widehat{\nu}$-a.s. for each $F \in \mathfrak{H}$, thus $\|{}^*\,({}^\circ F) - F\|_\mathbb{L} \approx 0$ $\widehat{\nu}$-a.s. In this case $\mathfrak{H}$ is called an **admissible set**.

Examples 3.1.
(I) $\overline{\mathfrak{H}} := \left\{F \in SL^2(\nu, \mathbb{L}) \mid {}^\circ F \text{ exists } \widehat{\nu}\text{-a.s.}\right\}$ is admissible.
(II) $\mathfrak{H}_{C_\mathbb{H}} := \left\{F \in \overline{\mathfrak{H}} \mid {}^\circ F = f \circ st \ \widehat{\nu}\text{-a.s. for some } f \in L^2(\lambda, \mathbb{H})\right\}$ is admissible.
(III) $\mathfrak{H}_\mathbb{H} := \{F \in \mathbb{L} \mid {}^\circ F \text{ exists}\}$ is admissible. Here we identify $\mathbb{L}$ with the set of constant functions from T into $\mathbb{L}$.

If a function g is $\mathcal{M}$-measurable, we will often write $g \in \mathcal{M}$. For each admissible set $\mathfrak{H}$ we define σ-algebras $\mathcal{W}^T_\mathfrak{H} \subseteq L_\nu(T)$ and $\mathcal{W}^\Omega_\mathfrak{H} \subseteq L_\Gamma(\mathcal{B})$:

$$\mathcal{W}^T_\mathfrak{H} := \sigma\left\{{}^\circ F \mid F \in \mathfrak{H}\right\} \vee \mathcal{N}_{\widehat{\nu}},$$

the σ-algebra generated by the set ${}^\circ\mathfrak{H}$ and the set $\mathcal{N}_{\widehat{\nu}}$ of all $\widehat{\nu}$-nullsets. Set

$$\mathcal{W}^\Omega_\mathfrak{H} := \sigma\left\{I({}^\circ F) \mid {}^\circ F \in \mathcal{W}^T_\mathfrak{H}\right\} \vee \mathcal{N}_{\widehat{\Gamma}}.$$

We will use the notation $\mathcal{W}^x_{C_\mathbb{H}}, \mathcal{W}^x_\mathbb{H}$ if $\mathfrak{H} = \mathfrak{H}_{C_\mathbb{H}}, \mathfrak{H} = \mathfrak{H}_\mathbb{H}$.

Examples 3.2.
(I) If $\mathfrak{H} = \overline{\mathfrak{H}}$, then $\mathcal{W}^T_\mathfrak{H} = L_\nu(T)$.
(II) If $\mathfrak{H} = \mathfrak{H}_{C_\mathbb{H}}$, then

$$\mathcal{W}^T_\mathfrak{H} = \left\{st^{-1}[B] \mid B \subseteq [0,1] \text{ is Lebesgue-measurable}\right\} \vee \mathcal{N}_{\widehat{\nu}}.$$

(III) If $\mathfrak{H} = \mathfrak{H}_\mathbb{H}$, then $\mathcal{W}^T_\mathfrak{H} = \{\emptyset, T\} \vee \mathcal{N}_{\widehat{\nu}}$.

A function f, defined on T^n or $[0,1]^n$ with values in n-fold tensor products, is called **symmetric** if $f_{(t_1,\dots,t_n)}(a_1,\dots,a_n) = f_{(t_{\sigma_1},\dots,t_{\sigma_n})}(a_{\sigma_1},\dots,a_{\sigma_n})$ for all permutations σ of $\{1,\dots,n\}$.

Fix an admissible set $\mathfrak{H}$. A sequence $(\mathfrak{H}_n)_{n\in\mathbb{N}_0}$ of vector spaces $\mathfrak{H}_n$ over $\mathbb{R}$ of symmetric nearstandard functions in $SL^2(\nu^n, \mathbb{L}^{\otimes_n})$ with $\mathfrak{H}_0 := \mathbb{R}$ is called an **admissible sequence over** $\mathfrak{H}$ if
(EW 1) ${}^\circ F \in \left(\mathcal{W}^T_\mathfrak{H}\right)^n$ for each $F \in \mathfrak{H}_n$.
(EW 2) ${}^\circ I_n(F) \in \mathcal{W}^\Omega_\mathfrak{H}$ for each $F \in \mathfrak{H}_n$. Recall that ${}^\circ I_n(F) = I_n({}^\circ F)$.
(EW 3) ${}^\circ\mathfrak{H}_n$ is a closed subspace of $L^2\left(\widehat{\nu^n}, \mathbb{H}^{\otimes_n}\right)$.
(EW 4) If $F_1,\dots,F_n \in SL^2(\nu, \mathbb{L})$ such that all ${}^\circ F_i$ are $\mathcal{W}^T_\mathfrak{H}$-measurable, then $(F_1 \odot \dots \odot F_n)^s \in \mathfrak{H}_n$, where G^s denotes the **symmetrization of** a function $G : T^n \to \mathbb{L}^{\otimes_n}$, i.e.,

$$G^s(t)(a) := \sum_\sigma G(t_{\sigma_1},\dots,t_{\sigma_n})(a_{\sigma_1},\dots,a_{\sigma_n}),$$

where σ runs through the permutations of $1,\dots,n$; $F_1 \odot \dots \odot F_n$ denotes the **tensor product** of the F_i, i.e.,

$$(F_1 \odot \dots \odot F_n)(t_1,\dots,t_n)(a_1,\dots,a_n) := F_1(t_1)(a_1) \cdot \ldots \cdot F_n(t_n)(a_n).$$

It follows that $\mathfrak{H}_1 = \left\{F \in SL^2(\nu) \mid {}^\circ F \in \mathcal{W}^T_\mathfrak{H}\right\}$. Set

$$L^p_\mathfrak{H}(\widehat{\Gamma}) := L^p(\Omega, \mathcal{W}^\Omega_\mathfrak{H}, \widehat{\Gamma}) \text{ and } L^p_\mathfrak{H}(\widehat{\nu^n}) := L^p(T^n, \left(\mathcal{W}^T_\mathfrak{H}\right)^n \vee \mathcal{N}_{\widehat{\nu^n}}, \widehat{\nu^n}).$$

Analogous notations and are used for L^p-spaces of Bochner p-times integrable functions with values in $\mathbb{H}^{\otimes n}$, which are denoted by $L^p_{\mathfrak{H}}(y,\mathbb{H}^{\otimes n})$, with $y=\widehat{\Gamma}$ or $y=\widehat{\nu^n}$. We will use the notation $L^p_{C_{\mathbb{H}}}(\cdot)$, $L^p_{\mathbb{H}}(\cdot)$ if $\mathfrak{H}=\mathfrak{H}_{C_{\mathbb{H}}},\mathfrak{H}=\mathfrak{H}_{\mathbb{H}}$. Set

$$SL^2_s(\nu^n,\mathbb{L}^{\otimes n}):=\left\{F\in SL^2(\nu^n,\mathbb{L}^{\otimes n})\mid F\text{ is symmetric}\right\}.$$

Examples 3.3.

(I) Set $\mathfrak{H}_0:=\mathbb{R}$ and for $n>0$

$$\mathfrak{H}_n:=\left\{F\in SL^2_s(\nu^n,\mathbb{L}^{\otimes n})\mid \ {}^\circ F\in(L_\nu(T))^n\vee\mathcal{N}_{\widehat{\nu^n}}\text{ exists }\widehat{\nu^n}\text{-a.s. }\right\}.$$

Then $(\mathfrak{H}_n)$ is an admissible sequence over $\overline{\mathfrak{H}}$. We have $L^p_{\overline{\mathfrak{H}}}(\widehat{\nu^n})=L^p(\widehat{\nu}^n)$, where

$$L^p(\widehat{\nu}^n):=L^p(T^n,(L_\nu(T))^n\vee\mathcal{N}_{\widehat{\nu^n}},\widehat{\nu}^n),$$

which is a strict extension of the Lebesgue space $L^p(\lambda^n)$. Moreover, $L^p(\widehat{\nu}^n)$ is a strict subspace of $L^p(\widehat{\nu^n})$, a result due to Y. Sun [30]).

(II) Set $\mathfrak{H}_0:=\mathbb{R}$ and for $n>0$

$$\mathfrak{H}_n:=\left\{F\in SL^2_s(\nu^n,\mathbb{L}^{\otimes n})\mid \ {}^\circ F=f\circ st\ \widehat{\nu^n}\text{-a.s. for some }f\in L^2(\lambda^n,\mathbb{H}^{\otimes n})\right\}.$$

Then $(\mathfrak{H}_n)$ is an admissible sequence over $\mathfrak{H}_{C_{\mathbb{H}}}$. Note that $L^p_{C_{\mathbb{H}}}(\widehat{\Gamma})=L^p_{\mathcal{W}_C}(\widehat{\Gamma})$. By Corollary 2.3, $L^p_{\mathcal{W}_C}(\widehat{\Gamma})$ can be identified with $L^p(W_{C_{\mathbb{F}}})$. In a similar way, $L^p_{C_{\mathbb{H}}}(\widehat{\nu^n})$ can be identified with $L^p(\lambda^n)$: $\iota:L^p(\lambda^n)\to L^p_{C_{\mathbb{H}}}(\widehat{\nu^n})$, $\iota(\varphi)(t):=\varphi({}^\circ t)$ defines a homeomorphic isomorphism from $L^p(\lambda^n)$ onto $L^p_{C_{\mathbb{H}}}(\widehat{\nu^n})$.

