J. Math. Kyoto Univ. (JMKYAZ)
47-4 (2007), 781–808

Foxby equivalence over associative rings

By

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Abstract

We extend the definition of a semidualizing module to general associative rings. This enables us to define and study Auslander and Bass classes with respect to a semidualizing bimodule C. We then study the classes of C-flats, C-projectives, and C-injectives, and use them to provide a characterization of the modules in the Auslander and Bass classes. We extend Foxby equivalence to this new setting. This paper contains a few results which are new even in the commutative, noetherian setting.

Introduction

Over a commutative, noetherian local ring, semidualizing modules provide a common generalization of a dualizing (canonical) module and a free module of rank one. Foxby [13] first defined them (PG-modules of rank one), while Golod [16] (suitable modules) and Vasconcelos [26] (spherical modules) furthered their study. Recently, Araya-Takahashi-Yoshino [1] extended this definition to a pair of non-commutative, but noetherian rings, while White [28] extended the definition to the non-noetherian, but commutative, setting. In this paper (see Section 2), we define and study semidualizing (S, R)-bimodules, where R and S are arbitrary associative rings, thereby encompassing all of the aforementioned definitions.

Over a commutative noetherian ring, Avramov and Foxby [2], [13] and Enochs-Jenda-Xu [11] connected the study of (semi)dualizing modules to associated Auslander and Bass classes for (semi)dualizing modules, $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$, which are subcategories of the category of R-modules. This paper furthers this study, which in our setting involves an Auslander class $\mathcal{A}_C(R)$ over R, and a Bass class $\mathcal{B}_C(S)$ over S.

Those familiar with this area may wonder why we do not deal with derived Auslander and Bass classes in this paper. The short answer is that the non-commutative situation is more subtle. A longer answer is contained in Remark 2.

Many results for Auslander and Bass classes associated to a semidualizing

Received December 18, 2006

*This paper was written while HH was working at the University of Aarhus, Denmark.

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module over a commutative noetherian ring, carry over to an associative ring in a straightforward manner. However, some complications do arise. Thus, in Section 3, we define *faithfully* semidualizing bimodules, see Definition 3.1. Over a commutative ring, all semidualizing modules are faithfully semidualizing; see Theorem 3.1, but it is unknown to the authors if this is true in the non-commutative setting.

Section 4 contains basic properties of the Auslander and Bass classes. For example, the Auslander class $\mathcal{A}_C(R)$ contains all flat R-modules, and the Bass class $\mathcal{B}_C(S)$ contains all injective S-modules; see Lemma 4.1. Also, both are closed under summands, products, coproducts, and filtered colimits; see Proposition 4.2.

To further the study of the classes $\mathcal{A}_C(R)$ and $\mathcal{B}_C(S)$, in Section 5 we consider the classes of modules $\mathcal{F}_C(S)$, $\mathcal{P}_C(S)$, and $\mathcal{I}_C(R)$, consisting of the C-flats, C-projectives, and C-injectives, respectively. When C is free of rank one, these are the flats, projectives, and injectives, respectively. The following is Theorem 5.1.

Theorem 1 (Foxby equivalence). *Let ${}_SC_R$ be a semidualizing bimodule. There are* (*horizontal*) *equivalences of categories*

$$\begin{tikzcd}
\mathcal{P}_R(R) \arrow[r, shift left, "C\otimes_R -"] \arrow[d, hook] & \mathcal{P}_C(S) \arrow[l, shift left, "\mathrm{Hom}_S(C,-)", "\sim"'] \arrow[d, hook] \\
\mathcal{F}_R(R) \arrow[r, shift left, "C\otimes_R -"] \arrow[d, hook] & \mathcal{F}_C(S) \arrow[l, shift left, "\mathrm{Hom}_S(C,-)", "\sim"'] \arrow[d, hook] \\
\mathcal{A}_C(R) \arrow[r, shift left, "C\otimes_R -"] & \mathcal{B}_C(S) \arrow[l, shift left, "\mathrm{Hom}_S(C,-)", "\sim"'] \\
\mathcal{I}_C(R) \arrow[r, shift left, "C\otimes_R -"] \arrow[u, hook] & \mathcal{I}_S(S). \arrow[l, shift left, "\mathrm{Hom}_S(C,-)", "\sim"'] \arrow[u, hook]
\end{tikzcd}$$

Propositions 5.1 and 5.2, and Lemma 5.2 discuss closure properties of the classes above. Proposition 5.3 pertains to their (pre)covering and (pre)enveloping properties and includes some results that are new even in the commutative, noetherian setting. For example, when R is commutative noetherian the C-flats are preenveloping and the C-injectives are precovering.

Section 6 contains two main results of the paper, which provide alternative characterizations of the modules in the Auslander and Bass classes in terms of the C-injectives, C-projectives, and C-flats. Here is one.

Theorem 2. *Let ${}_SC_R$ be a semidualizing bimodule. An R-module M belongs to $\mathcal{A}_C(R)$ if and only if there exists a complex of R-modules*

$$X = \ \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow U^0 \longrightarrow U^1 \longrightarrow \cdots$$

that satisfies the following conditions

(a) *The complex X is exact*;
(b) *Each P_i is R-projective (or R-flat)*;
(c) *Each U^i is C-injective*;
(d) *There is an isomorphism $M \cong \operatorname{Coker}(P_1 \longrightarrow P_0)$; and*
(e) *The complex $C \otimes_R X$ is exact.*

Moreover, if $M \in \mathcal{A}_C(R)$ then any complex constructed by splicing together an augmented projective (or flat) resolution of M and an augmented proper C-injective coresolution of M will satisfy the above properties.

When C is faithfully semidualizing, there is a clear analogy with the local setting. First, the Auslander class contains the modules of finite flat dimension, while the Bass class contains the modules of finite injective dimension. Second, if any two modules in a short exact sequence are in the Auslander (respectively, Bass) class, then so is the third; see Corollary 6.3.

1. Background

This section introduces a number of assumptions, definitions, notions, and results which will be used throughout the paper.

1.1. Throughout this work, R and S are fixed associative rings with unities and all R- or S-modules are understood to be *unital left* R- or S-modules. Right R- or S-modules are identified with left modules over the *opposite rings* R^{op} or S^{op}.

The suggestive notation ${}_SM_R$ is used to denote that M is an (S,R)-*bimodule*. This means that M is both a left S- and a right R-module, and that these structures are compatible in the sense that $s(xr) = (sx)r$ for all $s \in S$, $r \in R$, and $x \in M$.

Finally, if $\mathcal{X}$ is a class of, say, R-modules, then $\mathcal{X}^f$ is the subclass of all finitely generated R-modules in $\mathcal{X}$. Throughout this background section, $\mathcal{X}$ will denote a fixed class of R-modules.

1.2. An *R-complex* is a sequence of R-module homomorphisms

$$X = \cdots \xrightarrow{\partial^X_{n+1}} X_n \xrightarrow{\partial^X_n} X_{n-1} \xrightarrow{\partial^X_{n-1}} \cdots$$

such that $\partial^X_{n-1}\partial^X_n = 0$ for each integer n.

In this paper, all resolutions will be built from precovers, which we now discuss.

1.3. Let M be an R-module. A homomorphism $\phi\colon X \to M$ with $X \in \mathcal{X}$ is an *$\mathcal{X}$-precover* of M if for every homomorphism $\psi\colon Y \to M$ with $Y \in \mathcal{X}$, there exists a homomorphism $f\colon Y \to X$ such that $\phi f = \psi$. If every R-module admits an $\mathcal{X}$-precover, then we say that the class $\mathcal{X}$ is *precovering.*

An *$\mathcal{X}$-cover* of M is an $\mathcal{X}$-precover $\phi\colon X \to M$ with the additional property that any endomorphism $f\colon X \to X$ with $\phi = \phi f$ must be an automorphism. If every R-module admits an $\mathcal{X}$-cover, then we say that the class $\mathcal{X}$ is *covering.*

Preenvelopes and *envelopes* are defined dually; see [10] for further details.

1.4. If the class $\mathcal{X}$ is precovering, then for any R-module M, there exists an *augmented proper $\mathcal{X}$-resolution* of M, that is, a complex

$$X^+ = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \to M \to 0$$

such that $\mathrm{Hom}_R(Y, X^+)$ is exact for all $Y \in \mathcal{X}$. The truncated complex

$$X = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \to 0$$

is a *proper $\mathcal{X}$-resolution* of M.

Note that X^+ need not be exact. However, if $\mathcal{X}$ contains the projectives, then any augmented proper $\mathcal{X}$-resolution of M is exact. *Augmented proper $\mathcal{X}$-coresolutions* are defined dually, and they must be exact if the class $\mathcal{X}$ contains the injectives.

1.5. A *degreewise finite projective resolution* of an R-module M is a projective resolution P of M such that each P_i is finitely generated (projective).

1.6. When $\mathcal{X}$ is precovering, the *$\mathcal{X}$-projective dimension* of M is

$$\mathcal{X}\text{-}\mathrm{pd}_R(M) = \inf\left\{\sup\{n \mid X_n \neq 0\} \,\middle|\, X \text{ is an } \mathcal{X}\text{-resolution of } M\right\}.$$

The modules of $\mathcal{X}$-projective dimension zero are the non-zero modules in $\mathcal{X}$. When $\mathcal{X}$ is preenveloping, the *$\mathcal{X}$-injective dimension*, denoted $\mathcal{X}\text{-}\mathrm{id}(-)$, is defined dually.

1.7. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. The class $\mathcal{X}$ is *closed under extensions* if it has the property that if M' and M'' are in $\mathcal{X}$, then so is M. The class $\mathcal{X}$ is *closed under kernels of epimorphisms* if whenever M and M'' are in $\mathcal{X}$, then so is M'. Finally, $\mathcal{X}$ is *closed under cokernels of monomorphisms* if whenever M' and M are in $\mathcal{X}$, then so is M''.

