

Lusternik-Schnirelmann category of stunted quasi-projective spaces

By

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Abstract

We determine the L-S category of stunted quasi-projective space $Q_{n,m} = Q_n/Q_m$ for $n \leq 4m + 3$. As a special case of our main result, the L-S category of Q_3 is determined to satisfy $\text{cat } Q_3 = 3$, which answers the question raised by Fernández-Suárez, Gómez-Tato and Tanré [4].

1. Introduction

In this paper, each space is assumed to have the homotopy type of a CW-complex. The (normalized) Lusternik-Schnirelmann category of X , denoted $\text{cat } X$ is the least number m such that there is a covering of X by $m + 1$ open subsets each of which is contractible in X .

Let \mathbb{H} be the quaternion and $S(\mathbb{H}^n)$ be the unit sphere in \mathbb{H}^n . In [8], James has defined the (quaternionic) quasi-projective space Q_n is as follows:

$$Q_n = S(\mathbb{H}^n) \times S(\mathbb{H}) / \sim,$$

where \sim is an equivalence relation given by

$$(u, q) \sim (uz, z^{-1}qz) \text{ for } z \in S(\mathbb{H}) \text{ and } (u, 1) \sim (v, 1).$$

Then Q_n is a CW-complex having one cell e^{4r-1} of dimension $4r - 1$ for $r = 1, \dots, n$ [8]. Hence, for $m < n$, Q_m is a subcomplex of Q_n and we denote by $Q_{n,m} = Q_n/Q_m$ the stunted quasi-projective space. There is the following result for $\text{cat } Q_{n,m}$.

Theorem 1.1 (Kishimoto and Kono).

$$\text{cat } Q_{n,m} = \begin{cases} 1 & m + 1 \leq n \leq 2m + 1 \\ 2 & 2m + 2 \leq n \leq 3m + 2 \end{cases}$$

and

$$\text{cat } Q_{n,m} \geq 3 \text{ for } n \geq 4m + 4.$$

This results obtained by Kishimoto and Kono to give a lower bound of L-S category of quaternionic Stiefel manifolds [9].

The main result of this paper is as follows:

Theorem 1.2. $\text{cat } Q_{n,m} = 3$ for $3m + 3 \leq n \leq 4m + 3$.

Corollary 1.1. When $n = 3$ and $m = 0$, $\text{cat } Q_3 = 3$.

This result answers the question in Fernández-Suárez, Gómez-Tato and Tanré [4], in which they asked whether $\text{cat } Q_3 = 2$ or 3.

In this paper, we follow the notation in Iwase [5]: let h^* be a multiplicative generalized cohomology. The cup-length of X with the cohomology theory h^* is the least number m such that all $(m + 1)$ -fold cup products vanish in the reduced cohomology $\tilde{h}^*(X)$. We denote this number by $\text{cup}(X; h)$. To obtain Theorem 1.2, we use the following fact due to Ganea [3] (see Iwase, Mimura and Nishimoto [7] for details).

Fact 1.1. Let X be an $(n - 1)$ -connected CW-complex and h^* be a multiplicative generalized cohomology. Then

$$\text{cup}(X; h) \leq \text{cat } X \leq \frac{\dim X}{n}.$$

2. Proof of Theorem 1.2

To obtain the lower bounds for $\text{cat } Q_{n,m}$, we use the cohomology theory introduced by Iwase and Mimura [6]: Let (X, A) be a pair of space. The cohomology theory h^* is defined by

$$h^*(X, A) = \{X/A, \mathcal{S}[0, 2]\},$$

where $\mathcal{S}[0, 2]$ is the spectrum obtained from the sphere spectrum \mathcal{S} by killing all homotopy groups of dimensions > 2 . Then h^* is an additive and multiplicative cohomology theory with the coefficient ring

$$h^* = h^*(pt) \cong \mathbb{Z}[\varepsilon]/(\varepsilon^3, 2\varepsilon), \text{deg } \varepsilon = -1,$$

where $\varepsilon \in h^{-1} = \pi_0^{\mathcal{S}}(\Sigma^{-1}\mathcal{S}) \cong \pi_1^{\mathcal{S}}(\mathcal{S})$ corresponds to the Hopf element η .

Since all the cells in $Q_{n,m}$ are concentrated in dimensions 3 modulo 4, we have

$$h^*(Q_{n,m}) \cong h^*\{1, x_{4m+3}, x_{4m+7}, \dots, x_{4n-1}\},$$

where $\text{deg } x_{4i-1} = 4i - 1$ for $m + 1 \leq i \leq n$. We need to show the following:

Proposition 2.1. $x_{4\ell+3}^2 = \varepsilon \cdot x_{8\ell+7} \in h^{8\ell+6}(Q_{2\ell+2,\ell})$ for any $\ell \geq 0$.