(III) Set $\mathfrak{H}_0:=\mathbb{R}$ and for $n>0$

$$\mathfrak{H}_n:=\left\{F\in\mathbb{L}^{\otimes n}\mid F\text{ is symmetric and }{}^\circ F\in\mathbb{H}^{\otimes n}\right\}.$$

Then $(\mathfrak{H}_n)$ is an admissible sequence over $\mathfrak{H}_{\mathbb{H}}$, and $L^p_{\mathbb{H}}(\widehat{\Gamma})=L^p_{\mathcal{W}}(\widehat{\Gamma})$ can be identified with the space $L^p(W_{\mathbb{F}})$. Note that, if $F\in\mathfrak{H}_n$, then

$$I_n(F)(X)=\sum_{t_1<\ldots<t_n}F(X_{t_1},...,X_{t_n}).$$

(IV) To introduce a forth example and also later on it is convenient to use the following shorthand. For internal functions $F:T^{n+1}\to\mathbb{L}^{\otimes n+1}$ we set

$$\widetilde{F}(t_1,...,t_{n+1})(a_1,...,a_{n+1}):=\sum_{i=1}^{n+1}F(t_1,...,t_{i-1},t_{i+1},...,t_{n+1},t_i)(a_1,...,a_{i-1},a_{i+1},...,a_{n+1},a_i).$$

Note that $\widetilde{F}$ is symmetric if F is symmetric in the first n components, and note that $\widetilde{F}\in SL^2(\nu^{n+1},\mathbb{L}^{\otimes n+1})$ if $F\in SL^2(\nu^{n+1},\mathbb{L}^{\otimes n+1})$. Let $(\mathfrak{H}_n)$ is an admissible sequence over $\mathfrak{H}$. Define

$$\widetilde{\mathfrak{H}}_n:=\left\{F\in\mathfrak{H}_n\mid\ {}^\circ\widetilde{F\odot G}\in\ {}^\circ\mathfrak{H}_{n+1}\text{ for each }G\in SL^2(\nu,\mathbb{L})\text{ with }{}^\circ G\in\mathcal{W}^T_{\mathfrak{H}}\right\}.$$

It is straightforward to prove that $\left(\widetilde{\mathfrak{H}}_n\right)$ is an admissible sequence over $\mathfrak{H}$.

The following result is of great importance for the following investigations. Its proof is similar to the usual proof of chaos decomposition results.

Theorem 3.4. *Let $(\mathfrak{H}_n)$ be an admissible sequence over $\mathfrak{H}$. Fix $\varphi \in L^2_{\mathfrak{H}}(\widehat{\Gamma})$. Then φ has the decomposition*

$$\varphi = \sum_{n=0}^{\infty} {}^\circ I_n(F_n) = \sum_{n=0}^{\infty} I_n({}^\circ F_n) \text{ in } L^2_{\mathfrak{H}}(\widehat{\Gamma}),$$

where $F_n \in \widetilde{\mathfrak{H}}_n$ (see Example (IV)). Moreover, ${}^\circ I_0(F_0) = {}^\circ F_0 = \mathbb{E}(\varphi)$. We may assume that $F_n(t) = 0$ if the components of t are not pairwise different.

The ***kernels*** *${}^\circ F_n$ of φ in this decomposition of φ are uniquely determined: if $\varphi = \sum_{n=0}^{\infty} {}^\circ I_n(G_n)$ converges in $L^2(\widehat{\Gamma})$ with symmetric G_n in $SL^2(\nu^n, \mathbb{L}^{\otimes_n})$, then $\int_{T^n} \|F_n - G_n\|^2_{\mathbb{L}^n} d\nu^n \approx 0$, thus*

$${}^\circ F_n = {}^\circ G_n \text{ in } L^2\left(\widehat{\nu^n}, \mathbb{H}^{\otimes_n}\right).$$

Proof. Let M be the set of all $\varphi \in L^2_{\mathfrak{H}}(\widehat{\Gamma})$ having the claimed decomposition. Since M is a closed subspace of $L^2_{\mathfrak{H}}(\widehat{\Gamma})$, we get $M = L^2_{\mathfrak{H}}(\widehat{\Gamma})$ if $\varphi = 0$, whenever $\varphi \perp M$. Fix $\varphi \perp M$. Then $\varphi \perp {}^\circ I_n(F^{\odot_n})$ for all $F \in \mathfrak{H}_1$. Note that in $L^2_{\mathfrak{H}}(\widehat{\Gamma})$ (see [26] for the special case $\mathfrak{H} = \mathfrak{H}_{C_{\mathbb{H}}}$).

$${}^\circ I_n(F^{\odot_n}) \cdot {}^\circ I(F) = (n+1)\, {}^\circ I_{n+1}(F^{\odot_{n+1}}) + {}^\circ I_{n-1}(F^{\odot_{n-1}}) \cdot {}^\circ \int_T \|F\|_{\mathbb{L}}\, d\nu.$$

It follows that $H_n({}^\circ I(F)) = {}^\circ I_n(F^{\odot_n})$ for $F \in \mathfrak{H}_1$ with $\int_T \|F\|_{\mathbb{L}}\, d\nu \approx 1$, where H_n is the n^{th} Hermite polynomial $H_n(x) := \frac{(-1)^n}{n!} e^{\frac{1}{2}x^2} \frac{d^n}{(dx)^n} e^{-\frac{1}{2}x^2}$. Since each polynomial is a linear combination of Hermite polynomials, we have $\varphi \perp p({}^\circ I(F))$ for each polynomial p and each $F \in \mathfrak{H}_1$. It follows that $\varphi \perp e^{{}^\circ I(F)}$. Since φ belongs to the σ-algebra generated by the functions ${}^\circ I(F)$ with $F \in \mathfrak{H}_1$ and, we have $\varphi = 0$.

To prove the uniqueness, let $\varphi = \sum_{n=0}^{\infty} {}^\circ I_n(F_n) = \sum_{n=0}^{\infty} {}^\circ I_n(G_n)$. Then, by the pairwise orthogonality of the ${}^\circ I_n(F_n)$, $n \in \mathbb{N}_0$, and of the ${}^\circ I_n(G_n)$

$$0 = \mathbb{E}\left({}^\circ I_n(F_n - G_n)\right)^2 = {}^\circ \mathbb{E}\left(I_n(F_n - G_n)\right)^2 = {}^\circ \int_{T^n_<} \|F_n - G_n\|^2_{\mathbb{L}^n} d\nu^n,$$

where $T^n_< := \{(t_1, ..., t_n) \in T^n \mid t_1 < ... < t_n\}$. The result follows from the symmetry of the F_n, G_n. $\square$

This result shows that admissible sequences $(\mathfrak{H}_n)_{n\in\mathbb{N}}$ over $\mathfrak{H}$ are uniquely determined by $\mathfrak{H}$ in the following sense:

Corollary 3.5. *Let $(\mathfrak{H}_n)$ and $(\mathfrak{G}_n)$ be admissible sequences over $\mathfrak{H}$. Then for each $F \in \mathfrak{H}_n$ there exists an $G \in \mathfrak{G}_n$ with $\int_{T^n} \|F - G\|^2_{\mathbb{L}^n} d\nu^n \approx 0$, thus $F \approx_{\mathbb{L}^n} G$ $\widehat{\nu^n}$-a.s, thus ${}^\circ F = {}^\circ G$ $\widehat{\nu^n}$-a.s, thus ${}^\circ F_n = {}^\circ G_n$ in $L^2\left(\widehat{\nu^n}, \mathbb{H}^{\otimes_n}\right)$.*

Proof. Fix $F \in \mathfrak{H}_n$. Since ${}^\circ I_n(F) \in L^2_{\mathfrak{H}}(\widehat{\Gamma})$, by Theorem 3.4, ${}^\circ I_n(F)$ has a chaos decomposition ${}^\circ I_n(F) = \sum_{k=0}^{\infty} {}^\circ I_k(G_k)$ with $G_k \in \mathfrak{G}_k$. By Theorem 3.4, $\int_{T^n} \|F - G_n\|^2_{\mathbb{L}^n} \, d\nu^n \approx 0$. □

If $\mathfrak{H} = \mathfrak{H}_{C_{\mathbb{H}}}$, $\mathfrak{H} = \mathfrak{H}_{\mathbb{H}}$, then, since $L^2_{\mathfrak{H}}(\widehat{\Gamma}) = L^2(W_{C_{\mathbb{F}}})$, $L^2_{\mathfrak{H}}(\widehat{\Gamma}) = L^2(W_{\mathbb{F}})$, Theorem 3.4 yields a chaos decomposition of the standard spaces $L^2(W_{C_{\mathbb{F}}})$, $L^2(W_{\mathbb{F}})$, respectively.