1.8. The class $\mathcal{X}$ is *projectively resolving* if

(a) The class $\mathcal{X}$ contains every projective R-module, and
(b) For every exact sequence of R-modules $0 \to M' \to M \to M'' \to 0$ with $M'' \in \mathcal{X}$, one has $M \in \mathcal{X}$ if and only if $M' \in \mathcal{X}$.

The notion of *injectively resolving* is defined dually.

The class $\mathcal{X}$ is *finite projectively resolving* if

(a) The class $\mathcal{X}$ consists entirely of finitely generated R-modules,
(b) The class $\mathcal{X}$ contains every finitely generated projective R-module, and
(c) For every exact sequence of finitely generated R-modules $0 \to M' \to M \to M'' \to 0$ with $M'' \in \mathcal{X}$, one has $M \in \mathcal{X}$ if and only if $M' \in \mathcal{X}$.

The next result will be used in the proof of Theorem 3.1.

1.9. Vasconcelos [26, (4.3)] proves the following corollary of a theorem of Gruson: Let R be commutative and let C be a finitely generated R-module with $\mathrm{Ann}_R(C) = 0$. If M is any R-module such that $C \otimes_R M = 0$, then $M = 0$.

Remark 1. Let ${}_SC_R$ be an (S, R)-bimodule. Unless otherwise mentioned, an Ext group of the form $\mathrm{Ext}^*_S({}_SM, C)$ will be computed by resolving ${}_SM$ with a projective resolution. This has the consequence that $\mathrm{Ext}^*_S({}_SM, C)$ inherits a right R-structure. Similar remarks are to be said about the computation of other derived functors such as $\mathrm{Ext}^*_{R^{\mathrm{op}}}(M_R, C)$ or $\mathrm{Tor}^*_R(C, {}_RM)$.

We conclude this section with some necessary results about two important homomorphisms. The proofs are straightforward, keeping Remark 1 in mind.

1.10. For modules ${}_SM$, ${}_SN_R$, and ${}_RF$, the *tensor evaluation* morphism

$$\omega_{MNP} \colon \mathrm{Hom}_S(M, N) \otimes_R F \longrightarrow \mathrm{Hom}_S(M, N \otimes_R F)$$

is defined by $\omega_{MNF}(\psi \otimes f)(m) = \psi(m) \otimes f$. It is straightforward to verify that this is an isomorphism when M is a finitely generated projective. In general, ω_{MNF} is just an abelian group homomorphism. However, if ${}_SM$ has an additional right R-structure compatible with the given left S-structure, then ω_{MNF} becomes R-linear.

Lemma 1.1. *Let ${}_SM$, ${}_SN_R$, and ${}_RF$ be modules such that M admits a degreewise finite S-projective resolution, and let F be flat. Let i be an integer.*

(a) *The map ω_{MNF} induces an isomorphism of abelian groups*

$$\mathrm{Ext}^i_S(M, N) \otimes_R F \cong \mathrm{Ext}^i_S(M, N \otimes_R F).$$

(b) *If $\mathrm{Ext}^i_S(M, N) = 0$, then $\mathrm{Ext}^i_S(M, N \otimes_R F) = 0$.*
(c) *If F is faithfully flat and $\mathrm{Ext}^i_S(M, N \otimes_R F) = 0$, then $\mathrm{Ext}^i_S(M, N) = 0$.*

1.11. For modules M_R, ${}_SN_R$, and ${}_SI$, the *Hom-evaluation* morphism

$$\theta_{MNI} \colon M_R \otimes_R \mathrm{Hom}_S(N, I) \longrightarrow \mathrm{Hom}_S(\mathrm{Hom}_{R^{\mathrm{op}}}(M, N), I)$$

is defined by $\theta_{MNI}(m \otimes \phi)(\varphi) = (\phi \circ \varphi)(m)$. It is easily verified that this is an isomorphism when M is a finitely generated projective. In general, θ_{MNI} is just a homomorphism of abelian groups, but if M_R has an additional left S-structure compatible with the right R-structure, then θ_{MNI} becomes S-linear.

Lemma 1.2. *Let M_R, ${}_SN_R$, and ${}_SI$ be modules such that M has a degreewise finite R^{op}-projective resolution, and I is injective. Let i be an integer.*

(a) *The map θ_{MCI} induces an isomorphism of abelian groups*

$$\mathrm{Tor}^R_i(M, \mathrm{Hom}_S(N, I)) \cong \mathrm{Hom}_S(\mathrm{Ext}^i_{R^{\mathrm{op}}}(M, N), I).$$

(b) *If $\mathrm{Ext}^i_{R^{\mathrm{op}}}(M, N) = 0$, then $\mathrm{Tor}^R_i(M, \mathrm{Hom}_S(N, I)) = 0$.*

(c) *If I is faithfully injective and* $\mathrm{Tor}_i^R(M, \mathrm{Hom}_S(N, I)) = 0$*, then there is an equality* $\mathrm{Ext}^i_{R^{\mathrm{op}}}(M, N) = 0$.

2. Semidualizing bimodules

We begin by extending the definition of a semidualizing module.

Definition 2.1. An (S, R)-bimodule $C = {}_SC_R$ is *semidualizing* if

(a1) ${}_SC$ admits a degreewise finite S-projective resolution.
(a2) C_R admits a degreewise finite R^{op}-projective resolution.
(b1) The homothety map ${}_SS_S \xrightarrow{{}_S\gamma} \mathrm{Hom}_{R^{\mathrm{op}}}(C, C)$ is an isomorphism.
(b2) The homothety map ${}_RR_R \xrightarrow{\gamma_R} \mathrm{Hom}_S(C, C)$ is an isomorphism.
(c1) $\mathrm{Ext}_S^{\geqslant 1}(C, C) = 0$.
(c2) $\mathrm{Ext}_{R^{\mathrm{op}}}^{\geqslant 1}(C, C) = 0$.

2.1. Unless otherwise stated, when $R = S$ is commutative, all semidualizing bimodules in this paper are *symmetric* in the sense that the two R-actions on C agree. In this case we will use the terminology "C is semidualizing over R". Note that when $R = S$ is commutative and noetherian, Definition 2.1 agrees with the established terminology; that is a finitely generated R-module C is *semidualizing* if the natural homothety map $R \xrightarrow{\gamma_R} \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$. Two examples are the free module of rank 1, and over a Cohen-Macaulay local ring, the dualizing (canonical) module, when it exists.

Observation 2.1.

(a) When S is left noetherian and R is right noetherian, conditions (a1) and (a2) reduce to ${}_SC$ and C_R being finitely generated, and therefore Definition 2.1 agrees with that of Araya-Takahashi-Yoshino [1, (2.1)].
(b) Let $R = S$ be commutative. Conditions (a1) and (a2) reduce to the statement that C admits a degreewise finite projective resolution, while conditions (b1) and (b2) reduce to $\mathrm{Hom}_R(C, C) \cong R$, and conditions (c1) and (c2) reduce to $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$. Thus, Definition 2.1 agrees with that of White [28, (1.8)].
(c) By the symmetry of the definition above it is clear that if C is a semidualizing (S, R)-bimodule, then C is also a semidualizing $(R^{\mathrm{op}}, S^{\mathrm{op}})$-bimodule, cf. our convention in 1.1.
(d) If $R = S$ is commutative and C is a semidualizing R-module, then $C_{\mathfrak{p}}$ is a semidualizing $R_{\mathfrak{p}}$-bimodule for all $\mathfrak{p} \in \mathrm{Spec}(R)$, cf. Example 2.1(a) below.

Note that Example 2.1(a) below can be applied to produce examples of semidualizing modules over a commutative, but not necessarily noetherian, ring.

Example 2.1.

(a) Let $Q \to R$ be a flat homomorphism between commutative rings. If E is semidualizing over Q, then $E \otimes_Q R$ is semidualizing over R, cf. Prop. 3.2.

(b) Assume the R and S are Morita equivalent rings in the sense of [27, (9.5)], that is, there exists bimodules ${}_RP_S$ and ${}_SQ_R$ with

$$ {}_RP_S \otimes_S {}_SQ_R \cong {}_RR_R \quad \text{and} \quad {}_SQ_R \otimes_R {}_RP_S \cong {}_SS_S. $$

By [27, (9.5.4), (9.5.2)], the bimodule ${}_SQ_R$ is semidualizing.

We close this section with a comparison of *derived Auslander and Bass classes* [5] and *module Auslander and Bass classes* [13] in the non-commutative setting.

Remark 2. Over a commutative, noetherian ring R, Christensen [5] generalized the notions of a semidualizing module and a dualizing complex to that of a semidualizing complex C. He then connected their study to the associated *derived* Auslander and Bass classes, $\mathsf{A}_C(R)$ and $\mathsf{B}_C(R)$, which are subcategories of the derived category of R. When C is a semidualizing module, Foxby [13] studied the *module* Auslander and Bass classes, $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$, which are subcategories of the category of R-modules (see Definition 4.1).

Thus, for a semidualizing module C, a natural question arises: Does $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ consist of the modules belonging to $\mathsf{A}_C(R)$ and $\mathsf{B}_C(R)$?

In the commutative noetherian setting, the answer is "yes", and thus the existing literature tends to focus on the more general derived Auslander and Bass classes. However, for non-commutative rings the question is much more complicated:

Dualizing complexes of bimodules over a non-commutative but two-sided noetherian ring have been given several different definitions, e.g. [14], [24], [30]. In [6] the authors use a variant of Miyachi's definition [24] of a dualizing complex of bimodules to consider a special case of the derived Auslander and Bass classes. However, this definition requires the existence of a so-called *biprojective resolution* of the dualizing complex of bimodules, and such a resolution is only known to exist in certain special cases. Even if the semidualizing bimodule admits a biprojective resolution, so that the derived Auslander and Bass classes can be defined, it is not known to these authors if the modules in $\mathsf{A}_C(R)$ or $\mathsf{B}_C(R)$ belong to $\mathcal{A}_C(R)$ or $\mathcal{B}_C(R)$?

3. Faithfully semidualizing bimodules

This section focuses on *faithfully* semidualizing bimodules. Over a commutative ring, all semidualizing modules are faithfully semidualizing; see Theorem 3.1.