Then we obtain $x_{8m+7}^2 = \varepsilon \cdot x_{16m+15} \in h^{16m+14}(Q_{4m+4,2m+1})$ and

$$\begin{aligned} x_{4m+3}^2 x_{8m+7} &= (\varepsilon \cdot x_{8m+7}) x_{8m+7} = \varepsilon \cdot x_{8m+7}^2 \\ &= \varepsilon^2 \cdot x_{16m+15} \in h^{16m+13}(Q_{4m+4,m}). \end{aligned}$$

Hence we have

$$\begin{aligned} 0 \neq x_{4m+3}x_{8m+7} &= \varepsilon \cdot x_{12m+11} \\ &\in h^{12m+10}(Q_{4m+4,m}) \cong h^{12m+10}(Q_{3m+3,m}) \cong \mathbb{Z}/2. \end{aligned}$$

Thus we obtain

$$\begin{aligned} x_{4m+3}^3 &= (\varepsilon \cdot x_{8m+7})x_{4m+3} = \varepsilon \cdot x_{4m+3}x_{8m+7} \\ &= \varepsilon^2 x_{12m+11} \in h^{12m+9}(Q_{3m+3,m}) \end{aligned}$$

and $3 \leq \text{cup}(Q_{3m+3,m}; h)$. By Fact 1.1, we have

$$3 \leq \text{cup}(Q_{3m+3,m}; h) \leq \text{cup}(Q_{n,m}; h) \leq \text{cat } Q_{n,m} \text{ for } 3m+3 \leq n,$$

and

$$\text{cat } Q_{n,m} \leq \text{cat } Q_{4m+3,m} \leq \frac{16m+11}{4m+3} < 4 \text{ for } n \leq 4m+3.$$

So, we obtain Theorem 1.2. □

3. Proof of Proposition 2.1

Let $\gamma : S^{8\ell+6} \rightarrow Q_{2\ell+1,\ell}$ be the attaching map of the $(8\ell+7)$ -cell of $Q_{2\ell+2,\ell}$. There exist a CW-complex $Q'_{2\ell+1,\ell}$ such that $\Sigma Q'_{2\ell+1,\ell} = Q_{2\ell+1,\ell}$, since $Q_{2\ell+1,\ell}$ is $(4\ell+2)$ -connected and $\dim Q_{2\ell+1,\ell} = 8\ell+3$. We need the relation between the attaching map γ and the cup product in the cohomology theory h^* . By the parallel argument to Lemma 3.6 of [6], we obtain the following.

Lemma 3.1. *Let h^* be any multiplicative generalized cohomology theory and let $K = \Sigma Q \cup_f e^q$ for a given map f from S^{q-1} to a suspension of a space Q . Let x and y be the elements of $h^*(K)$ such that y corresponds to the generator of $h^*(S^q)$. Then*

$$x^2 = \pm \bar{H}_1^h(f) \cdot y \text{ in } h^*(K),$$

where $\pm \bar{H}_1^h$ is the composition $\rho^h \circ \lambda_2$ of the Boardman-Steer Hopf invariant $\lambda_2 : \pi_{q-1}(\Sigma Q) \rightarrow \pi_q(\Sigma^2 Q \wedge Q)$ (Boardman and Steer [2]) with the Hurewicz homomorphism $\rho^h : \pi_q(\Sigma^2 Q \wedge Q) \rightarrow h^{2r}(S^q) \cong h^{2r-q}$ given by $\rho^h(g) = \Sigma_*^{-q} g^*(i^*(x) \otimes i^*(x))$ ($i : \Sigma Q \rightarrow K$ is the inclusion).

Since $Q'_{2\ell+1,\ell}$ is $(4\ell+1)$ -connected and $\dim Q'_{2\ell+1,\ell} = 8\ell+2$, $Q'_{2\ell+1,\ell}$ has the homotopy type of the suspension. Hence, by [2], we have the equation

$$\lambda_2(\gamma) = \Sigma h_2^J(\gamma),$$

where h_2^J is the 2nd James Hopf invariant.

We consider the adjoint map $\text{ad}(\gamma) : S^{8\ell+5} \rightarrow \Omega Q_{2\ell+1,\ell}$ in

$$\Omega Q_{2\ell+1,\ell} \subset \Omega Q_{2\ell+2,\ell} \subset \Omega V_{2\ell+2,\ell},$$

where ΩX is a loop space of a space X and $V_{n,m} = Sp(n)/Sp(m)$. We recall from [10] that the homology group of $\Omega Sp(n)$ with coefficients in $\mathbb{Z}/2$ has the ring structure: $H_*(\Omega Sp(n); \mathbb{Z}/2) \cong \mathbb{Z}/2[u_2, u_6, \dots, u_{4n-2}]$, where $\deg u_{4i-2} = 4i-2$ for $1 \leq i \leq n$, and these generators satisfy the relations $u_{4j+2}Sq_2 = u_{2j}^2$ for $1 \leq j \leq n-1$, where Sq_* is the dual operation of the Steenrod operation. So, we have the homology of $\Omega V_{2\ell+2,\ell}$:

$$H_*(\Omega V_{2\ell+2,\ell}; \mathbb{Z}_2) \cong \mathbb{Z}_2[u_{4\ell+2}, u_{4\ell+6}, \dots, u_{8\ell+2}, u_{8\ell+6}]$$