Our aim now is to extend the preceding results to stochastic processes. To this end fix an admissible sequence $(\mathfrak{H}_n)$ over some admissible set $\mathfrak{H}$. Define

$$L^p_{\mathfrak{H}}(\widehat{\Gamma \otimes \nu}) := \left(\Omega \times T, \left(\mathcal{W}^{\Omega}_{\mathfrak{H}} \otimes \mathcal{W}^{T}_{\mathfrak{H}}\right) \vee \mathcal{N}_{\widehat{\Gamma \otimes \nu}}, \widehat{\Gamma \otimes \nu}\right).$$

In case $\mathfrak{H} = \mathfrak{H}_{C_{\mathbb{H}}}$, $\mathfrak{H} = \mathfrak{H}_{\mathbb{H}}$, we shall write $L^p_{C_{\mathbb{H}}}(\widehat{\Gamma \otimes \nu})$, $L^p_{\mathbb{H}}(\widehat{\Gamma \otimes \nu})$. Note that $L^p_{C_{\mathbb{H}}}(\widehat{\Gamma \otimes \nu})$ can be identified with the standard Lebesgue space

$$L^p(W_{C_{\mathbb{F}}} \otimes \lambda) := L^p(C_{\mathbb{F}} \otimes [0,1], \mathcal{B}(C_{\mathbb{F}}) \otimes Leb[0,1], W_{C_{\mathbb{F}}} \otimes \lambda);$$

the homeomorphic isomorphism is defined by $\iota(\varphi)(X,t) = \varphi(b_{\mathbb{F}}(X,\cdot), {}^\circ t)$. Analogous notations and results hold for L^p-spaces of Bochner p-times integrable functions with values in $\mathbb{H}^{\otimes n}$, which are denoted by $L^p_x(y, \mathbb{H}^{\otimes n})$.

In order to obtain a suitable orthogonal decomposition of the spaces $L^2_{\mathfrak{H}}(\widehat{\Gamma \otimes \nu}, \mathbb{H})$, we need the following abbreviations. For internal functions $F : T^{n+1} \to \mathbb{L}^{\otimes n+1}$ we set

$$I_{n,1}(F)(X,t) := \sum_{t_1 < \ldots < t_n} F(t_1, ..., t_n, t)(X_{t_1}, ..., X_{t_n}, \cdot) \in \mathbb{L}' = \mathbb{L}.$$

For each $t \in T$ we set $[1,t] := \{s \in T \mid s \leq t\}$ and define

$$(1_{[1,t]} \cdot_i F)(t_1, ..., t_{n+1})(\cdot) := 1_{[1,t]}(t_i) \cdot F(t_1, ..., t_{n+1})(\cdot).$$

Theorem 3.6. *Let $\varphi \in L^2_{\mathfrak{H}}(\widehat{\Gamma \otimes \nu}, \mathbb{H})$. Then there exists a sequence $(F_n)_{n \in \mathbb{N}}$ of internal functions $F_n : T^{n+1} \to \mathbb{L}^{\otimes n+1}$ with the following five properties*:

(1) *F_n is symmetric in the first n variables, $F_n \in SL^2(\nu^{n+1}, \mathbb{L}^{\otimes n+1})$ and ${}^\circ F_n \in L^2_{\mathfrak{H}}(\widehat{\nu^{n+1}}, \mathbb{H}^{\otimes n+1})$.*

(2) *$I_{n,1}(F_n) \in SL^2(\Gamma \otimes \nu, \mathbb{L})$ and ${}^\circ I_{n,1}(F_n) \in L^2_{\mathfrak{H}}(\widehat{\Gamma \otimes \nu}, \mathbb{H})$.*

(3) *$I_{n+1}(\widetilde{F_n}) \in SL^2(\Gamma)$ and ${}^\circ I_{n+1}(\widetilde{F_n}) \in L^2_{\mathfrak{H}}(\widehat{\Gamma})$.*

(4) *$I_{n+1}\left(\widetilde{1_{[1,\cdot]} \cdot_{n+1} F_n}\right) \in SL^2(\Gamma \otimes \nu)$, thus $t \to {}^\circ I_{n+1}\left(\widetilde{1_{[1,t]} \cdot_{n+1} F_n}\right)$ is in $L^2(\widehat{\Gamma \otimes \nu})$. If $[0,t] \in \mathcal{W}^T_{\mathfrak{H}}$ for all $t \in T$, then ${}^\circ I_{n+1}\left(\widetilde{1_{[1,t]} \cdot_{n+1} F_n}\right)$ is in $L^2_{\mathfrak{H}}(\widehat{\Gamma \otimes \nu})$.*

(5) *$\varphi = \sum_{n=0}^{\infty} {}^\circ I_{n,1}(F_n)$ converges in $L^2_{\mathfrak{H}}(\widehat{\Gamma \otimes \nu}, \mathbb{H})$.*

We may further assume that $F(t_1, ..., t_{n+1}) = 0$ *if the* t_i *are not pairwise different.*

If $\varphi = \sum_{n=0}^{\infty} {}^\circ I_{n,1}(K_n)$ *in* $L^2(\widehat{\Gamma \otimes \nu}, \mathbb{H})$ *and* $K_n \in SL^2(\nu^{n+1}, \mathbb{L}^{\otimes_{n+1}})$ *is symmetric in the first n variables, then* $F_n \approx_{\mathbb{L}^{n+1}} K_n$ $\widehat{\nu^{n+1}}$*-a.s.*

For $\mathfrak{H} = \mathfrak{H}_{C_{\mathbb{H}}}$*, this decomposition also yields a corresponding decomposition of functionals in* $L^2(W_{C_{\mathbb{F}}} \otimes \lambda, \mathbb{H})$.

Proof. Let M be the set of all $\varphi \in L^2_{\mathfrak{H}}(\widehat{\Gamma \otimes \nu}, \mathbb{H})$ such that there exists a sequence $(F_n)_{n \in \mathbb{N}}$ of internal functions $F_n : T^{n+1} \to \mathbb{L}^{\otimes_{n+1}}$ with the properties (1),...,(5). Obviously, M is a linear subspace of $L^2_{\mathfrak{H}}(\widehat{\Gamma \otimes \nu}, \mathbb{H})$ and M is closed. Therefore, it suffices to prove that $\varphi := 1_{B \times C} \odot \langle a, \cdot \rangle = 1_B \odot 1_C \odot \langle a, \cdot \rangle \in M$ for all $B \in \mathcal{W}^{\Omega}_{\mathfrak{H}}$, $C \in \mathcal{W}^T_{\mathfrak{H}}$ and $a \in \mathbb{H}$. According to Theorem 3.4, 1_B has the decomposition

$$1_B = \mathbb{E}(1_B) + \sum_{n=1}^{\infty} {}^\circ I_n(G_n) = \mathbb{E}(1_B) + \sum_{n=1}^{\infty} I_n({}^\circ G_n) \text{ in } L^2_{\mathfrak{H}}(\widehat{\Gamma}).$$

By Example (IV), we may assume that $G_n \in \widetilde{\mathfrak{H}}_n$. Since $C \in L_\nu(T)$, there exists an internal subset $A \subset T$ with $\widehat{\nu}(A \bigtriangleup C) = 0$. We set for $n \in \mathbb{N}$ and all $t \in T^n$, $s \in T$ and $y \in \mathbb{L}$

$$F_n := G_n \odot 1_A \odot a : (t, s, y) \mapsto G_n(t) \cdot 1_A(s) \cdot \langle a, y \rangle \, .$$

$$F_0 := \mathbb{E}(1_B) \cdot 1_A \odot a.$$

If the elements in the vector $r \in T^{n+1}$ are not pairwise different, set $F_n(r) := 0$. Note that $F_n \in SL^2(\nu^{n+1}, \mathbb{L}^{\otimes_{n+1}})$ and that Conditions (1),...,(5) are true.

The proof of the uniqueness is similar to the corresponding proof under Theorem 3.4. □

Using saturation, we obtain the following lifting theorem, which will be important in Section 6 and follows from Theorem 3.6.

Proposition 3.7. *Fix* $\varphi = \sum_{n=0}^{\infty} {}^\circ I_{n,1}(F_n) \in L^2_{\mathfrak{H}}(\widehat{\Gamma \otimes \nu}, \mathbb{H})$. *Then there exists an internal extension* $(F_n)_{n \in {}^*\mathbb{N}}$ *of* $(F_n)_{n \in \mathbb{N}}$ *and an unlimited* $K \in {}^*\mathbb{N}$ *such that for each unlimited* $M \leq K$, $M \in {}^*\mathbb{N}$,

$$\Phi := \sum_{n=0}^{M} I_{n,1}(F_n) \in SL^2(\Gamma \otimes \nu, \mathbb{L}) \;\; \text{and} \;\; \Phi \approx_{\mathbb{L}} \varphi \; \widehat{\Gamma \otimes \nu}\text{-a.s.}$$

We may assume that all the F_n *are symmetric in the first n variables and we may further assume that* $F_n(t_1, ..., t_{n+1}) \neq 0$ *implies that the* (t_i) *are pairwise different.*

4. Skorohod integral processes

In this section we present a Kolmogorov continuity criterion, by which it is possible to convert an internal process $F : \Omega \times T \to {}^*\mathbb{R}$ to a continuous process $f : \Omega \times [0,1] \to \mathbb{R}$, uniquely determined except for modifications. An application shows that Skorohod integral processes, defined on finite chaos levels, have continuous modifications. This is an extension of a result, due to Imkeller [15], for the classical Wiener space $C_{\mathbb{R}}$. Our proof is distinct from his proof. While he uses deep results on isoperimetric inequalities, our proof is elementary and is based on a refinement of Kolmogorov's continuity theorem.