Definition 3.1. A semidualizing bimodule ${}_SC_R$ is *faithfully semidualizing* if it satisfies the following conditions for all modules ${}_SN$ and M_R.

(a) If $\mathrm{Hom}_S(C, N) = 0$, then $N = 0$.
(b) If $\mathrm{Hom}_{R^{\mathrm{op}}}(C, M) = 0$, then $M = 0$.

3.1. Note that when $R = S$ is commutative, these conditions are the same, and a semidualizing module C is faithfully semidualizing if and only if, for any R-module N, the condition $\mathrm{Hom}_R(C, N) = 0$ implies $N = 0$.

3.2. By left exactness of $\mathrm{Hom}_S(C, -)$ and $\mathrm{Hom}_{R^{\mathrm{op}}}(C, -)$, it suffices to verify that the conditions in Def. 3.1 hold for all *cyclic* modules N and M.

Lemma 3.1. *A semidualizing bimodule ${}_SC_R$ is faithfully semidualizing if and only if the following conditions hold for all modules N_S and ${}_RM$.*

(a) *If $N \otimes_S C = 0$, then $N = 0$.*
(b) *If $C \otimes_R M = 0$, then $M = 0$.*

Proof. Assume that $N \otimes_S C = 0$. This provides the equality below

$$\mathrm{Hom}_S(C, \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(N \otimes_S C, \mathbb{Q}/\mathbb{Z}) = 0$$

while the isomorphism follows from Hom-tensor adjointness. Since C is faithfully semidualizing, $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}) = 0$. The module $\mathbb{Q}/\mathbb{Z}$ is faithfully injective, which forces $N = 0$. Similarly, if $C \otimes_R M = 0$, then $M = 0$.

The converse is proved similarly, using Lemma 1.2(a). $\square$

3.3. When R is commutative and noetherian, it is straightforward to prove that every semidualizing module C is faithfully semidualizing. Indeed, if $M \neq 0$, then for any associated prime $\mathfrak{p}$ of M there is a nonzero map $C_{\mathfrak{p}} \to k(\mathfrak{p}) \to M_{\mathfrak{p}}$. It follows that there is a nonzero map $C \to M$.

The situation where R is commutative, but non-noetherian seems to require significantly more work (which is done behind the scenes in Gruson's Theorem; see 1.9), but the same result holds, as we show next. The authors are unaware if this result carries over to the non-commutative setting. However, Proposition 3.2 (applied to well-known flat algebras found in e.g. [17], [23]) provide many examples of faithfully semidualizing bimodules over a wide class of non-commutative rings.

Proposition 3.1. *Let R be a commutative ring. If C is a semidualizing R-module, then C is faithfully semidualizing.*

Proof. As $R \cong \mathrm{Hom}_R(C, C)$ we have $\mathrm{Ann}_R(C) = 0$. Therefore, Lemma 3.1 and paragraph 1.9 (Gruson) imply that C is faithfully semidualizing. $\square$

The next result is a non-commutative, module version of [5, (5.1)].

Proposition 3.2. *Let Q be a commutative ring and let R be an associative and flat Q-algebra. If E is a semidualizing Q-module, then ${}_RC_R = E \otimes_Q {}_RR_R$ is a faithfully semidualizing (R, R)-bimodule.*

Proof. We first prove that C is a semidualizing bimodule. We show only that it satisfies (a1), (b1), and (c1) of Definition 2.1, as the other parts are proved similarly.

Since E admits a degreewise finite Q-projective resolution P and R is Q-flat, $P \otimes_Q {}_RR$ is a degreewise finite projective resolution of ${}_RC = E \otimes_Q {}_RR$.

As $Q \xrightarrow{\gamma_Q} \mathrm{Hom}_Q(E,E)$ is an isomorphism, the commutative diagram

$$\begin{tikzcd}
R \arrow[r, "{}_R\gamma"] \arrow[d, "\cong"] & \mathrm{Hom}_{R^{\mathrm{op}}}(C,C) \arrow[r, equal] & \mathrm{Hom}_{R^{\mathrm{op}}}(E \otimes_Q R, E \otimes_Q R) \arrow[d, "\cong"', "\text{adjointness}"] \\
Q \otimes_Q R \arrow[d, "\gamma_Q \otimes_Q R"', "\cong"] & & \mathrm{Hom}_Q(E, \mathrm{Hom}_{R^{\mathrm{op}}}(R, E \otimes_Q R)) \arrow[d, "\cong"'] \\
\mathrm{Hom}_Q(E,E) \otimes_Q R \arrow[rr, "\cong", "\omega_{EER}"'] & & \mathrm{Hom}_Q(E, E \otimes_Q R)
\end{tikzcd}$$

shows that $R \xrightarrow{{}_R\gamma} \mathrm{Hom}_R(C,C)$ is an isomorphism.

Finally, let P be a Q-projective resolution of E. For $i > 0$, the first and fourth isomorphisms below are by definition while the second is adjointness

$$\begin{aligned}
\mathrm{Ext}^i_R(C,C) &= \mathrm{H}_{-i}\,\mathrm{Hom}_R(P \otimes_Q R, E \otimes_Q R) \\
&\cong \mathrm{H}_{-i}\,\mathrm{Hom}_Q(P, \mathrm{Hom}_R(R, E \otimes_Q R)) \\
&\cong \mathrm{H}_{-i}\,\mathrm{Hom}_Q(P, E \otimes_Q R) \\
&\cong \mathrm{Ext}^i_Q(E, E \otimes_Q R) = 0.
\end{aligned}$$

The third isomorphism is standard, and the last equality is by Lemma 1.1(b), as $\mathrm{Ext}^i_Q(E,E) = 0$ and R is Q-flat. Next, note that for modules N_R and ${}_RM$ there are standard isomorphisms

$$N \otimes_R C = N \otimes_R (E \otimes_Q R) \cong (N \otimes_R R) \otimes_Q E \cong N \otimes_Q E$$

and similarly $C \otimes_R M \cong E \otimes_Q M$. As E is faithfully semidualizing by Proposition 3.1, it follows from Lemma 3.1 that C is faithfully semidualizing. ☐

4. Auslander and Bass classes with respect to C

In this section we introduce and investigate properties of the Auslander and Bass classes with respect to a semidualizing (S,R)-bimodule $C = {}_SC_R$. Over a commutative noetherian ring the following definition can be found in [13, sec. 1].

Definition 4.1. The *Auslander class* $\mathcal{A}_C(R)$ with respect to C consists of all R-modules M satisfying

(A1) $\mathrm{Tor}^R_{\geqslant 1}(C,M) = 0$,

(A2) $\mathrm{Ext}^{\geqslant 1}_S(C, C \otimes_R M) = 0$, and

(A3) The natural evaluation homomorphism $\mu_M \colon M \longrightarrow \mathrm{Hom}_S(C, C \otimes_R M)$ is an isomorphism (of R-modules).

The *Bass class* $\mathcal{B}_C(S)$ with respect to C consists of all S-modules N satisfying

(B1) $\mathrm{Ext}_S^{\geqslant 1}(C, N) = 0$,
(B2) $\mathrm{Tor}^R_{\geqslant 1}(C, \mathrm{Hom}_S(C, N)) = 0$, and
(B3) The natural evaluation homomorphism $\nu_N \colon C \otimes_R \mathrm{Hom}_S(C, N) \longrightarrow N$ is an isomorphism (of S-modules).

It is straightforward to check the following:

Observation 4.1. Given modules ${}_RM$ and ${}_SN$, the homomorphisms

$$C \otimes_R \mathrm{Hom}_S(C, C \otimes_R M) \underset{C \otimes_R \mu_M}{\overset{\nu_{(C \otimes_R M)}}{\rightleftarrows}} C \otimes_R M, \text{ and}$$
$$\mathrm{Hom}_S(C, N) \underset{\mathrm{Hom}_S(C, \nu_N)}{\overset{\mu_{\mathrm{Hom}_S(C,N)}}{\rightleftarrows}} \mathrm{Hom}_S(C, C \otimes_R \mathrm{Hom}_S(C, N))$$

from Definition 4.1 yield identities

$$\nu_{(C \otimes_R M)} \circ (C \otimes_R \mu_M) = \mathrm{id}_{(C \otimes_R M)}, \text{ and}$$
$$\mathrm{Hom}_S(C, \nu_N) \circ \mu_{\mathrm{Hom}_S(C,N)} = \mathrm{id}_{\mathrm{Hom}_S(C,N)}.$$

The next result is a component of the Foxby equivalence; see Theorem 5.1.

Proposition 4.1. *Let ${}_SC_R$ be a semidualizing bimodule. There are equivalences of categories*

$$\mathcal{A}_C(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C \otimes_R -}{\underset{\sim}{\rightleftarrows}}} \mathcal{B}_C(S).$$

Proof. To see that the functor $C \otimes_R -$ maps $\mathcal{A}_C(R)$ to $\mathcal{B}_C(S)$, let $M \in \mathcal{A}_C(R)$ and note that the following hold by definition of the class $\mathcal{A}_C(R)$. First, for all $i > 0$ there is an equality $\mathrm{Ext}^i_S(C, C \otimes_R M) = 0$. Second, for $i > 0$ one has

$$0 = \mathrm{Tor}^R_i(C, M) \cong \mathrm{Tor}^R_i(C, \mathrm{Hom}_S(C, C \otimes_R M)).$$

Finally, as μ_M is an isomorphism, so is $C \otimes_R \mu_M$, Thus, Observation 4.1 implies that $\nu_{(C \otimes_R M)}$ is also an isomorphism and the inclusion $C \otimes_R M \in \mathcal{B}_C(S)$ follows.

The proof that $\mathrm{Hom}_S(C, -)$ maps $\mathcal{B}_C(S)$ to $\mathcal{A}_C(R)$ is similar. To conclude, note that if $M \in \mathcal{A}_C(R)$ and $N \in \mathcal{B}_C(S)$, then there are natural isomorphisms

$$\mu_M \colon M \xrightarrow{\cong} \mathrm{Hom}_S(C, C \otimes_R M) \quad \text{and} \quad \nu_N \colon C \otimes_R \mathrm{Hom}_S(C, N) \xrightarrow{\cong} N.$$

The desired equivalences of categories now follows. $\square$

The next lemma, together with Corollary 6.2 (in view of Proposition 3.1), extend [13, (1.2)].