with the relation

$$(3.1) \quad u_{8\ell+6}Sq_2 = u_{4\ell+2}^2.$$

Hence $(8\ell+6)$ -skeleton of $\Omega(V_{2\ell+2,\ell})$ has a cell decomposition:

$$(3.2) \quad (\Omega V_{2\ell+2,\ell})^{(8\ell+6)} \simeq (S^{4\ell+2} \cup e^{4\ell+6} \cup \dots \cup e^{8\ell+2}) \cup_{S^{4\ell+2}} (S^{4\ell+2} \cup_{[\iota_{4\ell+2}, \iota_{4\ell+2}]} e^{8\ell+4}) \cup e^{8\ell+6},$$

where $[\iota_{4\ell+2}, \iota_{4\ell+2}] : S^{8\ell+3} \rightarrow S^{4\ell+2}$ is the Whitehead product of two copies of identity map $\iota_{4\ell+2} : S^{4\ell+2} \rightarrow S^{4\ell+2}$. By the relation (3.1), we have

$$(3.3) \quad (\Omega V_{2\ell+2,\ell})^{(8\ell+6)} / (\Omega V_{2\ell+2,\ell})^{(8\ell+2)} = S^{8\ell+4} \cup_{\eta_{8\ell+4}} e^{8\ell+6},$$

where η_k is a $(k-2)$ -fold suspension of the Hopf map $\eta_2 : S^3 \rightarrow S^2$ for $k \geq 2$. By the cell decomposition:

$$\begin{aligned} \Omega Q_{2\ell+1,\ell} &\simeq (S^{4\ell+2} \cup e^{4\ell+6} \cup \dots \cup e^{8\ell+2}) \\ &\cup_{S^{4\ell+2}} (S^{4\ell+2} \cup_{[\iota_{4\ell+2}, \iota_{4\ell+2}]} e^{8\ell+4}) \\ &\cup (\text{cells in dimensions } \geq 8\ell+8) \end{aligned}$$

and

$$(\Omega Q_{2\ell+1,\ell})^{(8\ell+4)} = (\Omega Q_{2\ell+2,\ell})^{(8\ell+4)} = (\Omega V_{2\ell+2,\ell})^{(8\ell+4)},$$

we identify $\text{ad}(\gamma)$ with a map:

$$\text{ad}(\gamma) : S^{8\ell+5} \rightarrow (\Omega Q_{2\ell+1,\ell})^{(8\ell+4)} = (\Omega V_{2\ell+2,\ell})^{(8\ell+4)}.$$

In consideration of $(8\ell+6)$ -skeleton:

$$\begin{aligned} (\Omega Q_{2\ell+1,\ell})^{(8\ell+4)} \cup_{\text{ad}(\gamma)} e^{8\ell+6} &= (\Omega Q_{2\ell+2,\ell})^{(8\ell+6)} \\ &= (\Omega V_{2\ell+2,\ell})^{(8\ell+6)}, \end{aligned}$$

the attaching map of $(8\ell+6)$ -cell of (3.2) is equal to $\text{ad}(\gamma)$. So, we have the

following commutative diagram:

$$\begin{array}{ccc}
 S^{8\ell+6} & \xrightarrow{=} & S^{8\ell+6} \\
 \downarrow \Sigma \text{ad}(\gamma) & & \downarrow \Sigma \text{ad}(\gamma) \\
 (\Sigma\Omega\Sigma Q'_{2\ell+1,\ell})^{(8\ell+5)} & \xrightarrow{=} & (\Sigma\Omega V_{2\ell+2,\ell})^{(8\ell+5)} \\
 \downarrow = & & \downarrow \text{proj} \\
 (\Sigma J^2)^{(8\ell+5)} & & \\
 \downarrow \text{proj} & & \\
 (\Sigma J^2)^{(8\ell+5)} / (\Sigma J^1)^{(8\ell+4)} & \xrightarrow{=} & (\Sigma\Omega V_{2\ell+2,\ell})^{(8\ell+5)} / (\Sigma\Omega V_{2\ell+2,\ell})^{(8\ell+4)} \\
 \downarrow = & & \downarrow = \\
 \Sigma(Q'_{2\ell+1,\ell} \wedge Q'_{2\ell+1,\ell})^{(8\ell+5)} & \xrightarrow{=} & S^{8\ell+5} \\
 \downarrow \text{incl} & & \\
 \Sigma(Q'_{2\ell+1,\ell} \wedge Q'_{2\ell+1,\ell}), & &
 \end{array}$$

where J^k is the k -stage James reduced product of $Q'_{2\ell+1,\ell}$ and proj and incl are the projection and the inclusion, respectively. The left column is the definition of the 2nd James Hopf invariant and the right column is equal to $\eta_{8\ell+5} : S^{8\ell+6} \rightarrow S^{8\ell+5}$, by (3.3). Thus, we have

$$h_2^J(\gamma) = (\text{incl}) \circ \eta_{8\ell+5}.$$

And using Lemma 3.1, we obtain the relation $x_{4\ell+3}^2 = \varepsilon \cdot x_{8\ell+7} \in h^{8\ell+6}(Q_{2\ell+2,\ell})$. This completes the proof. \square

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