The Skorohod integral process $\delta^{pr}(f) \in L^2\left(\widehat{\Gamma \otimes \nu}\right)$ of a process f in a dense subspace of $L^2_{\mathfrak{H}}\left(\widehat{\Gamma \otimes \nu}, \mathbb{H}\right)$ can be defined to be the standard part of the internal Riemann Stieltjes integral $\int_1^t F\Delta B$ with respect to the internal Brownian motion B, defined in Section 2. Here $F \in SL^2\left(\Gamma \otimes \nu, \mathbb{L}\right)$ is a lifting of f and

$$\int_1^t F\Delta B(X) := \int F\Delta B(X,t) := \sum_{s \leq t, s \in T} F(X,s)(X_s) \in {}^*\mathbb{R},$$

where we identify $\mathbb{L}$ with its dual $\mathbb{L}'$. Note that

$$\int_1^t I_{n,1}(F)\Delta B(X) = I_{n+1}\left(\widetilde{1_{[1,t]} \cdot_{n+1} F}\right),$$

if $F(t_1, ..., t_{n+1}) \neq 0$ implies that the t_i are pairwise different and F_n is symmetric in the first n arguments.

To define the **Skorohod integral process** δ^{pr}, fix $\varphi \in L^2_{\mathfrak{H}}(\widehat{\Gamma \otimes \nu}, \mathbb{H})$ with $\varphi = \sum_{n=0}^{\infty} {}^\circ I_{n,1}(F_n)$ according to Theorem 3.6. We define $\delta^{pr}\varphi : \Omega \times T \to {}^*\mathbb{R}$ by Riemann Stieltjes integration under the sum and under the standard part map:

$$(\delta^{pr}\varphi)(\cdot, t) = \sum_{n=0}^{\infty} \left({}^\circ \int_1^t I_{n,1}(F_n)\Delta B\right) = \sum_{n=0}^{\infty} {}^\circ I_{n+1}(\widetilde{1_{[1,t]} \cdot_{n+1} F_n})$$

if this series converges in $L^2(\widehat{\Gamma \otimes \nu})$, i.e.,

$$\sum_{n=0}^{\infty} \int_{\Omega \times T} ({}^\circ I_{n+1}(\widetilde{1_{[1,t]} \cdot_{n+1} F_n}))^2 d\widehat{\Gamma \otimes \nu} < \infty.$$

The densely defined linear operator $\delta^{pr} : L^2_{\mathfrak{H}}(\widehat{\Gamma \otimes \nu}, \mathbb{H}) \to L^2(\widehat{\Gamma \otimes \nu})$ is called the **Skorohod integral process**. The **Skorohod integral** $\delta\varphi$ of φ is defined by setting $\delta\varphi := \delta^{pr}\varphi(\cdot, H)$. Then

$$\delta\varphi = \sum_{n=0}^{\infty} {}^\circ I_{n+1}(\widetilde{F_n}).$$

Since $\delta\varphi$ is $\mathcal{W}_{\mathfrak{H}}^{\Omega}$-measurable, $\delta\varphi$ exists iff $\sum_{n=0}^{\infty}\int_{\Omega}({}^{\circ}I_{n+1}(\widetilde{F_n}))^2 d\widehat{\Gamma} < \infty$. The densely defined linear operator $\delta : L^2_{\mathfrak{H}}(\widehat{\Gamma\otimes\nu}, \mathbb{H}) \to L^2_{\mathfrak{H}}(\widehat{\Gamma})$ is called the **Skorohod integral**.

Since we may identify for $\mathfrak{H} = \mathfrak{H}_{C_{\mathbb{H}}}$ the spaces $L^2_{\mathfrak{H}}(\widehat{\Gamma})$ with $L^2(W_{C_{\mathbb{F}}})$ and $L^2_{\mathfrak{H}}(\widehat{\Gamma\otimes\nu}, \mathbb{H})$ with $L^2(W_{C_{\mathbb{F}}}\otimes\lambda, \mathbb{H})$, the Skorohod integral and the Skorohod integral process are defined for these standard spaces. In the same way, we may identify for $\mathfrak{H} = \mathfrak{H}_{\mathbb{H}}$, the spaces $L^2_{\mathfrak{H}}(\widehat{\Gamma})$ with $L^2(W_{\mathbb{F}})$ and $L^2_{\mathfrak{H}}(\widehat{\Gamma\otimes\nu}, \mathbb{H})$ with $L^2_{\mathfrak{H}}(\widehat{\Gamma}, \mathbb{H})$ and $L^2(W_{\mathbb{F}}, \mathbb{H})$. Therefore, the Skorohod integral and the Skorohod integral process are also defined for these standard spaces.

Using saturation we have the following lifting theorem for the Skorohod integral:

Proposition 4.1. *Fix* $\varphi = \sum_{n=0}^{\infty}{}^{\circ}I_{n,1}(F_n) \in L^2_{\mathfrak{H}}(\widehat{\Gamma\otimes\nu}, \mathbb{H})$. *Then,we have*: *if* $\delta\varphi$ *exists, then we can choose* Φ *and an unlimited* $K \in {}^*\mathbb{N}$ *such that the properties of Proposition* 3.7 *are fulfilled and such that, in addition,*

$$\delta\Phi = \int_1^H \Phi\Delta B = \sum_{n=0}^{M} I_{n+1}(\widetilde{F_n}) \in SL^2(\Gamma)$$

for all unlimited $M \leq K$ *and*

$$\delta\Phi \approx \delta\varphi\ \widehat{\Gamma}\text{-a.s. } \text{ in } \mathbb{R}.$$

Our aim is now to prove that Skorohod integral processes, defined on finite chaos levels, have continuous modifications. It is not possible to copy the proof of the continuity of the Itô integral, based on martingale theory. The Itô integral is a martingale, because the integrands are non time-anticipating. Since the Skorohod integral is also defined for time-anticipating integrands, the martingale properties fail in general.

To obtain continuous modifications in a more general context, assume first that $F : T \to {}^*\mathbb{R}$ is internal and that $D \subseteq T$ is S-**dense**, i.e., in each ε-neighborhood of each $r \in [0,1]$ there exists a $t \in D$ with $\left|\frac{t}{H} - r\right| < \varepsilon$. Suppose that F is uniformly S-continuous and nearstandard on D. Let $(d_n)_{n\in\mathbb{N}}$ be a sequence in D with $\lim_{n\to\infty}{}^{\circ}d_n = r$. Recall that, by convention, ${}^{\circ}d_n = {}^{\circ}\frac{d_n}{H}$. Then

$${}^{D}F : [0,1] \to \mathbb{R}, r \mapsto \lim_{n\to\infty}{}^{\circ}F(d_n)$$

is well defined and continuous. The function ${}^{D}F$ is called a **continuous modification of** F. The following example shows that this concept is closely related to the corresponding standard concept.

Proposition 4.2. *Suppose that* $f : \Omega\times[0,1] \to \mathbb{R}$ *is a process, which has a continuous modification* $g : \Omega\times[0,1] \to \mathbb{R}$ *in the standard sense. Then for each lifting* F *of* f *there exists an* S*-dense subset of* T *such that* ${}^{D}F = g$.

Proof. Let F be a lifting of f. Then, by Keisler's Fubini Theorem, $\widehat{\nu}\left(\widetilde{D}\right) = 1$, where

$$\widetilde{D} := \left\{ t \in T \mid {}^{\circ}F(\cdot, t) = f(\cdot, {}^{\circ}t)\ \widehat{\Gamma}\text{-a.s} \right\},$$

thus, $\widetilde{D}$ is S-dense in T. Note that there exists a countable S-dense subset $D \subset \widetilde{D}$ in T. Therefore, there exists a set $\widetilde{\Omega} \subset \Omega$ of $\widehat{\Gamma}$ measure 1 such that for all $X \in \widetilde{\Omega}$ and all $t \in D$,

$${}^{\circ}F(X, t) = f(X, {}^{\circ}t) = g(X, {}^{\circ}t).$$

Note that F is uniformly S-continuous and nearstandard on D and ${}^{D}F = g$ $\widehat{\Gamma}$-a.s. $\square$

In general, there may exist many continuous modifications of F. In order to obtain uniqueness, fix an internal process $F : \Omega \times T \to {}^{*}\mathbb{R}$ which is $\widehat{\Gamma}$-a.s. uniformly S-continuous and nearstandard on an S-dense $D \subseteq T$. A continuous modification ${}^{D}F : \Omega \times [0, 1] \to \mathbb{R}$ of F is called a **uniquely determined modification of** F if each other continuous modification ${}^{E}F$ of F is a modification of ${}^{D}F$ in the standard sense.

Let us call an internal process $F : \Omega \times T \to {}^{*}\mathbb{R}$ **weakly S-continuous on** T if for all $s, t \in T$ with $\frac{s}{H} \approx \frac{t}{H}$ there exists a set $U_{s,t} \subseteq \Omega$ of $\widehat{\Gamma}$-measure 1 such that $F(X, s) \approx F(X, t)$ for all $X \in U_{s,t}$.