Lemma 4.1. *Let ${}_SC_R$ be a semidualizing bimodule. The class $\mathcal{A}_C(R)$ contains all flat R-modules, and $\mathcal{B}_C(S)$ contains all injective S-modules.*

Proof. If F is R-flat then $\operatorname{Tor}^R_{\geqslant 1}(C,F)=0$, and Lemma 1.1(b) implies that $\operatorname{Ext}^{\geqslant 1}_S(C, C\otimes_R F)=0$. Finally, Lemma 1.1(a) implies that ω_{CCF} is an isomorphism of abelian groups. It is also R-linear, and so an R-module isomorphism. Thus, the commutative diagram

$$\begin{tikzcd} R\otimes_R F \arrow[r, "\gamma_R\otimes_R F", "\cong"'] \arrow[d, "\cong"'] & \operatorname{Hom}_S(C,C)\otimes_{R\ R}F \arrow[d, "\cong"', "\omega_{CCF}"] \\ F \arrow[r, "\mu_F"'] & \operatorname{Hom}_S(C, C\otimes_R F) \end{tikzcd}$$

implies that μ_F is an isomorphism. The other proof is dual. $\square$

We conclude this section by investigating how the Auslander and Bass classes behave with respect to various natural operations.

Proposition 4.2. *Let ${}_SC_R$ be a semidualizing bimodule.*

(a) *The classes $\mathcal{A}_C(R)$ and $\mathcal{B}_C(S)$ are closed under direct summands, products, coproducts, and filtered colimits.*
(b) *The classes $\mathcal{A}^f_C(R)$ and $\mathcal{B}^f_C(S)$ are closed under finite direct sums and direct summands.*

Proof. (a) It is straightforward to verify that $\mathcal{A}_C(R)$ is closed under direct summands and finite direct sums, as the functors Ext, Tor, and Hom are additive. We prove that $\mathcal{A}_C(R)$ is closed under filtered colimits, from which it follows that $\mathcal{A}_C(R)$ is closed under coproducts. To this end, let $\{M_\lambda\}$ be a filtered direct system of R-modules. Since tensor products and taking homology commute with filtered colimits, for each $i\geqslant 0$ there is an isomorphism of abelian groups

$$(\dagger)\qquad \operatorname{Tor}^R_i(C, \varinjlim M_\lambda)\cong \varinjlim \operatorname{Tor}^R_i(C, M_\lambda)$$

and, when $i=0$, an isomorphism of S-modules

$$(\dagger\dagger)\qquad C\otimes_R (\varinjlim M_\lambda)\cong \varinjlim (C\otimes_R M_\lambda).$$

As ${}_SC$ admits a degreewise finite S-projective resolution, $\operatorname{Ext}^i_S(C,-)$ commutes with filtered colimits. In particular, for $i\geqslant 0$ the isomorphism in $(\dagger\dagger)$ gives an isomorphism of abelian groups

$$(\dagger\dagger\dagger)\qquad \operatorname{Ext}^i_S(C, C\otimes_R(\varinjlim M_\lambda))\cong \varinjlim \operatorname{Ext}^i_S(C, C\otimes_R M_\lambda)$$

and, when $i=0$, an isomorphism of R-modules

$$\operatorname{Hom}_S(C, C\otimes_R(\varinjlim M_\lambda))\cong \varinjlim \operatorname{Hom}_S(C, C\otimes_R M_\lambda).$$

This isomorphism fits into the following commutative diagram

$$\begin{tikzcd} \varinjlim M_\lambda \arrow[rr, "\mu_{(\varinjlim M_\lambda)}"] \arrow[dr, "\varinjlim(\mu_{M_\lambda})"'] & & \mathrm{Hom}_S(C, C \otimes_R (\varinjlim M_\lambda)) \\ & \varinjlim \mathrm{Hom}_S(C, C \otimes_R M_\lambda) \arrow[ur, "\cong"'] & \end{tikzcd}$$

Since $\varinjlim(\mu_{M_\lambda})$ is an isomorphism if each μ_{M_λ} is, the diagram above, together with the isomorphisms (†) and (†††), imply that $\mathcal{A}_C(R)$ is closed under filtered colimits.

A proof similar to the one above, using the proof of [10, (3.2.26)], shows that $\mathcal{A}_C(R)$ is closed under products. Similar arguments show that the Bass class $\mathcal{B}_C(S)$ is closed under direct summands, coproducts, products, and filtered colimits.

(b) Since finite direct sums and direct summands of finitely generated modules are finitely generated, this follows from (a). □

5. C-flats, C-projectives, and C-injectives

In this section we study flat, projective, and injective modules with respect to a semidualizing bimodule $C = {}_SC_R$, investigate basic properties of these classes, and prove a form of Foxby equivalence. In the commutative noetherian setting, the following classes of modules already appear in, for example, [11], [12], [21], and [29].

Definition 5.1. An S-module is *C-flat* (resp., *C-projective*) if it has the form $C \otimes_R F$ for some flat (resp., projective) module ${}_RF$. An R-module is *C-injective* if it has the form $\mathrm{Hom}_S(C, I)$ for some injective module ${}_SI$. Set

$$\begin{aligned} \mathcal{F}_C &= \mathcal{F}_C(S) = \{C \otimes_R F \mid {}_RF \text{ is flat}\}, \\ \mathcal{P}_C &= \mathcal{P}_C(S) = \{C \otimes_R P \mid {}_RP \text{ is projective}\}, \\ \mathcal{I}_C &= \mathcal{I}_C(R) = \{\mathrm{Hom}_S(C, I) \mid {}_SI \text{ is injective}\}. \end{aligned}$$

It is straightforward to prove the following:

Lemma 5.1. *Let ${}_SC_R$ be a semidualizing bimodule. For modules ${}_RU$ and ${}_SV$ the following hold.*

(a) $V \in \mathcal{F}_C(S) \iff V \in \mathcal{B}_C(S)$ *and* $\mathrm{Hom}_S(C, V)$ *is flat over* R.
(b) $V \in \mathcal{P}_C(S) \iff V \in \mathcal{B}_C(S)$ *and* $\mathrm{Hom}_S(C, V)$ *is projective over* R.
(c) $U \in \mathcal{I}_C(R) \iff U \in \mathcal{A}_C(R)$ *and* $C \otimes_R U$ *is injective over* S.

The next result is a non-commutative, non-noetherian version of [13, (1.4)].

Theorem 5.1 (Foxby equivalence). *Let ${}_SC_R$ be a semidualizing bimodule. There are equivalences of categories as illustrated in Theorem* 1 *from the introduction.*

Proof. The equivalence between $\mathcal{A}_C(R)$ and $\mathcal{B}_C(S)$ was established in Proposition 4.1. The vertical containments are either trivial or follow from Lemmas 4.1 and 5.1. The horizontal equivalences follow from Lemma 5.1. $\square$

Remark 3. When C is faithfully semidualizing, a stronger version of the above theorem holds, proved using Corollaries 6.1 and 6.2. Specifically, replace the classes $\mathcal{P}_R(R)$, $\mathcal{F}_R(R)$ and $\mathcal{I}_S(S)$ with the classes of modules of finite R-projective, R-flat and S-injective dimension and replace the classes $\mathcal{P}_C(S)$, $\mathcal{F}_C(S)$ and $\mathcal{I}_C(R)$ with the classes of modules of finite $\mathcal{P}_C$-projective dimension over S, finite $\mathcal{F}_C$-projective dimension over S and finite $\mathcal{I}_C$-injective dimension over R; see 1.6.

We now prove some additional properties of the classes $\mathcal{F}_C$, $\mathcal{P}_C$, and $\mathcal{I}_C$. When R is commutative and noetherian, part (a) appears in [21, (2.14)].

Proposition 5.1. *Let ${}_SC_R$ be a semidualizing bimodule.*

(a) *The class $\mathcal{F}_C(S)$ is closed under coproducts, filtered colimits and summands. If R is right coherent, $\mathcal{F}_C(S)$ is also closed under products.*
(b) *The class $\mathcal{P}_C(S)$ is closed under coproducts and summands.*
(c) *The class $\mathcal{I}_C(R)$ is closed under products and summands. If S is left noetherian, then $\mathcal{I}_C(R)$ is also closed under coproducts and filtered colimits.*

Proof. We only prove (a), as (b) and (c) are similar. To prove that $\mathcal{F}_C(S)$ is closed under summands, consider a split exact sequence of S-modules

$$X = 0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0,$$

with $V \in \mathcal{F}_C(S)$. Lemma 5.1(a) and Proposition 4.2(a) imply $V', V'' \in \mathcal{B}_C(S)$. The complex $\operatorname{Hom}_S(C, X)$ is split exact and contains the module $\operatorname{Hom}_S(C, V)$ as a middle term. By Lemma 5.1(a), this module is R-flat. The flat R-modules are closed under summands, so the modules $\operatorname{Hom}_S(C, V')$ and $\operatorname{Hom}_S(C, V'')$ are R-flat. An application of Lemma 5.1(a) shows that $V', V'' \in \mathcal{F}_C(S)$.

The class $\mathcal{F}_C(S)$ is closed under finite direct sums, so in order to prove that it is closed under arbitrary coproducts, it suffices to prove that $\mathcal{F}_C(S)$ is closed under filtered colimits. Let $\{N_\lambda\}$ be a filtered direct system of C-flat S-modules. By Lemma 5.1(a) and Proposition 4.2(a) it follows that $\varinjlim N_\lambda \in \mathcal{B}_C(S)$. Lemma 5.1(a) implies that $\{\operatorname{Hom}_S(C, N_\lambda)\}$ is a filtered direct system of flat R-modules. Since the flat R-modules are closed under filtered colimits, the module

$$\operatorname{Hom}_S(C, \varinjlim N_\lambda) \cong \varinjlim \operatorname{Hom}_S(C, N_\lambda)$$

is R-flat. The above isomorphism comes from the fact that that $\operatorname{Hom}_S(C, -)$ commutes with filtered colimits since ${}_SC$ is finitely presented. An application of Lemma 5.1(a) implies $\varinjlim N_\lambda \in \mathcal{F}_C(S)$.

Finally, if R is right coherent then the class of flat (left) R-modules is closed under products by [10, (3.2.24)]. Since C_R is finitely presented, $C \otimes_R -$

commutes with products by [10, (3.2.22)]. Thus, arguments similar to the ones above show that $\mathcal{F}_C(S)$ is closed under products. $\square$

The next result shows, in particular, that the classes $\mathcal{P}_C(S)$, $\mathcal{F}_C(S)$, and $\mathcal{I}_C(R)$ are closed under extensions.

Proposition 5.2. *Let ${}_SC_R$ be a semidualizing bimodule. Consider the following exact sequences of S- and R-modules, respectively*

$$\begin{aligned} X = \ & 0 \longrightarrow W' \longrightarrow W \longrightarrow W'' \longrightarrow 0, \\ Y = \ & 0 \longrightarrow U' \longrightarrow U \longrightarrow U'' \longrightarrow 0. \end{aligned}$$

The following assertions hold.

(a) *If $W', W'' \in \mathcal{F}_C(S)$, then $W \in \mathcal{F}_C(S)$.*

(b) *If $W', W'' \in \mathcal{P}_C(S)$, then X splits and $W \in \mathcal{P}_C(S)$.*

(c) *If $U', U'' \in \mathcal{I}_C(R)$, then Y splits and $U \in \mathcal{I}_C(R)$.*

Proof. We prove only (b), as (c) is dual and (a) requires only minor adjustments. If P' is R-projective, then Lemma 1.1(b) implies $\mathrm{Ext}^1_S(C, C \otimes_R P') = 0$, and hence $\mathrm{Ext}^1_S(C, W') = 0$ since W' is C-projective. This, together with Lemma 5.1(b), forces $\mathrm{Hom}_S(C, X)$ to be a split exact sequence of R-projectives. Thus, the left column in the following commutative diagram, $C \otimes_R \mathrm{Hom}_S(C, X)$, is a split exact sequence of C-projective S-modules.

$$(*) \qquad \begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ C \otimes_R \mathrm{Hom}_S(C, W') \arrow[r, "\nu_{W'}", "\cong"'] \arrow[d] & W' \arrow[d] \\ C \otimes_R \mathrm{Hom}_S(C, W) \arrow[r, "\nu_W"] \arrow[d] & W \arrow[d] \\ C \otimes_R \mathrm{Hom}_S(C, W'') \arrow[r, "\nu_{W''}", "\cong"'] \arrow[d] & W'' \arrow[d] \\ 0 & 0 \end{tikzcd}$$

Lemma 5.1(b) implies that $\nu_{W''}$ and $\nu_{W'}$ are isomorphisms, and the five lemma forces ν_W to be an isomorphism as well. Thus, $V \in \mathcal{P}_C(S)$ and the right column in $(*)$ is split exact, as desired. $\square$

Corollary 5.1. *Let ${}_SC_R$ be a semidualizing bimodule. The classes $\mathcal{P}_C^f(S)$, $\mathcal{F}_C^f(S)$, and $\mathcal{I}_C^f(R)$ are closed under extensions* (*see notation in* 1.1)*.*

5.1. Recall that a short exact sequence of S-modules

$$X = 0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

is *pure exact* if $A \otimes_S X$ is exact for all S^{op}-modules A, equivalently, if $\mathrm{Hom}_S(B, X)$ is exact for all finitely presented S-modules B. When X is pure exact, N' is a *pure submodule* of N, and N'' a *pure quotient* of N, cf. [22, appendix].

The next result shows how $\mathcal{F}_C(S)$ and $\mathcal{I}_C(R)$ behave with respect to pure submodules and quotients. In the commutative noetherian setting, this is [9, (3.9)].

Lemma 5.2. *Let ${}_SC_R$ be a faithfully semidualizing bimodule.*

(a) *The class $\mathcal{F}_C(S)$ is closed under pure submodules and pure quotients.*

(b) *When S is left noetherian, the class $\mathcal{I}_C(R)$ is closed under pure submodules and pure quotients.*

Proof. (a). Consider a pure exact sequence, X, as in 5.1, with $N \in \mathcal{F}_C(S)$. Since C is finitely presented over S, the complex $\mathrm{Hom}_S(C, X)$ is an exact sequence of R-modules. We claim that $\mathrm{Hom}_S(C, X)$ is pure exact. To this end, let Q be a finitely presented R-module. Since C is finitely presented over S and $C \otimes_R -$ is right exact, the S-module $C \otimes_R Q$ is finitely presented. By Hom-tensor adjointness

$$\mathrm{Hom}_R(Q, \mathrm{Hom}_S(C, X)) \cong \mathrm{Hom}_S(C \otimes_R Q, X).$$

It remains to note that the latter complex (and hence also the first) is exact since X is pure exact and $C \otimes_R Q$ is finitely presented.

In the pure exact sequence of R-modules $\mathrm{Hom}_S(C, X)$, the module $\mathrm{Hom}_S(C, N)$ is R-flat by Lemma 5.1(a), since N is C-flat. Since the class of R-flat modules is closed under pure submodules and pure quotients, $\mathrm{Hom}_S(C, N')$ and $\mathrm{Hom}_S(C, N'')$ are also R-flat. Thus, if we can prove

$$N' \cong C \otimes_R \mathrm{Hom}_S(C, N') \quad \text{and} \quad N'' \cong C \otimes_R \mathrm{Hom}_S(C, N'')$$

then $N', N'' \in \mathcal{F}_C(S)$, as desired. Since $\mathrm{Hom}_R(C, X)$ is pure exact, there is a commutative diagram with exact columns

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & C \otimes_R \mathrm{Hom}_S(C, N') \arrow[r, "\nu_{N'}"] \arrow[d] & N' \arrow[d] \\
 & C \otimes_R \mathrm{Hom}_S(C, N) \arrow[r, "\nu_N", "\cong"'] \arrow[d] & N \arrow[d] \\
 & C \otimes_R \mathrm{Hom}_S(C, N'') \arrow[r, "\nu_{N''}"] \arrow[d] & N'' \arrow[d] \\
 & 0 & 0.
\end{tikzcd}
$$

Lemma 5.1(a) implies $N \in \mathcal{B}_C(S)$, so ν_N is an isomorphism. The snake lemma gives that $\nu_{N'}$ is injective, $\nu_{N''}$ is surjective, and that Ker $\nu_{N''} \cong$ Coker $\nu_{N'}$. Thus, it suffices to argue that Ker $\nu_{N''} = 0$. Since C is faithfully semidualizing, it is enough to prove that $\mathrm{Hom}_S(C, \mathrm{Ker}\ \nu_{N''}) = 0$. Applying $\mathrm{Hom}_S(C, -)$ to

$$0 \longrightarrow \mathrm{Ker}\ \nu_{N''} \longrightarrow C \otimes_R \mathrm{Hom}_S(C, N'') \xrightarrow{\nu_{N''}} N'' \longrightarrow 0$$

we see it is enough to show that $\mathrm{Hom}_S(C, \nu_{N''})$ is injective. We claim $\mathrm{Hom}_S(C, \nu_{N''})$ is an isomorphism. By Observation 4.1, it suffices to argue that $\mu_{\mathrm{Hom}_S(C,N'')}$ is an isomorphism. Since $\mathrm{Hom}_S(C, N'')$ is R-flat, this follows from Lemma 4.1.

The proof of (b) is dual to that of (a) — using that when S is left noetherian the class of injective S-modules is closed under pure submodules and pure quotients. $\square$

We conclude this section with a result on the (pre)covering and (pre)enveloping properties of the C-flats, C-projectives, and C-injectives. In the commutative noetherian setting, parts (a) and (c) below appear in [9, (3.5)].

Proposition 5.3. *Let ${}_SC_R$ be a semidualizing bimodule.*

(a) *The class $\mathcal{F}_C(S)$ is covering on the category of S-modules.*

(b) *The class $\mathcal{P}_C(S)$ is precovering on the category of S-modules.*

(c) *The class $\mathcal{I}_C(R)$ is enveloping on the category of R-modules.*

(d) *If R is right coherent and C is faithfully semidualizing, then the class $\mathcal{F}_C(S)$ is preenveloping on the category of S-modules.*

(e) *If S is left noetherian and C is faithfully semidualizing, then the class $\mathcal{I}_C(R)$ is covering on the category of R-modules.*

Proof. (a) By Bican-El Bashir-Enochs [3, (3)], the class of flat R-modules is covering. Thus, for any S-module N, the R-module $\mathrm{Hom}_S(C,N)$ has an R-flat cover $\alpha\colon F \longrightarrow \mathrm{Hom}_S(C,N)$. Define β to be the composite homomorphism

$$C \otimes_R F \xrightarrow{C\otimes_R \alpha} C \otimes_R \mathrm{Hom}_S(C,N) \xrightarrow{\nu_N} N.$$

This is an $\mathcal{F}_C(S)$-cover of N: To prove the precovering property, consider a homomorphism $\tau\colon C\otimes_R G \longrightarrow N$ where G is R-flat. We want $\psi\colon C\otimes_R G \longrightarrow C\otimes_R F$ such that $\tau = \beta\circ\psi$. The assignment ν from 4.1(B3) is natural. Also, Lemma 5.1(a) implies $C\otimes_R G \in \mathcal{B}_C(R)$. Observation 4.1 then gives rise to

$$(\dagger)\qquad \begin{tikzcd} C\otimes_R G \arrow[rr, "\tau"] \arrow[d, "\nu^{-1}_{(C\otimes_R G)}=C\otimes_R\mu_G"', "\cong"] & & N \\ C\otimes_R \mathrm{Hom}_S(C, C\otimes_R G) \arrow[rr, "C\otimes_R \mathrm{Hom}_S(C,\tau)"] & & C\otimes_R \mathrm{Hom}_S(C,N). \arrow[u, "\nu_N"'] \end{tikzcd}$$

Now, since α is an R-flat precover and G is R-flat there exists a homomorphism $\varphi\colon G \longrightarrow F$ making the following diagram commute

$$(\dagger\dagger)\qquad \begin{tikzcd} G \arrow[r, "\mu_G", "\cong"'] \arrow[d, dashed, "\varphi"'] & \mathrm{Hom}_S(C, C\otimes_R G) \arrow[d, "\mathrm{Hom}_S(C,\tau)"] \\ F \arrow[r, "\alpha"] & \mathrm{Hom}_S(C,N). \end{tikzcd}$$

Define $\psi = C\otimes_R \varphi$. The first "=" below comes from the commutativity of ($\dagger$)

$$\begin{aligned} \tau &= \nu_N \circ (C\otimes_R \mathrm{Hom}_S(C,\tau))\circ(C\otimes_R \mu_G) \\ &= \nu_N \circ (C\otimes_R \alpha)\circ(C\otimes_R \varphi) \\ &= \beta\circ\psi \end{aligned}$$

while the second comes the commutativity of the diagram induced by applying the functor $C\otimes_R -$ to the diagram ($\dagger\dagger$). The third holds by the definitions of β and ψ, and thus β is a precover of N, as desired.