Proposition 4.3. *Suppose that F is a weakly S-continuous process on T and D is an S-dense subset of T such that F is uniformly S-continuous and nearstandard on D $\widehat{\Gamma}$-a.s. Then ${}^{D}F$ is a uniquely determined continuous modification of F.*

Proof. We have already seen that ${}^{D}F$ is a continuous modification of F. To prove uniqueness, fix a second continuous modification ${}^{E}F$ of F. We have to prove that for all $k \in \mathbb{N}$ and all $r \in [0, 1]$

$$\widehat{\Gamma}\left\{ \left| {}^{D}F_r - {}^{E}F_r \right| \geq \frac{1}{k} \right\} = 0.$$

To this end let $(d_n), (e_n)$ be sequences in D, E, respectively, with $\lim {}^{\circ}d_n = r = \lim {}^{\circ}e_n$. It suffices to prove that for all $N \in \mathbb{N}$

$$\widehat{\Gamma}\left\{ \forall n \geq N \left| F_{d_n} - F_{e_n} \right| \geq \frac{1}{3k} \right\} = 0.$$

Let $(d_n)_{n \in {}^{*}\mathbb{N}}, (e_n)_{n \in {}^{*}\mathbb{N}}$ be internal extensions of $(d_n), (e_n)$, respectively, with $d_n, e_n \in T$ for all $n \in {}^{*}\mathbb{N}$. Then there exists an unlimited $M \in {}^{*}\mathbb{N}$ such that $\frac{d_K}{H} \approx r \approx \frac{e_K}{H}$ for all unlimited $K \in {}^{*}\mathbb{N}$, $K \leq M$. It suffices to prove that for all unlimited $K \in {}^{*}\mathbb{N}$, $K \leq M$

$$\Gamma\left\{ \left| F_{r_K} - F_{s_K} \right| \geq \frac{1}{3k} \right\} \approx 0.$$

This equality follows from the weak S-continuity of F and from $\frac{d_K}{H} \approx \frac{e_K}{H}$. $\square$

The following result is relevant for finding uniquely determined continuous modifications of internal processes. To prove it one may use a suitable time transformation (see [23]). Here we prefer a slight refinement of Kolmogorov's Continuity Theorem (see I. Karatzas and S. Shreve [16]).

Theorem 4.4. *Suppose that $F : \Omega \times T \to {}^*\mathbb{R}$ is an internal process. Set $F(\cdot, 0) := 0$. Suppose that there exists a continuous monotone increasing function $\alpha : [0,1] \to \mathbb{R}_0^+$ with $\alpha(0) = 0$ and a constant $c \in \mathbb{R}^+$ such that for all $t, s \in T \cup \{0\}$*

$$^\circ\mathbb{E}\left(\left(F_t - F_s\right)^4\right) \leq c\left(\alpha(^\circ t) - \alpha(^\circ s)\right)^2 \ \left(^\circ t := {}^\circ\frac{t}{H}\right).$$

(We may replace 4 by any number $p \geq 1$ and 2 by any number $q > 1$.)

Then F has a uniquely determined modification.

Proof. By normalization, we may assume that $\alpha(1) = 1$. For all $n \in \mathbb{N}$ and all $i = 0, ..., 2^n$ choose $t_i^n \in T \cup \{0\}$ such that $\alpha(^\circ t_i^n) = \frac{i}{2^n}$. Set $t_0^n = 0$ and $t_{2^n}^n = H$. We further assume that $t_{2^m \cdot k}^{n+m} = t_k^n$ for each $k \in \{0, ..., 2^n\}$. Set

$$D_n := \{t_i^n \mid i = 0, ..., 2^n\} \cup \left\{\frac{i \cdot H}{m} \mid 1 \leq m \leq n, i = 1, ..., m-1\right\}.$$

Since each $n \in \mathbb{N}$ divides H, $D_n \subseteq T \cup \{0\}$. Let $D_n = \{b_i^n \mid i = 0, ..., \overline{n}\}$ with $0 = b_0^n < ... < b_{\overline{n}}^n$. Note that $D_n \subseteq D_{n+1}$. Since α is monotone increasing, $0 \leq \left(\alpha(^\circ b_i^n) - \alpha(^\circ b_{i-1}^n)\right) \leq \frac{1}{2^n}$. Set

$$D := \bigcup_{n \in \mathbb{N}} D_n.$$

Since D is countable, there exists a set $U_1 \subseteq \Omega$ of $\widehat{\Gamma}$-measure 1 such that $F_a(X) \approx F_b(X)$ is limited for all $X \in U_1$ and all $a, b \in D$ with $\alpha(^\circ b) = \alpha(^\circ a)$. By Tschebyschev's inequality, we obtain

$$\widehat{\Gamma}\left\{\left|F_{b_i^n} - F_{b_{i-1}^n}\right| \geq 2^{-\frac{1}{8}n}\right\} \leq c \cdot 2^{-\frac{3}{2}n}.$$

Since, by a very rough estimate, $\overline{n} \leq 2^n + n^2$, we obtain

$$\widehat{\Gamma}\left\{\max_{i=1,\ldots,\overline{n}}\left|F_{b_i^n} - F_{b_{i-1}^n}\right| \geq 2^{-\frac{1}{8}n}\right\} \leq c\left(2^{-\frac{1}{2}n} + n^2 2^{-\frac{3}{2}n}\right).$$

Since the last term is the general term of a convergent series, by the Borel Cantelli Lemma, there exists a set $U_2 \subseteq \Omega$ of $\widehat{\Gamma}$-measure 1 and a function $g : U_2 \to \mathbb{N}$ such that for all $X \in U_2$ and all $n \geq g(X)$

$$(1) \qquad \max_{i=1,\ldots,\overline{n}}\left|F_{b_i^n}(X) - F_{b_{i-1}^n}(X)\right| < 2^{-\frac{1}{8}n}.$$

Set $U := U_1 \cap U_2$. Fix $X \in U$ and $n \geq g(X)$. We will prove by induction on m that for all $m > n$ and for all $s, t \in D_m$, with $|\alpha(^\circ t) - \alpha(^\circ s)| < \frac{1}{2^n}$:

$$|F_t(X) - F_s(X)| < 2 \sum_{j=n+1}^{m} j^2 2^{-\frac{1}{8}j}. \tag{2}$$

Suppose that $s, t \in D_{n+1}$ with $s < t$ and $\alpha(^\circ t) - \alpha(^\circ s) < \frac{1}{2^n}$. Let $t_i^n \leq s < t_{i+1}^n$. Then $t < t_{i+2}^n$. Since between t_i^n and t_{i+2}^n there are less than $2(n+1)^2$ elements of D_{n+1}, we obtain from (1)

$$|F_t(X) - F_s(X)| < 2(n+1)^2 2^{-\frac{1}{8}(n+1)}.$$

Assume that $M > n+1$ and (2) is true for all $m < M$. Suppose that $s, t \in D_M$ with $|\alpha(^\circ t) - \alpha(^\circ s)| < \frac{1}{2^n}$. We may assume $s < t$. Set $s_0 := \min\{a \in D_{M-1} \mid s \leq a\}$ and $t_0 := \max\{a \in D_{M-1} \mid a \leq t\}$. Assume first that $s_0 \leq t_0$. Since $|\alpha(^\circ t_0) - \alpha(^\circ s_0)| < \frac{1}{2^n}$ and between t_0 and t and s and s_0 there are at most M^2 many elements of D_M, we obtain from (1) and the induction hypothesis:

$$|F_t(X) - F_s(X)| \leq 2 \sum_{j=n+1}^{M} j^2 2^{-\frac{1}{8}j}.$$

If $t_0 < s_0$, the proof is similar.

To prove our final result, fix $s, t \in D$ with $0 \leq \alpha(^\circ t) - \alpha(^\circ s) < \frac{1}{2^{g(X)}}$. If $\alpha(^\circ t) = \alpha(^\circ s)$, then the inequalities under (3) below become true. Now assume $0 < \alpha(^\circ t) - \alpha(^\circ s)$. We select n such that $\frac{1}{2^{n+1}} \leq \alpha(^\circ t) - \alpha(^\circ s) < \frac{1}{2^n}$. Then $n \geq g(X)$ and $\left(\frac{1}{2^{n+1}}\right)^{\frac{1}{16}} \leq (\alpha(^\circ t) - \alpha(^\circ s))^{\frac{1}{16}}$. We obtain

$$\begin{aligned} {}^\circ|F_t(X) - F_s(X)| &\leq 2 \sum_{j=n+1}^{\infty} j^2 2^{-\frac{1}{16}j} (\alpha(^\circ t) - \alpha(^\circ s))^{\frac{1}{16}} \\ &\leq \left(2 \sum_{j=0}^{\infty} j^2 2^{-\frac{1}{16}j}\right) (\alpha(^\circ t) - \alpha(^\circ s))^{\frac{1}{16}}. \end{aligned} \tag{3}$$

We have proved that there exists a countable dense subset $D \subset T$, a constant d, a set $U \subseteq \Omega$ of $\widehat{\Gamma}$-measure 1 and a function $g : U \to \mathbb{N}$ such that for all $X \in U$ and all $s, t \in D$ with $0 \leq \alpha(^\circ t) - \alpha(^\circ s) < \frac{1}{2^{g(X)}}$

$${}^\circ|F_t(X) - F_s(X)| \leq d \cdot |\alpha(^\circ t) - \alpha(^\circ s)|^{\frac{1}{16}}.$$

It follows that F is uniformly S-continuous and nearstandard on D $\widehat{\Gamma}$-a.s. From the hypothesis we have immediately that F is weakly S-continuous. Now apply Proposition 4.3. □

Corollary 4.5. *Fix $F \in SL^2\left(\nu^{n+1}, \mathbb{L}^{\otimes_{n+1}}\right)$ and assume that F is symmetric in the first n arguments. We assume that $F = 1_{T_{\neq}^{n+1}} \cdot F \neq 0$. Then the* *internal integral of F,*

$$t \mapsto \int_1^t I_{n,1}(F)\Delta B,$$

has a uniquely determined continuous modification. It follows that $\delta^{pr}(\varphi)$ can be defined as a continuous process if φ belongs to finite chaos levels.