To see that β is a cover, let $G = F$, $\tau = \beta$, and $\beta\circ\psi = \beta$. We show ψ is an automorphism. It is straightforward to verify that the following diagram is commutative, and Lemma 4.1 imples μ_F is an isomorphism

$$\begin{tikzcd} F \arrow[rr, "\alpha"] \arrow[d, "\mu_F"', "\cong"] & & \mathrm{Hom}_S(C,N) \\ \mathrm{Hom}_S(C, C\otimes_R F) \arrow[rr, "\mathrm{Hom}_S(C, C\otimes_R\alpha)"] & & \mathrm{Hom}_S(C, C\otimes_R \mathrm{Hom}_S(C,N)). \arrow[u, "\mathrm{Hom}_S(C,\nu_N)"'] \end{tikzcd}$$

The equality $\beta\circ\psi=\beta$ implies

$$\mathrm{Hom}_S(C,\beta)\circ \mathrm{Hom}(C,\psi) = \mathrm{Hom}_R(C,\beta).$$

Using the diagram immediately above, one checks that the following diagram is commutative

$$\begin{tikzcd} & F \arrow[dl, dashed, "\mu_F^{-1}\circ\mathrm{Hom}_S(C,\psi)\circ\mu_F"'] \arrow[d, "\alpha"] \\ F \arrow[r, "\alpha"'] & \mathrm{Hom}_S(C,N). \end{tikzcd}$$

Since α is an R-flat cover, it follows that $\mu_F^{-1} \circ \mathrm{Hom}_S(C,\psi) \circ \mu_F$, and hence $\mathrm{Hom}_S(C,\psi)$ must be an automorphism. Finally, the commutative diagram

$$\begin{tikzcd} C \otimes_R F \arrow[rr, "\psi"] \arrow[d, "C\otimes_R\mu_F"', "\cong"] & & C \otimes_R F \arrow[d, "\cong"', "C\otimes_R\mu_F"] \\ C \otimes_R \mathrm{Hom}_S(C, C \otimes_R F) \arrow[rr, "\cong", "C\otimes_R\mathrm{Hom}_S(C,\psi)"'] & & C \otimes_R \mathrm{Hom}_S(C, C \otimes_R F) \end{tikzcd}$$

implies that ψ itself must be an automorphism.

(b) This is similar to the proof of (a), using the fact that the class of R-projective modules is precovering.

(c) This proof is dual to the proof of (a), using that the class of S-injective modules is enveloping by Xu [29, (1.2.11)] and Eckmann–Schopf [8].

(d) It suffices by [25, (3.5)(c)] (see also [20, (2.6)(ii)]) to prove that $\mathcal{F}_C(S)$ is closed under arbitrary products and pure submodules, which follows immediately from Proposition 5.1(a) and Lemma 5.2(a).

(e) By [20, (2.5)], it suffices to show that $\mathcal{I}_C(R)$ is closed under coproducts and pure quotients. This follows from Proposition 5.1(c) and Lemma 5.2(b). $\square$

6. Characterizations of $\mathcal{A}_C$ and $\mathcal{B}_C$ and applications

Before characterizing the modules in the Auslander and Bass classes in terms of the C-flats, C-projectives, and C-injectives, we note an immediate consequence of the adjoint isomorphisms

$$\mathrm{Hom}_S(C \otimes_R X, I) \cong \mathrm{Hom}_R(X, \mathrm{Hom}_S(C,I)), \text{ and}$$
$$\mathrm{Hom}_S(C \otimes_R P, Y) \cong \mathrm{Hom}_R(P, \mathrm{Hom}_S(C,Y)).$$

Lemma 6.1. *Let ${}_SC_R$ be a semidualizing bimodule, let X be a complex of R-modules, and let Y a complex of S-modules.*

(a) *If $C \otimes_R X$ is exact, then* $\mathrm{Hom}_R(X, \mathrm{Hom}_S(C,I))$ *is exact for all injective S-modules I. If I is faithfully S-injective and* $\mathrm{Hom}_R(X, \mathrm{Hom}_S(C,I))$ *is exact, then $C \otimes_R X$ is exact.*

(b) *If* $\mathrm{Hom}_S(C,Y)$ *is exact, then* $\mathrm{Hom}_S(C \otimes_R P, Y)$ *is exact for all projective R-modules P. If P is faithfully R-projective and* $\mathrm{Hom}_S(C \otimes_R P, Y)$ *is exact, then* $\mathrm{Hom}_S(C,Y)$ *is exact.*

In the commutative noetherian setting, Theorem 2 from the Introduction and Lemma 6.1 appear in [9, (3.6, 3.7)]; see also [29, (5.5.4, 5.5.5)]. We now prove Theorem 2 from the introduction.

Proof of Theorem 2. Assume $M \in \mathcal{A}_C(R)$ so that $\operatorname{Tor}^R_{\geqslant 1}(C, M) = 0$. An augmented projective resolution P^+ of M then gives rise to an exact sequence

$$C \otimes_R P^+ = \ \cdots \longrightarrow C \otimes_R P_1 \longrightarrow C \otimes_R P_0 \longrightarrow C \otimes_R M \longrightarrow 0.$$

By Proposition 5.3(c), the class of C-injective modules is preenveloping. Thus, M admits an augmented proper C-injective coresolution, that is, a complex

$$U^+ = \ 0 \longrightarrow M \longrightarrow U^0 \longrightarrow U^1 \longrightarrow \cdots$$

such that $\operatorname{Hom}_R(U^+, W)$ is exact for $W \in \mathcal{I}_C(R)$. In particular, if ${}_RI$ is faithfully injective, then $\operatorname{Hom}_R(U^+, \operatorname{Hom}_S(C, I))$ is exact. Thus, Lem. 6.1(a) implies

$$C \otimes_R U^+ = \ 0 \longrightarrow C \otimes_R M \longrightarrow C \otimes_R U^0 \longrightarrow C \otimes_R U^1 \longrightarrow \cdots$$

is exact. Therefore, we prove that U^+ is exact. The complex X, obtained by splicing together P^+ and U^+, then has the desired properties. By Lemma 5.1(c), $C \otimes_R U^i$ is injective for all $i \geqslant 0$, so $C \otimes_R U^+$ is an augmented injective resolution of $C \otimes_R M$. Since U_i and M are in $\mathcal{A}_C(R)$, there is an isomorphism

$$\operatorname{Hom}_S(C, C \otimes_R U^+) \cong U^+.$$

Since $\operatorname{Ext}_S^{\geqslant 1}(C, C \otimes_R M) = 0$, it follows that U^+ is exact.

Conversely, assume there is a complex X satisfying properties (a)–(e) of the theorem, where each P_i is R-flat. The complex X induces exact sequences P^+ and U^+, as depicted above, and property (e) implies that the complexes $C \otimes_R P^+$ and $C \otimes_R U^+$ are exact.

Since P_i is R-flat and $C \otimes_R P^+$ is exact, there is an equality $\operatorname{Tor}^R_{\geqslant 1}(C, M) = 0$. Also, as $C \otimes_R U^+$ is exact and the functor $\operatorname{Hom}_S(C, -)$ is left exact, the right column in the following commutative diagram is exact

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
M \arrow[r, "\mu_M"] \arrow[d] & \operatorname{Hom}_S(C, C \otimes_R M) \arrow[d] \\
U^0 \arrow[r, "\mu_{U^0}", "\cong"'] \arrow[d] & \operatorname{Hom}_S(C, C \otimes_R U^0) \arrow[d] \\
U^1 \arrow[r, "\mu_{U^1}", "\cong"'] & \operatorname{Hom}_S(C, C \otimes_R U^1)
\end{tikzcd}$$

By Lemma 5.1(c) one has $U^i \in \mathcal{A}_C(R)$ so μ_{U^0} and μ_{U^1} are isomorphisms. The five lemma implies that μ_M is an isomorphism. The μ_{U^i} and μ_M fit together to give an isomorphism of complexes $\mathrm{Hom}_S(C, C \otimes_R U^+) \cong U^+$. Since the complex U^+ is exact, it follows that $\mathrm{Hom}_S(C, C \otimes_R U^+)$ is exact. As $C \otimes_R U^+$ is an augmented S-injective resolution of $C \otimes_R M$, there is an equality

$$\mathrm{Ext}_S^{\geqslant 1}(C, C \otimes_R M) = 0.$$

Thus, M belongs to $\mathcal{A}_C(R)$. $\square$

The next result is proved in a similar manner.

Theorem 6.1. *Let ${}_SC_R$ be a semidualizing bimodule. An S-module N belongs to $\mathcal{B}_C(S)$ if and only if there exists a complex of S-modules*

$$Y = \ \cdots \longrightarrow W_1 \longrightarrow W_0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

that satisfies the following conditions

(a) *The complex Y is exact*;
(b) *Each I^i is S-injective*;
(c) *Each W_i is a C-projective* (*or C-flat*);
(d) *There is an isomorphism $N \cong \mathrm{Ker}(I^0 \longrightarrow I^1)$; and*
(e) *The complex $\mathrm{Hom}_S(C, Y)$ is exact.*

Moreover, if $N \in \mathcal{B}_C(S)$ then any complex constructed by splicing together an augmented injective coresolution of N and an augmented proper C-projective resolution of N will satisfy the above properties.

The next two theorems address the behavior of the classes $\mathcal{A}_C(R)$ and $\mathcal{B}_C(S)$ with respect to short exact sequences, see 1.8 for the terminology.

Theorem 6.2. *Let ${}_SC_R$ be a semidualizing bimodule. The classes $\mathcal{A}_C(R)$ and $\mathcal{A}_C^f(R)$ are projectively resolving, and $\mathcal{B}_C(S)$ is injectively resolving.*

Proof. We prove that $\mathcal{A}_C(R)$ is projectively resolving; the other proofs are similar. By Lemma 4.1, the class $\mathcal{A}_C(R)$ contains the R-projective modules. Thus, it suffices to show that, given an exact sequence of R-modules

$$X = 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

with $M'' \in \mathcal{A}_C(R)$, then $M' \in \mathcal{A}_C(R)$ if and only if $M \in \mathcal{A}_C(R)$. Since $M'' \in \mathcal{A}_C(R)$ we have $\mathrm{Tor}_1^R(C, M'') = 0$. In particular, the complex $C \otimes_R X$ is exact. By Lemma 6.1(a), the complex $\mathrm{Hom}_R(X, U)$ is exact for all $U \in \mathcal{I}_C(R)$. Moreover, the class $\mathcal{I}_C(R)$ is closed under finite direct sums by Proposition 5.1(c) and is preenveloping by Proposition 5.3(c). Thus, the Horseshoe Lemma for preenveloping classes [10, (8.2.2)] gives a commutative diagram

$$(\dagger)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U' & \longrightarrow & U' \oplus U'' & \longrightarrow & U'' & \longrightarrow & 0 \end{array}$$

with exact rows and where each vertical map gives rise to an augmented proper $\mathcal{I}_C(R)$-coresolution. Similarly, the Horseshoe Lemma for projective resolutions yields a commutative diagram with exact rows

$$(\dagger\dagger)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & P' & \longrightarrow & P' \oplus P'' & \longrightarrow & P'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

where each vertical map gives rise to an augmented projective resolution. Splicing together ($\dagger$) and ($\dagger\dagger$) provides a degreewise split exact sequence of complexes

$$(*)\qquad \begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P'_1 & \longrightarrow & P'_1 \oplus P''_1 & \longrightarrow & P''_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P'_0 & \longrightarrow & P'_0 \oplus P''_0 & \longrightarrow & P''_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U'_0 & \longrightarrow & U'_0 \oplus U''_0 & \longrightarrow & U''_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & U'_1 & \longrightarrow & U'_1 \oplus U''_1 & \longrightarrow & U''_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

where P'_i, P''_i are projective, $U'_i, U''_i \in \mathcal{I}_C(R)$, and where

$$M' = \operatorname{Coker}(P'_1 \longrightarrow P'_0)\ ,\ M'' = \operatorname{Coker}(P''_1 \longrightarrow P''_0),$$
$$M = \operatorname{Coker}(P'_1 \oplus P''_1 \longrightarrow P'_0 \oplus P''_0).$$

Two of the (nonzero) modules in the complex X are in $\mathcal{A}_C(R)$, and so by Theorem 2, two of the complexes in ($*$) are exact. The long exact sequence in homology implies that the third complex is also exact. Furthermore, if we apply $C \otimes_R -$ to ($*$), we get another degreewise split exact sequence of complexes, and again, since two of these complexes are exact (by assumption), so is the third. Another application of Theorem 2 completes the proof. $\square$

The next result follows from Lemma 5.1 and Theorem 6.2 by taking appropriate bounded resolutions and breaking them up into short exact sequences.

Corollary 6.1. *Let ${}_SC_R$ be a semidualizing bimodule. The class $\mathcal{A}_C(R)$ contains the R-modules of finite $\mathcal{I}_C$-injective dimension and the class $\mathcal{B}_C(S)$ contains the S-modules of finite $\mathcal{F}_C$-projective dimension and finite $\mathcal{P}_C$-projective dimension.*

Theorem 6.3. *Let ${}_SC_R$ be a faithfully semidualizing bimodule. The classes $\mathcal{A}_C(R)$ and $\mathcal{A}_C^f(R)$ are closed under cokernels of monomorphisms and $\mathcal{B}_C(S)$ is closed under kernels of epimorphisms.*

Proof. We only prove the statement for $\mathcal{A}_C(R)$, as the other statements are proved similarly. Consider a short exact sequence of R-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

with M' and M in $\mathcal{A}_C(R)$. The proof of Theorem 6.2 gives the desired conclusion provided $\mathrm{Tor}_1^R(C, M'') = 0$. To verify this vanishing, note that $\mathrm{Tor}_1^R(C, M) = 0$ as $M \in \mathcal{A}_C(R)$. Hence, there is an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^R(C, M'') \longrightarrow C \otimes_R M' \longrightarrow C \otimes_R M \longrightarrow C \otimes_R M'' \longrightarrow 0.$$

Using Remark 1, this is an exact sequence of S-modules homomorphisms. Applying the functor $\mathrm{Hom}_S(C, -)$ to this sequence provides the right-hand exact column in the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & \mathrm{Hom}_S(C, \mathrm{Tor}_1^R(C, M'')) \arrow[d] \\
M' \arrow[r, "\mu_{M'}", "\cong"'] \arrow[d] & \mathrm{Hom}_S(C, C \otimes_R M') \arrow[d] \\
M \arrow[r, "\mu_M", "\cong"'] & \mathrm{Hom}_S(C, C \otimes_R M)
\end{tikzcd}$$

As $M' \longrightarrow M$ is injective, a diagram chase shows that $\mathrm{Hom}_S(C, \mathrm{Tor}_1^R(C, M'')) = 0$. Since C is faithfully semidualizing, $\mathrm{Tor}_1^R(C, M'') = 0$, as desired. □

The next result—which is a non-commutative, non-noetherian version of [13, (1.2)]—follows from Theorem 6.3 by taking appropriate bounded resolutions and breaking them up into short exact sequences.

Corollary 6.2. *Let ${}_SC_R$ be a faithfully semidualizing bimodule. The class $\mathcal{A}_C(R)$ (resp., $\mathcal{A}_C^f(R)$) contains the R-modules (resp., finite R-modules) of finite flat dimension, and the class $\mathcal{B}_C(S)$ contains the S-modules of finite injective dimension.*

Theorems 6.2 and 6.3 immediately give the following non-commutative, non-noetherian version of [13, (1.3)] and [29, (5.5.6), (5.5.7)].

Corollary 6.3. *Let ${}_SC_R$ be a faithfully semidualizing bimodule. The classes $\mathcal{A}_C(R)$ and $\mathcal{B}_C(S)$ have the property that if two of three modules in a short exact sequence are in the class then so is the third.*

We also have the following result related to Proposition 5.2.

Corollary 6.4. *Let ${}_SC_R$ be a faithfully semidualizing bimodule. The classes $\mathcal{P}_C(S)$ and $\mathcal{F}_C(S)$ are projectively resolving and the class $\mathcal{I}_C(R)$ is injectively resolving.*

Proof. We only prove the claim for $\mathcal{P}_C(S)$. By Proposition 5.2, we only need to argue that $\mathcal{P}_C(S)$ is closed under kernels of epimorphisms. Consider an exact sequence of S-modules

$$0 \longrightarrow W' \longrightarrow W \longrightarrow W'' \longrightarrow 0$$

with $W, W'' \in \mathcal{P}_C(S)$. By Lemma 5.1(b), one has $W, W'' \in \mathcal{B}_C(S)$ and so Theorem 6.3 implies that $W' \in \mathcal{B}_C(S)$. It follows that $\mathrm{Ext}^1_S(C, W') = 0$. The same technique as in the proof of Proposition 5.2 shows that $W' \in \mathcal{P}_C(S)$. $\square$

The next result is well-known with a straightforward proof.

Lemma 6.2. *Let ${}_SC_R$ be a semidualizing bimodule, and let M be an R-module. If $\mathrm{Ext}^{\geqslant 1}_S(C, C \otimes_R M) = 0$, then $\mathrm{Ext}^{\geqslant 1}_S(C \otimes_R P, C \otimes_R M) = 0$ for all projective R-modules P.*

The following result can be thought of as a derived version of Foxby equivalence. For derived Auslander and Bass classes over a commutative noetherian ring, cf. Remark 2, related results can be found in e.g. [5, (4.5)].

Theorem 6.4. *Let M and M' be R-modules, let N and N' be S-modules, let $\tilde{N}$ be an S^{op}-module, and let $i \geqslant 0$.*

(a) *If $M \in \mathcal{A}_C(R)$ and $\mathrm{Tor}^R_{\geqslant 1}(C, M') = 0$ (e.g., if $M' \in \mathcal{A}_C(R)$), then*

$$\mathrm{Ext}^i_R(M', M) \cong \mathrm{Ext}^i_S(C \otimes_R M', C \otimes_R M).$$

(b) *If $N \in \mathcal{B}_C(S)$ and $\mathrm{Ext}^{\geqslant 1}_S(C, N') = 0$ (e.g., if $N' \in \mathcal{B}_C(S)$), then*

$$\mathrm{Ext}^i_S(N, N') \cong \mathrm{Ext}^i_R(\mathrm{Hom}_S(C, N), \mathrm{Hom}_S(C, N')).$$

(c) *If $N \in \mathcal{B}_C(S)$ and $\mathrm{Tor}^S_{\geqslant 1}(\tilde{N}, C) = 0$, then*

$$\mathrm{Tor}^S_i(\tilde{N}, N) \cong \mathrm{Tor}^R_i(\tilde{N} \otimes_S C, \mathrm{Hom}_S(C, N)).$$

Each isomorphism defined above is a natural isomorphism of abelian groups.