Proof. Fix a symmetric $G \in SL^2\left(\nu^n, \mathbb{L}^{\otimes_n}\right)$. We introduce the following notation: we identify $n \in \mathbb{N}$ with $\{1, ..., n\}$. If $p \in n \cup \{0\}$, and $\sigma : (n-p) \nearrow 2(n-p)$ is strictly monotone increasing, then $\overline{\sigma} : (n-p) \nearrow 2(n-p)$ denotes the enumeration of $2(n-p) \setminus \{\sigma_1, ..., \sigma_{n-p}\}$. If $t = (t_1, ..., t_{n-p})$, then we set $t_\sigma := \left(t_{\sigma_1}, ..., t_{\sigma_{n-p}}\right)$. Now define $Q^G_{p,\sigma} : T^{2(n-p)} \to \mathbb{L}^{\otimes_{2(n-p)}}$ by setting

$$Q^G_{p,\sigma}(t)(a) := \int_{T^p_<} \langle G(t_\sigma, s)(a_\sigma, \cdot), G(t_{\overline{\sigma}}, s)(a_{\overline{\sigma}}, \cdot)\rangle_{\mathbb{L}^p}\, d\nu^p(s).$$

It is proved in [26] that $Q^G_{p,\sigma} \in SL^2\left(\nu^{2(n-p)}, \mathbb{L}^{\otimes_{2(n-p)}}\right)$ and

$$\mathbb{E}\left((I_n(G))^2 - \sum_{p=0}^{n} \sum_{\sigma:(n-p)\nearrow 2(n-p)} I_{2(n-p)}(Q^G_{p,\sigma})\right)^2 \approx 0.$$

Now, by the properties of Gaußian measures, we obtain for $\sigma, \rho : (n-p) \nearrow 2(n-p)$:

$$\begin{aligned}
&\left|\mathbb{E}\left(I_{2(n-p)}(Q^G_{p,\sigma}) \cdot I_{2(n-p)}(Q^G_{p,\rho})\right)\right|^2 \\
&= \left|\int_{T^{2(n-p)}_<} \left\langle Q^G_{p,\sigma}, Q^G_{p,\rho}\right\rangle_{\mathbb{L}^{2(n-p)}} d\nu^{2(n-p)}\right|^2 \\
&\leq \int_{T^{2(n-p)}_<} \left\|Q^G_{p,\sigma}\right\|^2_{\mathbb{L}^{2(n-p)}} d\nu^{2(n-p)} \cdot \int_{T^{2(n-p)}_<} \left\|Q^G_{p,\rho}\right\|^2_{\mathbb{L}^{2(n-p)}} d\nu^{2(n-p)} \\
&\leq \left(\int_{T^n} \|G\|^2_{\mathbb{L}^n} d\nu^n\right)^2 \cdot \left(\int_{T^n} \|G\|^2_{\mathbb{L}^n} d\nu^n\right)^2.
\end{aligned}$$

It follows that $\left|\mathbb{E}\left(I_{2(n-p)}(Q^G_{p,\sigma}) \cdot I_{2(n-p)}(Q^G_{p,\rho})\right)\right| \leq \left(\int_{T^n} \|G\|^2_{\mathbb{L}^n} d\nu^n\right)^2$ and therefore,

$$\mathbb{E}\left(\sum_{p=0}^{n} \sum_{\sigma:(n-p)\nearrow 2(n-p)} I_{2(n-p)}(Q^G_{p,\sigma})\right)^2 \leq c_n \cdot \left(\int_{T^n} \|G\|^2_{\mathbb{L}^n} d\nu^n\right)^2$$

with $c_n := \sum_{p=0}^n \left(\begin{array}{c} 2n-2p \\ n-p \end{array}\right)^2$. This proves that

$$^\circ\mathbb{E}\left(I_n(G)^4\right) \leq c_n \cdot \left(^\circ\int_{T^n} \|G\|^2_{\mathbb{L}^n} d\nu^n\right)^2.$$

In order to finish the proof, we define

$$\alpha({}^\circ t) := \sum_{i=1}^{n+1} {}^\circ \int_1^t \int_{T^n} \left\| F_{(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{n+1},t_i)} \right\|_{\mathbb{L}^{n+1}}^2 d\nu^n d\nu(t_i).$$

Since $\|F\|_{\mathbb{L}^{n+1}}^2 \in SL^1(\nu^{n+1})$, by Keisler's Fubini Theorem, $\alpha : [0,1] \to \mathbb{R}$ is well defined and a monotone increasing continuous function. Now we obtain

$$\begin{aligned}
&{}^\circ\mathbb{E}\left(\left(\int_s^t I_{n,1}(F)\Delta B\right)^4\right) = {}^\circ\mathbb{E}\left(\left(\int I_{n,1}(1_{]s,t]} \cdot_{n+1} F)\Delta B\right)^4 (\cdot, H)\right) \\
&\le c_{n+1}\left({}^\circ \int_{T^{n+1}} \left\| \sum_{i=1}^{n+1} 1_{]s,t]}(t_i) \cdot F_{(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{n+1},t_i)} \right\|_{\mathbb{L}^{n+1}}^2 d\nu^{n+1}\right)^2 \\
&\le (n+1)^2 c_{n+1}\left({}^\circ \int_{T^{n+1}} \sum_{i=1}^{n+1} 1_{]s,t]}(t_i) \left\| F_{(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{n+1},t_i)} \right\|_{\mathbb{L}^{n+1}}^2 d\nu^{n+1}\right)^2 \\
&= (n+1)^2 c_{n+1}\left(\sum_{i=1}^{n+1} {}^\circ \int_s^t \int_{T^n} \left\| F_{(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{n+1},t_i)} \right\|_{\mathbb{L}^{n+1}}^2 d\nu^n d\nu(t_i)\right)^2 \\
&= (n+1)^2 c_{n+1} \left(\alpha({}^\circ t) - \alpha({}^\circ s)\right)^2.
\end{aligned}$$

Now apply Theorem 4.4. □

5. The Malliavin derivative and the Clark Ocone formula

The Clark Ocone formula here is a martingale representation theorem for functionals $\varphi \in L^2_{\mathfrak{H}}\left(\widehat{\Gamma}\right)$. In case φ is Malliavin differentiable, then φ is the Itô integral of the conditional expectation of the Malliavin derivative of φ. In order to state the Clark Ocone formula one needs the time line to be able to define the notion "non time anticipating". The Clark Ocone formula has important applications in finance mathematics (see Aase et al. [1] or Di Nunno et al. [10]).

This formula has been proved by Clark [7] and more general by Ocone [24] for the classical Wiener space $C_{\mathbb{R}}$. In this case, a simple proof, using saturated models of mathematics, can be found in Cutland and Ng [9]. Berger [4] proved the Clark Ocone formula for the abstract Wiener space, using the Üstünel-Zakai-Itô integral, based on a resolution of the identity on $\mathbb{H}$. In [27] there is a proof of this formula for the space $C_{\mathbb{F}}$. Recently, Mayer-Wolf and Zakai [22] extended the Clark Ocone formula to Banach space valued random variables, defined on $L^2(W_{\mathbb{F}})$. Using Cutland and Ng's approach, we will now prove the Clark Ocone formula for functionals, defined on the non separable space $L^2_{\mathfrak{H}}\left(\widehat{\Gamma}\right)$.

In order to define the Malliavin derivative in our setting, we use the following admissible sequence. The proof is straightforward.

Lemma 5.1. *Fix an admissible sequence* $(\mathfrak{H}_n)$ *over* $\mathfrak{H}$. *Set* $\mathfrak{H}'_0 := \mathbb{R}$ *and for* $n \geq 1$ *let* $\mathfrak{H}_n^{'}$ *be the set of all* $F \in \mathfrak{H}_n$ *such that for* $\widehat{\Gamma \otimes \nu}$*-almost all* $(\cdot, t) \in \Omega \times T$ *and*

$$I_{n-1,1}(F_n) \in SL^2(\Gamma \otimes \nu, \mathbb{L}) \text{ and } {}^{\circ}I_{n-1,1}(F_n) \in L^2_{\mathfrak{H}}(\widehat{\Gamma \otimes \nu}, \mathbb{H}).$$

Then $(\mathfrak{H}'_n)$ *is an admissible sequence over* $\mathfrak{H}$.

Using Theorem 3.4, we define the Malliavin derivative d on a dense subspace of $L^2_{\mathfrak{H}}(\widehat{\Gamma})$ with values in $L^2_{\mathfrak{H}}(\widehat{\Gamma \otimes \nu}, \mathbb{L})$, where $\mathfrak{H}$ is an arbitrary admissible vector space. Fix $\varphi \in L^2_{\mathfrak{H}}(\widehat{\Gamma})$ with chaos decomposition

$$\varphi = \mathbb{E}\varphi + \sum_{n=0}^{\infty} {}^{\circ}I_n(F_n) \text{ with } F_n \in \mathfrak{H}'_n.$$

Now the Malliavin derivative d of φ is nothing but finite dimensional differentiation under the standard part map and the sum, i.e., d is defined on a dense subspace of $L^2_{\mathfrak{H}}(\widehat{\Gamma})$ by

$$d(\varphi)(X,t) := \sum_{n=0}^{\infty} {}^{\circ}\frac{\partial}{\partial X_t} I_n(F_n) := \sum_{n=0}^{\infty} {}^{\circ}I_{n-1,1}(F_n)(X,t)$$

if $\sum_{n=0}^{\infty} {}^{\circ}I_{n-1,1}(F_n)$ converges in $L^2_{\mathfrak{H}}(\widehat{\Gamma \otimes \nu}, \mathbb{H}^n)$, which is the case if and only if $\sum_{n=1}^{\infty} \sqrt{n} \cdot {}^{\circ}I_n(F_n)$ converges in $L^2_{\mathfrak{H}}(\widehat{\Gamma})$. Therefore, d is densely defined.