Proof. (a) We proceed by induction on i. For $i = 0$, the first isomorphism below holds since $M \in \mathcal{A}_C(R)$ while the second is Hom-tensor adjointness

$$\begin{aligned}\operatorname{Hom}_R(M', M) &\cong \operatorname{Hom}_R(M', \operatorname{Hom}_S(C, C \otimes_R M)) \\ &\cong \operatorname{Hom}_S(C \otimes_R M', C \otimes_R M).\end{aligned}$$

Moreover, these isomorphisms are natural in M' and M. Next, assume that $i > 0$. The induction hypothesis implies that for $j < i$ there exist isomorphisms

$$(\dagger) \qquad \operatorname{Ext}^j_R(L', L) \xrightarrow{\cong} \operatorname{Ext}^j_S(C \otimes_R L', C \otimes_R L),$$

which are natural for all R-modules $L \in \mathcal{A}_C(R)$ and L' with $\operatorname{Tor}^R_{\geqslant 1}(C, L') = 0$. Now, consider $M \in \mathcal{A}_C(R)$ and M' such that $\operatorname{Tor}^R_{\geqslant 1}(C, M') = 0$. There is a projective R-module P' which gives rise to an exact sequence

$$X = 0 \longrightarrow K' \longrightarrow P' \longrightarrow M' \longrightarrow 0.$$

The equalities $\operatorname{Tor}^R_{\geqslant 1}(C, M') = 0 = \operatorname{Tor}^R_{\geqslant 1}(C, P')$ and the appropriate long exact sequence imply $\operatorname{Tor}^R_{\geqslant 1}(C, K') = 0$. Thus, we may apply the induction hypothesis to the modules $L' = K'$ (or $L' = P'$) and $L = M$. Since $\operatorname{Tor}^R_1(C, M') = 0$, the complex X induces the exact sequence of S-modules

$$C \otimes_R X \;=\; 0 \longrightarrow C \otimes_R K' \longrightarrow C \otimes_R P' \longrightarrow C \otimes_R M' \longrightarrow 0.$$

The long exact sequence coming from the complexes $\operatorname{Hom}_R(X, M)$ and $\operatorname{Hom}_S(C \otimes_R X, C \otimes_R M)$ give rise to a commutative diagram with exact columns

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\operatorname{Ext}^{i-1}_R(P', M) \arrow[r, "\cong"] \arrow[d] & \operatorname{Ext}^{i-1}_S(C \otimes_R P', C \otimes_R M) \arrow[d] \\
\operatorname{Ext}^{i-1}_R(K', M) \arrow[r, "\cong"] \arrow[d] & \operatorname{Ext}^{i-1}_S(C \otimes_R K', C \otimes_R M) \arrow[d] \\
\operatorname{Ext}^{i}_R(M', M) \arrow[d] & \operatorname{Ext}^{i}_S(C \otimes_R M', C \otimes_R M) \arrow[d] \\
\operatorname{Ext}^{i}_R(P', M) \arrow[d, equal] & \operatorname{Ext}^{i}_S(C \otimes_R P', C \otimes_R M) \arrow[d, equal] \\
0 & 0.
\end{tikzcd}$$

The right zero follows from Lemma 6.2, and the two isomorphisms come from the induction hypothesis. Diagram chasing provides a unique isomorphism

$$(\star) \qquad \operatorname{Ext}^i_R(M', M) \xrightarrow{\cong} \operatorname{Ext}^i_S(C \otimes_R M', C \otimes_R M)$$

making the induced diagram commutative. It is straightforward to verify that $(\star)$ is natural in M and M'. Parts (b) and (c) have similar proofs. $\square$

7. Auslander and Bass classes over commutative rings

For noetherian rings the next two results are in [4, (3.2.9)], [5, (5.8), (5.9)].

Proposition 7.1. *Let R be a commutative ring and C a semidualizing R-module. Assume M and N are R-modules and $\mathfrak{p}$ is a prime ideal of R.*

(a) *If $M \in \mathcal{A}_C(R)$, then $M_\mathfrak{p} \in \mathcal{A}_{C_\mathfrak{p}}(R_\mathfrak{p})$.*
(b) *If $N \in \mathcal{B}_C(R)$, then $N_\mathfrak{p} \in \mathcal{B}_{C_\mathfrak{p}}(R_\mathfrak{p})$.*

Proof. We prove only (a), as (b) is similar. By Observation 2.1(d), the $R_\mathfrak{p}$-module $C_\mathfrak{p}$ is semidualizing. If $M \in \mathcal{A}_C(R)$, then for $i > 0$

$$\begin{aligned}\operatorname{Tor}_i^{R_\mathfrak{p}}(C_\mathfrak{p}, M_\mathfrak{p}) &\cong \operatorname{Tor}_i^R(C,M)_\mathfrak{p} = 0,\\ \operatorname{Ext}^i_{R_\mathfrak{p}}(C_\mathfrak{p}, C_\mathfrak{p}\otimes_{R_\mathfrak{p}} M_\mathfrak{p}) &\cong \operatorname{Ext}^i_R(C, C\otimes_R M)_\mathfrak{p} = 0;\end{aligned}$$

where the second row uses the assumption that C admits a degreewise finite R-projective resolution. Furthermore, the commutative diagram

$$\begin{tikzcd} M_\mathfrak{p} \arrow[r, "\mu_{(M_\mathfrak{p})}"] \arrow[d, "(\mu_M)_\mathfrak{p}"', "\cong"] & \operatorname{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p}, C_\mathfrak{p}\otimes_{R_\mathfrak{p}} M_\mathfrak{p}) \arrow[d, "\cong"] \\ \operatorname{Hom}_R(C, C\otimes_R M)_\mathfrak{p} \arrow[r, "\cong"'] & \operatorname{Hom}_{R_\mathfrak{p}}(C_\mathfrak{p}, (C\otimes_R M)_\mathfrak{p}) \end{tikzcd}$$

shows that $\mu_{(M_\mathfrak{p})}$ is an isomorphism. Thus, one has $M_\mathfrak{p} \in \mathcal{A}_{C_\mathfrak{p}}(R_\mathfrak{p})$. $\square$

Proposition 7.2. *If R is commutative and C is a semidualizing R-module, then the following hold for all R-modules M and N.*

(a) *$M \in \mathcal{A}_C(R) \iff \operatorname{Hom}_R(M,I) \in \mathcal{B}_C(R)$ for all injective R-modules I.*
(b) *$N \in \mathcal{B}_C(R) \iff \operatorname{Hom}_R(N,I) \in \mathcal{A}_C(R)$ for all injective R-modules I.*
(c) *$M \in \mathcal{A}_C(R) \iff M\otimes_R F \in \mathcal{A}_C(R)$ for all flat R-modules F.*
(d) *$N \in \mathcal{B}_C(R) \iff N\otimes_R F \in \mathcal{B}_C(R)$ for all flat R-modules F.*

Proof. We prove (a), as (b), (c), and (d) are similar. If I is injective then Hom-tensor adjointness gives an isomorphism

$$\operatorname{Hom}_R(\operatorname{Tor}_i^R(C,M), I) \cong \operatorname{Ext}^i_R(C, \operatorname{Hom}_R(M,I)).$$

Thus, $\operatorname{Tor}_i^R(C,M) = 0$ if and only if $\operatorname{Ext}^i_R(C,\operatorname{Hom}_R(M,I)) = 0$ for all injective modules I. The first isomorphism below follows from Lemma 1.2(a)

$$\begin{aligned}\operatorname{Hom}_R(\operatorname{Ext}^i_R(C, C\otimes_R M), I) &\cong \operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C\otimes_R M, I))\\ &\cong \operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C, \operatorname{Hom}_R(M,I)))\end{aligned}$$

while the second is by Hom-tensor adjointness. Hence, $\mathrm{Ext}^i_R(C, C \otimes_R M) = 0$ if and only if $\mathrm{Tor}_i^R(C, \mathrm{Hom}_R(C, \mathrm{Hom}_R(M, I))) = 0$ for all injective modules I. Finally, there is a commutative diagram

$$\begin{tikzcd}
C \otimes_R \mathrm{Hom}_R(C \otimes_R M, I) \arrow[r, "C\otimes_R(\text{adjointness})", "\cong"'] \arrow[d, "\theta_{C(C\otimes_R M)I}"', "\cong"] & C \otimes_R \mathrm{Hom}_R(C, \mathrm{Hom}_R(M, I)) \arrow[d, "\nu_{\mathrm{Hom}_R(M,I)}"] \\
\mathrm{Hom}_R(\mathrm{Hom}_R(C, C \otimes_R M), I) \arrow[r, "\mathrm{Hom}_R(\mu_M, I)"'] & \mathrm{Hom}_R(M, I)
\end{tikzcd}$$

The left most vertical map is an isomorphism by Lemma 1.2(a). It follows that μ_M is an isomorphism if and only if $\nu_{\mathrm{Hom}_R(M,I)}$ is an isomorphism for all injective modules I. In conclusion, $M \in \mathcal{A}_C(R)$ if and only if $\mathrm{Hom}_R(M, I) \in \mathcal{B}_C(R)$ for all injective modules I. $\square$

Remark 4. Each of the statements in the above proposition has a third equivalent condition. For example, the statements in (a) are both equivalent to the following: $\mathrm{Hom}_R(M, I) \in \mathcal{B}_C(R)$ for some faithfully injective module I.

Acknowledgements. We thank Sean Sather-Wagstaff for his detailed reading of this manuscript and many thoughtful suggestions. We thank Hans-Bjørn Foxby and Søren Jøndrup for useful discussions about some of the material in Section 3, and Tom Marley for informing us of Gruson's Theorem. We also thank Frank Moore, Lars Winther Christensen, Luchezar Avramov, Srikanth Iyengar and the referee for their helpful comments.

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