Then the Malliavin derivative is also defined for standard spaces:

Examples 5.2.

(I) In case $\mathfrak{H} = \mathfrak{H}_{\mathbb{H}}$, d is a densely defined operator from $L^2(W_{\mathbb{F}})$ into $L^2(W_{\mathbb{F}}, \mathbb{H})$.

(II) In case $\mathfrak{H} = \mathfrak{H}_{C_{\mathbb{H}}}$, d is a densely defined operator from $L^2(W_{C_{\mathbb{F}}})$ into $L^2(W_{C_{\mathbb{F}}} \otimes \lambda, \mathbb{H})$.

(III) In case $\mathfrak{H} = \overline{\mathfrak{H}}$, the Malliavin derivative is an extension of the standard Malliavin derivative.

In order to define the Itô integral $\int f db_{\mathbb{F}}$, one needs non time-anticipating integrands f, and therefore, we take an admissible vector space $\mathfrak{H}$ such that

$$[1, t] \in \mathcal{W}^T_{\mathfrak{H}} \text{ for each } t \in T.$$

We use the internal filtration $(\mathcal{B}_n) := (\mathcal{B}_n)_{n \in T}$ on the internal Borel algebra $\mathcal{B}$ on $\Omega = \mathbb{L}^H$, where $\mathcal{B}_n$ is the family of cylinder sets

$$\mathcal{B}_n := \left\{ A \times \mathbb{L}^{H-n} \mid A \in \mathcal{B}\left(\mathbb{L}^n\right) \right\}, \ \mathcal{B}_0 := \{\emptyset, \Omega\}.$$

Set $t^- := t - 1$ for each $t \in T$.

Fix a $(\mathcal{B}_{n^-})$-adapted $F \in SL^2(\Gamma \otimes \nu, \mathbb{L})$. Using the fact that $\int F \Delta B$ is a $(\mathcal{B}_n)$-martingale, it is shown in [27], following Anderson's approach to the Itô integral [3] for finite dimensional Brownian motion, that $(X,t) \mapsto \sum_{s\le t} F_s(X_s)$ is S-continuous $\widehat{\Gamma}$-a.s, thus this function can be converted to a continuous process on the time line $[0,1]$, setting $\int F db_{\mathbb{F}}(\cdot, {}^\circ t) = {}^\circ \int F \Delta B(\cdot, t)$ for all $t \in T$ $\widehat{\Gamma}$-a.s.

If, in addition, F is a lifting of $f : \Omega \times T \to \mathbb{H}$ then we may set

$$\int f db_{\mathbb{F}} := \int F db_{\mathbb{F}}.$$

This integral is called the **Itô integral** of f. Note that the Itô integral integrates Hilbert space valued processes under a Fréchet space valued Brownian motion. In the work of Duncan (see Duncan [11]), for example, there is a standard approach to this integral. The following continuity result for $\int \cdot \, db_{\mathbb{F}}$ is a simple consequence of saturation:

Lemma 5.3. *Let (f_n) be a sequence of functions f_n, converging to f in $L^2_{\mathfrak{H}}(\widehat{\Gamma \otimes \nu}, \mathbb{H})$. If each f_n has a $(\mathcal{B}_{n^-})$-adapted lifting $F_n \in SL^2(\Gamma \otimes \nu, \mathbb{L})$, then f has a $(\mathcal{B}_{n^-})$-adapted lifting in $SL^2(\Gamma \otimes \nu, \mathbb{L})$ and*

$$\lim_{n\to\infty} \int f_n db_{\mathbb{F}} = \int f db_{\mathbb{F}} \text{ in } L^2(\widehat{\Gamma} \otimes \lambda).$$

In order to find relevant examples of an $\mathbb{H}$-valued process having a $(\mathcal{B}_{n^-})$-adapted lifting, we define a filtration $(\mathfrak{f}_n) = (\mathfrak{f}_n)_{n\in T\cup\{0\}}$ on $\mathcal{D}$, setting

$$\mathfrak{f}_n := \left(\left(\mathcal{B}_n \vee \mathcal{N}_{\widehat{\Gamma}} \right) \cap \mathcal{W}^{\Omega}_{\mathfrak{H}} \right).$$

note that the Loeb σ-algebra $L_\Gamma(\mathcal{B}_n)$ over $\mathcal{B}_n$ is a subset of $\mathcal{B}_n \vee \mathcal{N}_{\widehat{\Gamma}}$.

The following admissible sequence is used, the proof is straightforward.

Lemma 5.4. *Let $(\mathfrak{H}_n)$ be an admissible sequence over $\mathfrak{H}$. Set $\mathfrak{H}''_0 := \mathbb{R}$ and for $n \ge 1$, let $\mathfrak{H}''_n$ be the set of all $F \in \mathfrak{H}'_n$ ($\mathfrak{H}'_n$ is defined in Lemma 5.1) such that*

$${}^\circ\mathbb{E}^{\mathcal{B}_{t^-}}(I_{n-1,1}(F))(\cdot, t) = \mathbb{E}^{\mathfrak{f}_{t^-}}\, {}^\circ I_{n-1,1}(F)(\cdot, t),$$

thus $\left(\mathbb{E}^{\mathcal{B}_{t^-}}(I_{n-1,1}(F))_t\right)$ is a $(\mathcal{B}_{n^-})$-adapted lifting of $\left(\mathbb{E}^{\mathfrak{f}_{t^-}}\, {}^\circ I_{n-1,1}(F)_t\right)$. Then $(\mathfrak{H}''_n)$ is an admissible sequence over $\mathfrak{H}$.

Theorem 5.5. *Let $\varphi \in L^2_{\mathfrak{H}}(\widehat{\Gamma})$ Then*
(1) $\varphi = \mathbb{E}\varphi + \int \left(t \mapsto \mathbb{E}^{\mathfrak{f}_{t^-}} \sum_{n=1}^{\infty} {}^\circ I_{n-1,1}(F_n)(\cdot, t) \right) db_{\mathbb{F}}(\cdot, 1)$.
(2) *If φ is Malliavin differentiable, then*

$$\varphi = \mathbb{E}\varphi + \int \left(\mathbb{E}^{\mathfrak{f}_{t^-}} \, d\varphi(\cdot, t) \right) db_{\mathbb{F}}(\cdot, 1).$$

Proof. By Theorem 3.4, $\varphi = \mathbb{E}\varphi + \sum_{n=1}^{\infty} {}^\circ I_n(F_n)$ with $F_n \in \mathfrak{H}''_n$. Note that $I_n(F_n) = \int \left(t \mapsto \mathbb{E}^{\mathcal{B}_{t^-}} I_{n-1,1}(F_n)(\cdot, t) \right) \Delta B(\cdot, H)$. Using this fact and Lemmata 5.4 and 5.3, we obtain in $L^2_{\mathfrak{H}}(\widehat{\Gamma})$,

$$\begin{aligned}
\varphi - \mathbb{E}\varphi &= \sum_{n=1}^{\infty} {}^\circ \left(I_n(F_n)\right) \\
&= \sum_{n=1}^{\infty} {}^\circ \left(\int t \mapsto \mathbb{E}^{\mathcal{B}_{t^-}} I_{n-1,1}(F_n)(\cdot,t)\Delta B \right)(\cdot,H) \\
&= \sum_{n=1}^{\infty} \left(\int \left(t \mapsto \mathbb{E}^{\mathfrak{f}_{t^-}}\, {}^\circ I_{n-1,1}(F_n)(\cdot,t)\right) \right) db_{\mathbb{F}}(\cdot,1) \\
&= \int \left(t \mapsto \mathbb{E}^{\mathfrak{f}_{t^-}} \sum_{n=1}^{\infty} {}^\circ I_{n-1,1}(F_n)(\cdot,t) \right) db_{\mathbb{F}}(\cdot,1) =
\end{aligned}$$

in case, φ is Malliavin differentiable

$$\int \left(t \mapsto \mathbb{E}^{\mathfrak{f}_{t^-}}\, d\varphi\,(\cdot,t)\right) db_{\mathbb{F}}(\cdot,1).$$

This proves the result. □

6. Anticipative Girsanov transformations

In this part we apply the decomposition result for stochastic processes (Theorem 3.6) and the resulting finite dimensional representation (Propositions 3.7 and 4.1) to time-anticipating Girsanov transformations $\sigma : \Omega \to C_{\mathbb{F}}$ given by

$$\sigma(X)(r) := \sigma(X,r) = b_{\mathbb{F}}(X,r) + \int_0^r \varphi(X,s)d\widehat{\nu}(s),$$

where $\varphi : \Omega \times T \to \mathbb{H}$ is $\mathcal{W}^{\Omega}_{\mathfrak{H}} \otimes \mathcal{W}^{T}_{\mathfrak{H}}$-measurable and $\varphi\,(X,\cdot) \in L^2\,(\widehat{\nu})$ for $\widehat{\Gamma}$-almost all $X \in \Omega$. Then the mapping ${}^\circ t \mapsto \int_0^{{}^\circ t} \varphi(\cdot,s)d\widehat{\nu}(s)$ is a $\widehat{\Gamma}$-a.s. continuous mapping from $[0,1]$ into $\mathbb{H}$ if we define $\int_0^{{}^\circ t} \varphi(\cdot,s)d\widehat{\nu}(s) := \int_{[1,t]} \varphi(\cdot,s)d\widehat{\nu}(s)$. We demand from $\mathfrak{H}$ that $[1,t] \in \mathcal{W}^{T}_{\mathfrak{H}}$ for each $t \in T$. Let us address the following question:

When does there exist a probability measure P on $L_{\Gamma}(\mathcal{B})$, absolutely continuous or equivalent to $\widehat{\Gamma}$, such that the process σ follows the law of Brownian motion with respect to P ?

If φ is time-anticipating, then it is impossible for σ to become a Brownian motion, because the martingale property fails.

Example 6.1. If $\mathfrak{H} = \mathfrak{H}_{C_{\mathbb{F}}}$, then σ defines a shift on $C_{\mathbb{F}}$: fix $f \in C_{\mathbb{F}}$. Then there exists an $X \in \Omega$ with $b_{\mathbb{F}}(X,\cdot) = f$. Set $\sigma(f) := \sigma(X)$. Then, since $\varphi(\cdot,t)$ is measurable with respect to the Brownian motion, σ is a well defined mapping from $C_{\mathbb{F}}$ into $C_{\mathbb{F}}$ (see [26]).

Answers to related questions are given for example in the work of Buckdahn [6], Kusuoka [20], Enchev [13], Ramer [29]. It is often presumed that the kernel φ of the Girsanov transformation is Skorohod integrable or locally $\mathbb{L}_{1,2}$ or it is $\mathbb{H}$-continuously differentiable (see also Nualart [23]). (The space $\mathbb{L}_{1,2}$ is a subspace of the domain of the Skorohod integral, consisting of the Malliavin differentiable integrands.)

In our approach we use the fact that the probability space Ω is finite dimensional in the sense of a highly saturated model of mathematics set-up and that each L^2-functional on Ω is infinitely close to a continuously differentiable function (see Propositions 3.7 and 4.1). Therefore, it is not necessary to demand any smoothness conditions from φ. Moreover, we are able to apply the elementary transformation in finite dimensional analysis. Some of the results in [26] will be extended.

Since φ can be approximated by functions $\varphi_n \in L^2_{\mathfrak{H}}(\widehat{\Gamma \otimes \nu}, \mathbb{H})$, and each φ_n has a smooth lifting according to Proposition 3.7, by saturation, one can find a lifting Φ of φ such that $\Phi(X, \cdot) \in SL^2(\nu, \mathbb{L})$ for $\widehat{\Gamma}$-almost all $X \in \mathbb{L}^H$ and Φ is given by

$$\Phi(X,t) = \sum_{k=0}^{M} \sum_{t_1 < \ldots < t_k} F_k(t_1, ..., t_k, t)\,(X_{t_1}, ..., X_{t_k}, \cdot) \in \mathbb{L}' = \mathbb{L},$$

where M is an unlimited natural number.

If φ has a $(\mathcal{B}_{n^-})$-adapted lifting, we may assume that all F_k are nonanticipating, thus, Φ is $(\mathcal{B}_{n^-})_{n \in T}$-adapted.

Define the function $\widetilde{\Phi} : \mathbb{L}^H \to \mathbb{F}^H$ by setting:

$$\widetilde{\Phi}(X) := \left(X_t + \Phi(X,t)\frac{1}{H} \right)_{t \in T}.$$

Since $b_{\mathbb{F}}(\cdot, {}^\circ n) \approx_{\mathbb{F}} B(\cdot, n)$ and $\int_0^{{}^\circ n} \varphi(\cdot, s)(a) d\widehat{\nu}(s) \approx \int_1^n \Phi(\cdot, s)(a) d\nu(s)$ for all $a \in \mathbb{H}$ $\widehat{\Gamma}$-a.s., and since the dual space $\mathbb{F}'$ can be identified with a subspace of $\mathbb{H}$, we have an internal representation of σ:

Proposition 6.2. *For $\widehat{\Gamma}$-almost all X,*

$$a(\sigma(X)({}^\circ n)) \approx \langle B_n \circ \widetilde{\Phi}(X), a \rangle \text{ for each } a \in \mathbb{F}' \text{ and each } n \in T.$$

Since $\widetilde{\Phi} : \mathbb{L}^H \to \mathbb{L}^H$ is a *finite sum of multi-linear forms, $\widetilde{\Phi}$ is continuously differentiable. In analogy to the standard theory we call an internal function $\widetilde{\Phi} : \mathbb{L}^H \to \mathbb{L}^H$ **regular** if it is possible to apply the usual finite dimensional transformation rule: for each internal $\mathcal{B}$-measurable function $G : \mathbb{L}^H \to \mathbb{L}^H$ and each $B \in \mathcal{B}$ we have

$$\int_B G \circ \widetilde{\Phi} \cdot \left| \det \widetilde{\Phi}' \right| d\Gamma = \int_{\widetilde{\Phi}[B]} G d\Gamma,$$

if the integral of one of both sides exists. For example, $\widetilde{\Phi}$ is regular, if

(1) $\widetilde{\Phi}$ bijective and

(2) $\det \widetilde{\Phi}'_X \neq 0$ for all $X \in \mathbb{L}^H$.

Note that the derivative $\widetilde{\Phi}'_X$ of $\widetilde{\Phi}$ at X is an endomorphism on the finite dimensional Euclidean space $\mathbb{L}^H$. The associated matrix corresponds to the **Carleman Fredholm matrix** in the standard theory.

Now we shall give necessary and sufficient conditions for positive answers to our question: Define first the internal **Cameron Martin Girsanov function** $\Theta_\Phi : \mathbb{L}^H \to {}^*\mathbb{R}$ by setting

$$\Theta_\Phi(X) := e^{-\sum_{s\leq H}\left(\Phi(X,s)(X_s)+\frac{1}{2}\|\Phi(X,s)\|^2_{\mathbb{L}}\frac{1}{H}\right)}\left|\det \widetilde{\Phi}'_X\right|.$$

The function Φ is called **adequate**, if the following 3 conditions hold:

(A) $\widetilde{\Phi}$ is regular.

(B) $\det \widetilde{\Phi}' \not\approx 0$ $\widehat{\Gamma}$-a.s..

(C) $\Theta_\Phi \in SL^1(\Gamma)$.

If, in addition to (A), (B) and (C), we also have

(D) $e^{-\sum_{s\leq H}\left(\Phi(X,s)(X_s)+\frac{1}{2}\|\Phi(X,s)\|^2_{\mathbb{L}}\frac{1}{H}\right)} \not\approx 0$ for $\widehat{\Gamma}$ almost all $X \in \mathbb{L}^H$, then Φ is called **strongly adequate**.

It is easy to see that, if Φ is $(\mathcal{B}_{n^-})$-adapted, then conditions (A) and (B) are true, in fact, $\det \widetilde{\Phi}' = 1$. By Prop. 4.1, if φ is Skorohod integrable, then we can choose Φ such that (D) holds and the internal function $X \mapsto e^{-\sum_{s\leq H}\left(\Phi(X,s)(X_s)+\frac{1}{2}\|\Phi(X,s)\|^2_{\mathbb{L}}\frac{1}{H}\right)}$ is a lifting of $X \mapsto e^{-\delta\varphi-\frac{1}{2}\int_T\|\varphi(X,s)\|^2_{\mathbb{H}}d\widehat{\nu}(s)}$.

Now we obtain the following result, using transfer of the finite dimensional transformation rule (see [26]):

Theorem 6.3. *Assume that Φ is (strongly) adequate. Then $\widehat{\Theta_\Phi\Gamma}$ is (equivalent) absolutely continuous to $\widehat{\Gamma}$.*

Fix $r, t \in [0,1]$ with $r < t$. Then $a(\sigma(\cdot,t)-\sigma(\cdot,r))$ is $\mathcal{N}(0,t-r)$-distributed with respect to the measure $\widehat{\Theta_\Phi\Gamma}$ for each $a \in \mathbb{F}'$ with $\|a\|_{\mathbb{F}} = 1$.

Proof. The proof of the first part of the Theorem can be found in [26]. Now fix $a \in \mathbb{F}'$ with $\|a\|_{\mathbb{F}} = 1$. We extend $a = \mathfrak{e}_1$ to an internal orthonormal basis $\mathcal{E} = (\mathfrak{e}_i)_{i\leq\omega}$ of $\mathbb{L}$. Choose $n, m \in T$ such that $\frac{n}{H} \approx t$ and $\frac{m}{H} \approx r$. Then we have for each $\lambda \in \mathbb{R}$,

$$\begin{aligned}\int_\Omega e^{\lambda a(\sigma(\cdot,t)-\sigma(\cdot,r))}d\widehat{\Theta_\Phi\Gamma} &\approx \int_\Omega e^{\lambda\left\langle B_n\circ\widetilde{\Phi}-B_m\circ\widetilde{\Phi},a\right\rangle}d\Theta_\Phi\Gamma \\ &= \int_\Omega e^{\lambda\langle B_n-B_m,a\rangle}d\Gamma \approx \int_\Omega e^{\lambda a(b(\cdot,t)-b(\cdot,r))}d\widehat{\Gamma}.\end{aligned}$$

Since $a(b(\cdot,t)-b(\cdot,r))$ is $\mathcal{N}(0,t-s)$-distributed, also $a(\sigma(\cdot,t)-\sigma(\cdot,r))$ is $\mathcal{N}(0,t-s)$-distributed. □

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